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ERROR ANALYSIS OF THE hp–VERSION UNDER NUMERICAL INTEGRATIONS FOR NON-CONSTANT COEFFICIENTS

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Abstract. In this paper we consider the hp–version to solve non-constant coefficients elliptic equations on a bounded, convex polygonal domain Ω in R^2. A family $G_p = \{I_m\}$ of numerical quadrature rules satisfying certain properties can be used for calculating the integrals. When the numerical quadrature rules $I_m \in G_p$ are used for computing the integrals in the stiffness matrix of the variational form we will give its variational form and derive an error estimate of $\|u - \widetilde{u}_p^h\|_{1,\Omega}$.

1. Introduction

Let Ω be a bounded, convex polygonal domain in R^2 with the boundary Γ. To solve non-constant coefficients elliptic equations with Dirichlet boundary conditions on Ω we consider the hp–version with a quasi-uniform mesh and uniform p. Let $\mathcal{M} = \{\mathcal{J}^h\}, h \geqq 0$ be a quasi-uniform, regular family of meshes $\mathcal{J}^h = \{\Omega_k^h\}$ defined on Ω, where Ω_k^h is a closed quadrilateral, and

$$(1.1) \qquad \max_{\Omega^h \in \mathcal{J}^h} \operatorname{diam}(\Omega^h) = h \quad \text{for all } \Omega^h, \mathcal{J}^h \in \mathcal{M}.$$

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Further we assume that for each $\Omega_k^h \in \mathcal{J}^h$ there exists an invertible mapping $T_k^h : \widehat{\Omega} \to \Omega_k^h$ with the following correspondence:

$$\widehat{x} \in \widehat{\Omega} \longleftrightarrow x = T_k^h(\widehat{x}) \in \Omega_k^h, \tag{1.2}$$

and

$$\widehat{t} \in U_p(\widehat{\Omega}) \longleftrightarrow t = \widehat{t} \circ (T_k^h)^{-1} \in U_p(\Omega_k^h), \tag{1.3}$$

where $\widehat{\Omega}$ denotes the reference elements $\widehat{I}^2 = [-1,1]^2$ in R^2,

$$\begin{aligned} &U_p(\widehat{\Omega}) \\ &= \{\widehat{t} : \widehat{t} \text{is a polynomial of degree} \leqq p \text{in each variable on} \widehat{\Omega}\}, \end{aligned} \tag{1.4}$$

and

$$U_p(\Omega_k^h) \;=\; \{ t \;:\; \widehat{t} = t \circ T_k^h \in U_p(\widehat{\Omega}) \}. \tag{1.5}$$

We now consider the following model problem of non-constants elliptic equations :
Find $u \in H_0^1(\Omega)$, such that

$$-\text{div}(a\nabla u) = f \quad \text{in} \quad \Omega \subset R^2, \tag{1.6}$$

where two functions a and f satisfy a compatibility condition to ensure a solution exists, and

$$H_0^1(\Omega) = \{u \in H^1(\Omega) \;:\; u \;\text{ vanishes on }\; \Gamma\}. \tag{1.7}$$

For the sake of simplicity, we assume that

$$0 < A_1 \leqq a(x) \leqq A_2 \quad \text{for all} \quad x \in \Omega, \text{and} \tag{1.8}$$

$$f \in L_2(\Omega). \tag{1.9}$$

In addition, we also assume that there exists a constant $M \geqq 1$ such that

$$\|T_k^h\|_{m,\infty,\widehat{\Omega}}, \;\; \|(T_k^h)^{-1}\|_{m,\infty,\Omega_k^h} \leqq A \quad \text{for} \quad 0 \leqq m \leqq M, \tag{1.10}$$

$$(1.11) \qquad \|\widehat{J}_k^h\|_{m,\infty,\widehat{\Omega}}, \ \|(\widehat{J}_k^h)^{-1}\|_{m,\infty,\Omega_k^h} \leqq A \quad \text{for} \quad 0 \leqq m \leqq M-1,$$

where $\widehat{J}_k^h$ and $(\widehat{J}_k^h)^{-1}$ denote the Jacobians of T_k^h and $(T_k^h)^{-1}$ respectively.

Then, as seen in [8, Theorem 3.1.2], we obtain the following correspondence:
For any $\alpha \in [1,\infty]$, $0 \leqq m \leqq M$,

$$(1.12) \qquad \widehat{t} \in W^{m,\alpha}(\widehat{\Omega}) \longleftrightarrow t = \widehat{t} \circ (T_k^h)^{-1} \in W^{m,\alpha}(\Omega_k^h)$$

with norm equivalence

$$(1.13) \qquad C_1 h^{(m-\frac{2}{\alpha})} \|t\|_{m,\alpha,\Omega_k^h} \leqq \|\widehat{t}\|_{m,\alpha,\widehat{\Omega}} \leqq C_2 h^{(m-\frac{2}{\alpha})} \|t\|_{m,\alpha,\Omega_k^h},$$

with the subscript α omitted when $\alpha = 2$. Namely, we have

$$(1.14) \qquad C_1 h^{(m-1)} \|t\|_{m,\Omega_k^h} \leqq \|\widehat{t}\|_{m,\widehat{\Omega}} \leqq C_2 h^{(m-1)} \|t\|_{m,\Omega_k^h}.$$

Let us define

$$(1.15) \qquad S_p^h(\Omega) = \{u \in H^1(\Omega) \ : \ u_{\Omega_k^h} \circ (T_k^h) \in U_p(\widehat{\Omega}) \text{ for all } \Omega_k^h \in \mathcal{J}^h\}$$

where $u_{\Omega_k^h}$ denotes the restriction of $u \in H^1(\Omega)$ to $\Omega_k^h \in \mathcal{J}^h$, and

$$(1.16) \qquad S_{p,0}^h(\Omega) = S_p^h(\Omega) \cap H_0^1(\Omega).$$

Then, using the hp–version of the finite element method with the mesh $\mathcal{J}^h = \{\Omega_k^h\}$ we obtain the following discrete variational form of (1.6): Find $u_p^h \in S_{p,0}^h(\Omega)$ satisfying

$$(1.17) \qquad B(u_p^h, v_p^h) = (f, v_p^h)_\Omega \quad \text{for all} \quad v_p^h \in S_{p,0}^h(\Omega),$$

where

$$(1.18) \qquad B(u,v) = \int_\Omega a \nabla u \cdot \nabla v \, dx,$$

the usual inner product

$$(1.19) \qquad (f,v)_\Omega = \int_\Omega f v \, dx.$$

In [6], I. Babuška and M. Suri already obtained the following optimal estimate for the hp−version: For any $u \in H_0^\sigma(\Omega)(\sigma \geqq 1)$ we have

$$\|u - u_p^h\|_{1,\Omega} \leqq C\,p^{-(\sigma-1)} h^{min(p,\sigma-1)} \|u\|_{k,\Omega}, \tag{1.20}$$

where C is independent of u, h, and p [but depends on Ω and σ].

The above optimal result follows under the assumption that all integrations are performed exactly. In practice, to compute the integrals in the variational formulation of the discrete problem we need the numerical quadrature rule scheme. The integrals are seldom computed exactly. In this paper we consider a family $G_p = \{I_m\}$ of numerical quadrature rules satisfying certain properties, which can be used for calculating the integrals in the stiffness matrix of (1.17). Under the numerical quadrature rules we will give its variational form and derive an error estimate of $\|u - \widetilde{u}_p^h\|_{1,\Omega}$ where $\widetilde{u}_p^h$ is an approximation satisfying (2.6).

Let us now give some approximation results which will be used later.

Lemma 1.1. For each integer $l \geqq 0$, there exists a sequence of projections
$\Pi_p^l : H^l(\widehat{\Omega}) \to U_p(\widehat{\Omega}), \quad p = 1, 2, 3, \cdots$ such that

$$\Pi_p^l \widehat{v}_p = \widehat{v}_p \quad \text{for all} \quad \widehat{v}_p \in U_p(\widehat{\Omega}), \tag{1.21}$$

$$\begin{aligned} &\|\widehat{u} - \Pi_p^l \widehat{u}\|_{s,\widehat{\Omega}} \leqq C\,p^{-(r-s)} \|\widehat{u}\|_{r,\widehat{\Omega}} \quad \text{for all} \quad \widehat{u} \in H^r(\widehat{\Omega}) \\ &0 \leqq s \leqq l \leqq r. \end{aligned} \tag{1.22}$$

Proof. See [9, Lemma 3.1].

Lemma 1.2. Let $\widehat{u} \in H^r(\widehat{\Omega})$ with $r \geqq 2$. Then the projection Π_p^2 from Lemma 1.1 satisfies

$$\|\widehat{u} - \Pi_p^2 \widehat{u}\|_{0,\infty,\widehat{\Omega}} \leqq C\,p^{-(r-1)} \|\widehat{u}\|_{r,\widehat{\Omega}}\,. \tag{1.23}$$

Proof. By interpolation results (see [9, Theorem 3.2] and [7, Theorem 6.2.4]) we have that for $0 < \varepsilon \leqq \frac{1}{2}$,

$$\|\widehat{u} - \Pi_p^2 \widehat{u}\|_{0,\infty,\widehat{\Omega}} \leqq C \|\widehat{u} - \Pi_p^2 \widehat{u}\|_{1+\varepsilon,\widehat{\Omega}}^{\frac{1}{2}} \|\widehat{u} - \Pi_p^2 \widehat{u}\|_{1-\varepsilon,\widehat{\Omega}}^{\frac{1}{2}} . \tag{1.24}$$

We also have from Lemma 1.1 that

$$\|\widehat{u} - \Pi_p^2 \widehat{u}\|_{r,\widehat{\Omega}} \leqq C p^{-(s-r)} \|\widehat{u}\|_{s,\widehat{\Omega}} \text{for} 0 \leqq r \leqq 2 \leqq s. \tag{1.25}$$

Hence, taking $r = 1 + \varepsilon$ and $r = 1 - \varepsilon$ in (1.25) we obtain

$$\|\widehat{u} - \Pi_p^2 \widehat{u}\|_{1+\varepsilon,\widehat{\Omega}}^{\frac{1}{2}} \|\widehat{u} - \Pi_p^2 \widehat{u}\|_{1-\varepsilon,\widehat{\Omega}}^{\frac{1}{2}} \leqq C p^{-(s-1)} \|\widehat{u}\|_{s,\widehat{\Omega}}, \tag{1.26}$$

which completes the proof from (1.24).

Lemma 1.3. Suppose that $T_k^h : \widehat{\Omega} \longrightarrow \Omega_k^h$ is an invertible affine mapping. Then for any $u \in H^\sigma(\Omega), \sigma \geqq 0$ we have

$$\inf_{\widehat{v} \in U_p(\widehat{\Omega})} \|\widehat{u}_{\Omega_k^h} - \widehat{v}\|_{\sigma,\widehat{\Omega}} \leqq C h^\mu \|u_{\Omega_k^h}\|_{\sigma,\Omega_k^h}, \tag{1.27}$$

where $\mu = \min(p, \sigma - 1)$ and C is independent of h, p and u.

Proof. The proof is given in [6].

Lemma 1.4. For each $u \in H^\sigma(\Omega)$ and $\Omega_k^h \in \mathcal{J}^h$ there exists a sequence $z_p^h \in U_p(\Omega_k^h), p = 1, 2, \cdots$ such that for any $0 \leqq r \leqq \sigma$

$$\|u_{\Omega_k^h} - z_p^h\|_{r,\Omega_k^h} \leqq C h^{(\mu - r + 1)} p^{-(\sigma - r)} \|u_{\Omega_k^h}\|_{\sigma,\Omega_k^h} \text{ for all } \Omega_k^h \in \mathcal{J}^h, \tag{1.28}$$

where $\mu = \min(p, \sigma - 1)$ and C is independent of h, p and u.

Proof. See [6, Lemma 4.5].

2. $hp-$version under numerical quadrature rules

We consider numerical quadrature rules I_m defined on the reference element $\widehat{\Omega}$ by

$$(2.1)\qquad I_m(\widehat{f}) = \sum_{i=1}^{n(m)} \widehat{w}_i^m \, \widehat{f}(\widehat{x}_i^m) \sim \int_{\widehat{\Omega}} \widehat{f}(\widehat{x})\, d\widehat{x},$$

where m is a positive integer. Let $G_p = \{I_m\}$ be a family of quadrature rules I_m with respect to $U_p(\widehat{\Omega})$, $p = 1, 2, 3, \cdots$, satisfying the following properties : For each $I_m \in G_p$,

(K1) $\widehat{w}_i^m > 0$ and $\widehat{x}_i^m \in \widehat{\Omega}$ for $i = 1, \cdots, n(m)$.

(K2) $I_m(\widehat{f}^{\,2}) \leqq C_1 \|\widehat{f}\|^2_{0,\widehat{\Omega}}$ for all $\widehat{f} \in U_p(\widehat{\Omega})$.

(K3) $C_2 \|\widetilde{f}\|^2_{0,\widehat{\Omega}} \leqq I_m(\widetilde{f}^{\,2})$ for all $\widetilde{f} \in \widetilde{U}_p(\widehat{\Omega})$,
where $\widetilde{U}_p(\widehat{\Omega}) = \{ \frac{\partial \widehat{f}}{\partial \widehat{x}_i} \; : \; \widehat{f} \in U_p(\widehat{\Omega}) \} \subset U_p(\widehat{\Omega})$.

(K4) $I_m(\widehat{f}) = \int_{\widehat{\Omega}} \widehat{f}(\widehat{x})\, d\widehat{x}$ for all $\widehat{f} \in U_{d(m)}(\widehat{\Omega})$,
where $d(m) \geqq \widetilde{d}(p) > 0$.

We also get a family $G_{p,\Omega} = \{I_{m,\Omega}\}$ of numerical quadrature rules with respect to $S_p^h(\Omega)$, defined by

$$(2.2)\qquad \begin{aligned} I_{m,\Omega_k^h}(f_{\Omega_k^h}) &= \sum_{j=1}^{n(m)} w_j^k f_{\Omega_k^h}(x_j^m) = \sum_{j=1}^{n(m)} \widehat{w}_j^m \widehat{J}_k^h(\widehat{x}_j^m)(f_{\Omega_k^h} \circ T_k^h)(\widehat{x}_j^m) \\ &= I_m(\widehat{J}_k^h \, \widehat{f}_{\Omega_k^h}) \end{aligned}$$

and

$$(2.3)\qquad I_{m,\Omega}(f) = \sum_{\Omega_k^h \in \mathcal{J}^h} I_{m,\Omega_k^h}(f_{\Omega_k^h}).$$

In particular, one may be interested in Gauss-Legendre(G-L) quadrature rules. Let L_q denote the cross-products of $q-$point G-L rules along the $\widehat{x}_1$ and $\widehat{x}_2$ axes on $\widehat{\Omega} = \widehat{I} \times \widehat{I}$, given by

$$L_q(\widehat{f}) = \sum_{i=1}^{q} \sum_{j=1}^{q} \widehat{w}_i^q \, \widehat{w}_j^q \, \widehat{f}(\widehat{x}_{ij}^q) \text{for all} \widehat{f} \in L_2(\widehat{\Omega}),$$

where $\widehat{x}_{ij}^q = (\widehat{x}_i^q, \widehat{x}_j^q) \in \widehat{\Omega} = \widehat{I} \times \widehat{I}$ with the weights $\widehat{w}_i^q$ and $\widehat{w}_j^q$. We consider a family $\{L_q\}_{q \geqq l(p)}$ of G-L quadrature rules with respect to $U_p(\widehat{\Omega})$ such that $l(p) = p+1$. Then, $\{L_q\}_{q \geqq l(p)}$ satisfy the properties $(K1)-(K4)$. In fact, when $q \geqq p+1$ $L_q(\widehat{f})$ is exact for all $\widehat{f} \in U_{d(q)}(\widehat{\Omega})$ with $d(q) \geqq 2p+1 > 0$, so that (K2) and (K3) hold with $C_1 = C_2 = 1$.

Here, one may employ numerical quadrature rules schemes for computing the integrals in the discrete variational form (1.17). Especially, since the model problem (1.6) is a non-constant coefficients elliptic problem the numerical quadrature rules $I_m \in G_p$ can be used for calculating the integrals in the stiffness matrix. Thus, we denote by DF the 2×2 Jacobian matrix of $F : R^2 \to R^2$, and define two discrete inner products

$$(2.4)\quad (u,v)_{m,\Omega_k^h} = I_{m,\Omega_k^h}\big((uv)_{\Omega_k^h}\big) = I_m\big(\widehat{J_k^h}\widehat{(uv)_{\Omega_k^h}}\big) \text{ on } \Omega_k^h \in \mathcal{J}^h,$$

$$(2.5)\quad (u,v)_{m,\Omega} = \sum_{\Omega_k^h \in \mathcal{J}^h} (u,v)_{m,\Omega_k^h} \text{ on } \Omega.$$

Then, under the assumption that all integrations in the load vector of (1.17) are performed exactly, using the quadrature rules $I_m \in G_p$ for computing the integrals in the stiffness matrix of (1.17) we obtain the following actual problem of (1.17):
Find $\widetilde{u}_p^h \in S_{p,0}^h(\Omega)$, such that

$$(2.6)\qquad B_{m,\Omega}(\widetilde{u}_p^h, v_p^h) = (f, v_p^h)_\Omega \text{ for all } v_p^h \in S_{p,0}^h(\Omega),$$

where

$$B_{m,\Omega}(\widetilde{u}_p^h, v_p^h) = I_{m,\Omega}(a\nabla\widetilde{u}_p^h \cdot \nabla v_p^h)$$
$$= \sum_{\Omega_k^h \in \mathcal{J}^h} I_{m,\Omega_k^h}\big(a_{\Omega_k^h}\nabla(\widetilde{u}_p^h)_{\Omega_k^h} \cdot \nabla(v_p^h)_{\Omega_k^h}\big)$$
$$= \sum_{\Omega_k^h \in \mathcal{J}^h} I_m\Big(\widehat{J_k^h}\widehat{a_{\Omega_k^h}}\Big[\big(\widehat{DT_k^{h^{-1}}}\big)^t\big(\nabla\widehat{(\widetilde{u}_p^h)_{\Omega_k^h}}\big)\Big]^t\Big[\big(\widehat{DT_k^{h^{-1}}}\big)^t\big(\nabla\widehat{(v_p^h)_{\Omega_k^h}}\big)\Big]\Big).$$

Here, if we let $\left(DT_k^{h^{-1}}\right)\left(DT_k^{h^{-1}}\right)^t = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then $\widehat{(a_{ij})_{\Omega_k^h}} = \widehat{J_k^h}\widehat{(b_{ij})_{\Omega_k^h}}$ are the entries of the matrix $\widehat{J_k^h}\left(\widehat{DT_k^{h^{-1}}}\right)\left(\widehat{DT_k^{h^{-1}}}\right)^t$. For the simplicity of notation, if the restrictions $\widehat{a_{\Omega_k^h}}$, $\widehat{(a_{ij})_{\Omega_k^h}}$, $\widehat{(\widetilde{u}_p^h)_{\Omega_k^h}}$ and $\widehat{(v_p^h)_{\Omega_k^h}}$ are simply denoted by $\widehat{a}$, $\widehat{a_{ij}}$, $\widehat{\widetilde{u}_p^h}$ and $\widehat{v_p^h}$ respectively, then we have

$$
\begin{aligned}
&B_{m,\Omega}(\widetilde{u}_p^h, v_p^h) \\
&= \sum_{\Omega_k^h \in \mathcal{J}^h} I_m\left(\widehat{J_k^h}\widehat{a_{\Omega_k^h}}\left(\nabla\widehat{(\widetilde{u}_p^h)_{\Omega_k^h}}\right)^t\left(\widehat{DT_k^{h^{-1}}}\right)\left(\widehat{DT_k^{h^{-1}}}\right)^t\left(\nabla\widehat{(v_p^h)_{\Omega_k^h}}\right)\right) \\
&= \sum_{\Omega_k^h \in \mathcal{J}^h} I_m\left(\widehat{a}\begin{pmatrix} \frac{\partial\widehat{\widetilde{u}_p^h}}{\partial\widehat{x}_1} \\ \frac{\partial\widehat{\widetilde{u}_p^h}}{\partial\widehat{x}_2} \end{pmatrix}^t \begin{pmatrix} \widehat{a_{11}} & \widehat{a_{12}} \\ \widehat{a_{21}} & \widehat{a_{22}} \end{pmatrix} \begin{pmatrix} \frac{\partial\widehat{v_p^h}}{\partial\widehat{x}_1} \\ \frac{\partial\widehat{v_p^h}}{\partial\widehat{x}_2} \end{pmatrix}\right) \qquad (2.7) \\
&= \sum_{\Omega_k^h \in \mathcal{J}^h} \sum_{i,j=1}^{2} \left(\widehat{a}\widehat{a_{ij}} \frac{\partial\widehat{\widetilde{u}_p^h}}{\partial\widehat{x}_i} \frac{\partial\widehat{v_p^h}}{\partial\widehat{x}_j}\right)_{m,\widehat{\Omega}} \\
&= \sum_{\Omega_k^h \in \mathcal{J}^h} \sum_{i,j=1}^{2} \left(\widehat{a}\widehat{a_{ij}} \frac{\partial\widehat{\widetilde{u}_p^h}}{\partial\widehat{x}_i}, \frac{\partial\widehat{v_p^h}}{\partial\widehat{x}_j}\right)_{m,\widehat{\Omega}}.
\end{aligned}
$$

3. main results

Let us now derive an estimate of the error $\|u - \widetilde{u}_p^h\|_{1,\Omega}$ for the $hp-$ version under numerical quadrature rules I_m. In fact, $\|u - \widetilde{u}_p^h\|_{1,\Omega}$ depends on two separate terms. The first dependence is on the error $\|u - u_p^h\|_{1,\Omega}$ given in (1.20). Next, the error will depend upon the smoothness of a. We will start with the following lemma.

Lemma 3.1. Let u be the exact solution of (1.6) and u_p^h that of (1.17). Let $\widetilde{u}_p^h$ be an approximate solution of u which satisfies a discrete variational form (2.6). Then there exists a constant C independent of

m such that

(3.1)

$$\|u-\tilde{u}_p^h\|_{1,\Omega} \leqq C \inf_{v_p^h\in S_{p,0}^h(\Omega)} \{\|u-v_p^h\|_{1,\Omega} + \sup_{w_p^h\in S_{p,0}^h(\Omega)} \frac{|B(u_p^h,w_p^h)-B_{m,\Omega}(v_p^h,w_p^h)|}{\|w_p^h\|_{1,\Omega}}\}.$$

Proof. Let v_p^h be an arbitrary element in $S_{p,0}^h(\Omega)$. Then we have

$$\|u-\tilde{u}_p^h\|_{1,\Omega} \leqq \|u-v_p^h\|_{1,\Omega} + \|v_p^h-\tilde{u}_p^h\|_{1,\Omega}. \tag{3.2}$$

From the ellipticity of $B_{m,\Omega}(\cdot,\cdot)$, for a constant $C_1>0$

$$\begin{aligned} C_1\|v_p^h-\tilde{u}_p^h\|_{1,\Omega}^2 &\leqq B_{m,\Omega}(v_p^h-\tilde{u}_p^h, v_p^h-\tilde{u}_p^h) \\ &= |B_{m,\Omega}(v_p^h, v_p^h-\tilde{u}_p^h) - (f, v_p^h-\tilde{u}_p^h)| \\ &= |B_{m,\Omega}(v_p^h, v_p^h-\tilde{u}_p^h) - B(u_p^h, v_p^h-\tilde{u}_p^h)|. \end{aligned} \tag{3.3}$$

Hence, taking the infimum with respect to $v_p^h\in S_{p,0}^h(\Omega)$ we have

(3.4)

$$\|u-\tilde{u}_p^h\|_{1,\Omega} \leqq C \inf_{v_p^h\in S_{p,0}^h(\Omega)} \{\|u-v_p^h\|_{1,\Omega} + \frac{|B(u_p^h,v_p^h-\tilde{u}_p^h)-B_{m,\Omega}(v_p^h,v_p^h-\tilde{u}_p^h)|}{\|v_p^h-\tilde{u}_p^h\|_{1,\Omega}}\}.$$

The proof is completed by taking $w_p^h = v_p^h-\tilde{u}_p^h \in S_{p,0}^h(\Omega)$.

Lemma 3.2. Let $\widehat{u_p}, \widehat{w_p} \in U_p(\widehat{\Omega})$ and $\widehat{g}\in L_\infty(\widehat{\Omega})$. Then, for all $\widehat{v_q^1}, \widehat{v_q^2} \in U_q(\widehat{\Omega})$, $\widehat{g_r}\in U_r(\widehat{\Omega})$ with $0<q\leqq p$ and $r=d(m)-p-q>0$ we have

$$\begin{aligned} &|(\widehat{g}\,\widehat{u_p},\widehat{u_p})_{\widehat{\Omega}} - (\widehat{g}\,\widehat{u_p},\widehat{u_p})_{m,\widehat{\Omega}}| \\ &\quad\leqq C\{\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}}\|\widehat{u_p}-\widehat{v_q^1}\|_{0,\widehat{\Omega}}\|\widehat{u_p}-\widehat{v_q^2}\|_{0,\widehat{\Omega}} \\ &\qquad + \|\widehat{g}-\widehat{g_r}\|_{0,\infty,\widehat{\Omega}}\|\widehat{u_p}\|_{0,\widehat{\Omega}}\,\|\widehat{u_p}\|_{0,\widehat{\Omega}}\}, \end{aligned} \tag{3.5}$$

where C is independent of p,q and m.

Proof. For any $\widehat{g_r} \in U_r(\widehat{\Omega})$ we have

(3.6)

$$
\begin{aligned}
&|\,(\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}\,| \\
&\qquad \leqq \,|\,(\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}}\,| + |\,(\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}\,| \\
&\qquad\qquad + \,|\,(\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}} - (\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}\,|\,.
\end{aligned}
$$

Thank to (K4),

$$
\begin{aligned}
(3.7)\quad (\widehat{g_r}\widehat{v_q^1}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{v_q^1}, \widehat{u_p})_{m,\widehat{\Omega}} &= 0 \text{ for any } \widehat{v_q^1} \in U_q(\widehat{\Omega}), \text{ and} \\
(\widehat{g_r}\widehat{u_p}, \widehat{v_q^2})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{v_q^2})_{m,\widehat{\Omega}} &= 0 \text{ for any } \widehat{v_q^2} \in U_q(\widehat{\Omega}).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|\,(\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}\,| \\
(3.8)\qquad &\leqq \,|\,(\widehat{g_r}\widehat{u_p}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}}\,| \\
&\qquad + \,|\,(\widehat{g_r}\widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}}\,|.
\end{aligned}
$$

By the Schwarz inequality we obtain

$$
\begin{aligned}
&|\,(\widehat{g_r}\widehat{u_p}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}}\,| \\
(3.9)\qquad &\leqq \,(\widehat{g_r}(\widehat{u_p} - \widehat{v_q^1}), \widehat{g_r}(\widehat{u_p} - \widehat{v_q^1}))_{\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p} - \widehat{v_q^2}, \widehat{u_p} - \widehat{v_q^2})_{\widehat{\Omega}}^{\frac{1}{2}} \\
&\leqq \,C\,\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^1}\|_{0,\widehat{\Omega}}\,\|\widehat{u_p} - \widehat{v_q^2}\|_{0,\widehat{\Omega}}\,.
\end{aligned}
$$

Also, from (K2) we have

$$
\begin{aligned}
&|\,(\widehat{g_r}\widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}} \\
&\leqq (\widehat{g_r}(\widehat{u_p} - \widehat{v_q^1}), \widehat{g_r}(\widehat{u_p} - \widehat{v_q^1}))_{m,\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p} - \widehat{v_q^2}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}}^{\frac{1}{2}} \\
(3.10)\qquad &\leqq C\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}} (\widehat{u_p} - \widehat{v_q^1}, \widehat{u_p} - \widehat{v_q^1})_{m,\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p} - \widehat{v_q^2}, \widehat{u_p} - \widehat{v_q^2})_{m,\widehat{\Omega}}^{\frac{1}{2}} \\
&\leqq C\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^1}\|_{0,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^2}\|_{0,\widehat{\Omega}}\,.
\end{aligned}
$$

Hence, combining (3.9) and (3.10) we have

$$
\begin{aligned}
&|\,(\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g_r}\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}\,| \\
(3.11)\qquad &\leqq \,C\,\|\widehat{g_r}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^1}\|_{0,\widehat{\Omega}} \|\widehat{u_p} - \widehat{v_q^2}\|_{0,\widehat{\Omega}}\,.
\end{aligned}
$$

Similarly, since $\widehat{g} \in L_\infty(\widehat{\Omega})$ we obtain

$$\begin{aligned}(3.12)\qquad & |(\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}} - (\widehat{g}_r\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}}| \\ & \leqq ((\widehat{g} - \widehat{g}_r)\widehat{u_p}, (\widehat{g} - \widehat{g}_r)\widehat{u_p})_{\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p}, \widehat{u_p})_{\widehat{\Omega}}^{\frac{1}{2}} \\ & \leqq C\,\|\widehat{g} - \widehat{g}_r\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p}\|_{0,\widehat{\Omega}} \|\widehat{u_p}\|_{0,\widehat{\Omega}}\,,\end{aligned}$$

and

$$\begin{aligned}(3.13)\qquad & |(\widehat{g}_r\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}} - (\widehat{g}\,\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}| \\ & \leqq ((\widehat{g}_r - f)\widehat{u_p}, (\widehat{g}_r - \widehat{g})\widehat{u_p})_{m,\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}^{\frac{1}{2}} \\ & \leqq C\,\|\widehat{g}_r - \widehat{g}\|_{0,\infty,\widehat{\Omega}} (\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}^{\frac{1}{2}} (\widehat{u_p}, \widehat{u_p})_{m,\widehat{\Omega}}^{\frac{1}{2}} \\ & \leqq C\,\|\widehat{g}_r - \widehat{g}\|_{0,\infty,\widehat{\Omega}} \|\widehat{u_p}\|_{0,\widehat{\Omega}} \|\widehat{u_p}\|_{0,\widehat{\Omega}}\,.\end{aligned}$$

The lemma follows from (3.11), (3.12), (3.13) and (3.6).

As seen in Lemma 3.1, the last dependence of $\|u - \widetilde{u}_p^h\|_{1,\Omega}$ is on the smoothness of a. In this connection, we let

$$(3.14)\qquad M_{p,q} = \max_{\Omega_k^h \in \mathcal{J}^h} \max_{i,j} \|\widehat{a}\widehat{a_{ij}}\|_{p,q,\widehat{\Omega}}\,,$$

where the subscript q will be omitted when $q = 2$. Then, we obtain the following results which give an estimate for the last term of the right side in (3.1).

Lemma 3.3. Let $I_m \in G_p$ be a quadrature rule defined on $\widehat{\Omega} \subset R^2$, which satisfies $d(m) - p - 1 > 0$. Let $u \in H^\sigma(\Omega)$, $a \in H^\alpha(\Omega)$ and $\widehat{a}_{ij} \in H^\rho(\widehat{\Omega})$ for $i, j = 1, 2$, such that $\lambda = \min(\alpha, \rho) \geqq 2$. Then, for any $w_p^h \in S_{p,0}^h(\Omega)$ and an approximation u_p^h which satisfies (1.17) we have

(3.15)

$$\begin{aligned}& \frac{|\,B(u_p^h, w_p^h) - B_{m,\Omega}(u_p^h, w_p^h)\,|}{\|w_p^h\|_{1,\Omega}} \\ & \leqq C\{(r^{-(\lambda-1)}M_\lambda + M_{0,\infty})(\|u - u_p^h\|_{1,\Omega} + q^{-(\sigma-1)}h^{(\sigma-1)}\|u\|_{\sigma,\Omega}) \\ & \qquad + r^{-(\lambda-1)}M_\lambda\|u\|_{1,\Omega}\},\end{aligned}$$

where q is a positive integer such that $0 < q \leqq p$ and $r = d(m)-p-q > 0$.

Proof. For arbitrary $w_p^h \in S_{p,0}^h(\Omega)$ we have

$$\begin{aligned}(3.16)\quad & |\, B(u_p^h, w_p^h) - B_{m,\Omega}(u_p^h, w_p^h)\,| \\ & \leqq C \max_{\Omega_k^h \in \mathcal{J}^h} \max_{i,j} \left| \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a}_{ij} \frac{\partial \widehat{u}_p^h}{\partial \widehat{x}_i}, \frac{\partial \widehat{w}_p^h}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \right|.\end{aligned}$$

For any $\widehat{a}_{ij}$ $i,j = 1,2$ and $\Omega_k^h \in \mathcal{J}^h$ we let q be any integer such that $0 < q \leqq p$ and $r = d(m) - p - q > 0$. Then since $\widehat{a}\,\widehat{a}_{ij} \in L_\infty(\widehat{\Omega})$, due to Lemma 3.2 with $\widehat{v}_q^1 = \frac{\partial}{\partial \widehat{x}_i}(\Pi_q^1 \widehat{u}_p^h)$ $\widehat{v}_q^2 = \frac{\partial \Pi_q^1 \widehat{w_p^h}}{\partial \widehat{x}_j} \in U_q(\widehat{\Omega})$ and $\widehat{g}_r = \Pi_r^2(\widehat{a}\,\widehat{a}_{ij})$, we have

$$\begin{aligned}(3.17)\quad & \left| \left(\widehat{a}\,\widehat{a_{ij}} \frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i}, \frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j} \right)_{\widehat{\Omega}} - \left(\widehat{a}\,\widehat{a_{ij}} \frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i}, \frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j} \right)_{m,\widehat{\Omega}} \right| \\ & \leqq C\Big\{ \|\Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} \Big\| \frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i} - \frac{\partial \Pi_q^1 \widehat{u}}{\partial \widehat{x}_i} \Big\|_{0,\widehat{\Omega}} \Big\| \frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j} - \frac{\partial \Pi_q^1 \widehat{w_p^h}}{\partial \widehat{x}_j} \Big\|_{0,\widehat{\Omega}} \\ & + \|\widehat{a}\,\widehat{a_{ij}} - \Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} \Big\| \frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i} \Big\|_{0,\widehat{\Omega}} \Big\| \frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j} \Big\|_{0,\widehat{\Omega}} \Big\}.\end{aligned}$$

Since $\widehat{a}\,\widehat{a_{ij}} \in H^\lambda(\widehat{\Omega})$ with $\lambda = \min(\alpha, \rho) \geqq 2$ we obtain from Lemma 1.2 and (1.14) that

$$\begin{aligned}(3.18)\quad & \|\widehat{a}\,\widehat{a_{ij}} - \Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}} \Big\| \frac{\partial \widehat{u_p^h}}{\partial \widehat{x}_i} \Big\|_{0,\widehat{\Omega}} \Big\| \frac{\partial \widehat{w_p^h}}{\partial \widehat{x}_j} \Big\|_{0,\widehat{\Omega}} \\ & \leqq C\, r^{-(\lambda-1)} \|\widehat{a}\,\widehat{a_{ij}}\|_{\lambda,\widehat{\Omega}} (\|\widehat{u} - \widehat{u_p^h}\|_{1,\widehat{\Omega}} + \|\widehat{u}\|_{1,\widehat{\Omega}}) \|\widehat{w_p^h}\|_{1,\widehat{\Omega}} \\ & \leqq C\, r^{-(\lambda-1)} M_\lambda (\|u - u_p^h\|_{1,\Omega_k^h} + \|u\|_{1,\Omega_k^h}) \|w_p^h\|_{1,\Omega_k^h} \,.\end{aligned}$$

Further, it follows from Lemma 1.1, Lemma 1.2 and (1.14) that

$$
\begin{aligned}
&\|\Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}}\,\Big\|\frac{\partial\widehat{u_p^h}}{\partial\widehat{x}_i}-\frac{\partial\Pi_q^1\widehat{u}}{\partial\widehat{x}_i}\Big\|_{0,\widehat{\Omega}}\Big\|\frac{\partial\widehat{w_p^h}}{\partial\widehat{x}_j}-\frac{\partial\Pi_q^1\widehat{w_p^h}}{\partial\widehat{x}_j}\Big\|_{0,\widehat{\Omega}}\\
&\leqq C\left\{\|\widehat{a}\,\widehat{a_{ij}}-\Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}}\right.\\
&\qquad\left.+\|\widehat{a}\,\widehat{a_{ij}}\|_{0,\infty,\widehat{\Omega}}\right\}\|\widehat{u_p^h}-\Pi_q^1\widehat{u}\|_{1,\widehat{\Omega}}\|\widehat{w_p^h}-\Pi_q^1\widehat{w_p^h}\|_{1,\widehat{\Omega}}\\
&\leqq C\left\{\|\widehat{a}\,\widehat{a_{ij}}-\Pi_r^2(\widehat{a}\,\widehat{a_{ij}})\|_{0,\infty,\widehat{\Omega}}+M_{0,\infty}\right\}\left\{\|\widehat{u}-\widehat{u_p^h}\|_{1,\widehat{\Omega}}\right.\\
&\qquad\left.+\|\widehat{u}-\Pi_q^1\widehat{u}\|_{1,\widehat{\Omega}}\right\}\|\widehat{w_p^h}-\Pi_q^1\widehat{w_p^h}\|_{1,\widehat{\Omega}}\\
&\leqq C\left\{r^{-(\lambda-1)}\|\widehat{a}\,\widehat{a_{ij}}\|_{\lambda,\widehat{\Omega}}+M_{0,\infty}\right\}\left\{\|\widehat{u}-\widehat{u_p^h}\|_{1,\widehat{\Omega}}\right.\\
&\qquad\left.+q^{-(\sigma-1)}\|\widehat{u}\|_{\sigma,\widehat{\Omega}}\right\}\|\widehat{w_p^h}\|_{1,\widehat{\Omega}}\\
&\leqq C\left\{r^{-(\lambda-1)}M_\lambda+M_{0,\infty}\right\}\left\{\|u-u_p^h\|_{1,\Omega_k^h}\right.\\
&\qquad\left.+q^{-(\sigma-1)}h^{(\sigma-1)}\|u\|_{\sigma,\Omega_k^h}\right\}\|w_p^h\|_{1,\Omega_k^h},
\end{aligned}
\tag{3.19}
$$

where C is independent of p and q.

Thus, substituting (3.18) and (3.19) in (3.17) we have

(3.20)

$$
\begin{aligned}
&|\,B(u_p^h,w_p^h)-B_{m,\Omega}(u_p^h,w_p^h)\,|\\
&\leqq C\sum_{\Omega_k^h\in\mathcal{J}^h}\max_{i,j}\Big|\left(\widehat{a}\,\widehat{a_{ij}}\frac{\partial\widehat{u_p^h}}{\partial\widehat{x}_i},\frac{\partial\widehat{w_p^h}}{\partial\widehat{x}_j}\right)_{\widehat{\Omega}}-\left(\widehat{a}\,\widehat{a_{ij}}\frac{\partial\widehat{u_p^h}}{\partial\widehat{x}_i},\frac{\partial\widehat{w_p^h}}{\partial\widehat{x}_j}\right)_{m,\widehat{\Omega}}\Big|\\
&\leqq C\sum_{\Omega_k^h\in\mathcal{J}^h}\left\{(r^{-(\lambda-1)}M_\lambda+M_{0,\infty})(\|u-u_p^h\|_{1,\Omega_k^h}\right.\\
&\qquad+q^{-(\sigma-1)}h^{(\sigma-1)}\|u\|_{\sigma,\Omega_k^h})\\
&\qquad\left.+r^{-(\lambda-1)}M_\lambda(\|u-u_p^h\|_{1,\Omega_k^h}+\|u\|_{1,\Omega_k^h})\right\}\|w_p^h\|_{1,\Omega_k^h}\\
&\leqq C\left\{(r^{-(\lambda-1)}M_\lambda+M_{0,\infty})(\|u-u_p^h\|_{1,\Omega}+q^{-(\sigma-1)}h^{(\sigma-1)}\|u\|_{\sigma,\Omega})\right.\\
&\qquad\left.+r^{-(\lambda-1)}M_\lambda\|u\|_{1,\Omega}\right\}\|w_p^h\|_{1,\Omega}\,.
\end{aligned}
$$

The lemma follows from dividing by $\|w_p^h\|_{1,\Omega}$.

By a direct application of Lemma 3.3 and (1.20) to Lemma 3.1 we obtain the following main theorem which gives an asymptotic, $H^1(\Omega)$-norm estimate for the rate of convergence under numerical quadrature rules.

Theorem 3.4. *Let $I_m \in G_p$ be a quadrature rule defined on $\widehat{\Omega} \subset R^2$, which satisfies $d(m) - p - 1 > 0$. We assume that $u \in H^\sigma(\Omega), a \in H^\alpha(\Omega)$ and $\widehat{a}_{ij} \in H^\rho(\widehat{\Omega})$ for $i, j = 1, 2$ such that $\lambda = \min(\alpha, \rho) \geqq 2$. Then, for any positive integer q such that $0 < q \leqq p$, we have*

(3.21)

$$\|u - \widetilde{u}_p^h\|_{1,\Omega} \leqq C\{(r^{-(\lambda-1)}M_\lambda + M_{0,\infty})q^{-(\sigma-1)}h^\mu\|u\|_{\sigma,\Omega} + r^{-(\lambda-1)}M_\lambda\|u\|_{1,\Omega}\},$$

where $\mu = \min(p, \sigma - 1)$ and $r = d(m) - p - q$.

Proof. Taking $v_p^h \in S_{p,0}^h(\Omega)$ with an approximation u_p^h of u which satisfies (1.17), we obtain from Lemma 3.1 that

$$\|u - \widetilde{u}_p^h\|_{1,\Omega} \leqq C\{\|u - u_p^h\|_{1,\Omega} + \sup_{w_p^h \in S_{p,0}^h(\Omega)} \frac{|B(u_p^h, w_p^h) - B_{m,\Omega}(u_p^h, w_p^h)|}{\|w_p^h\|_{1,\Omega}}\}. \tag{3.22}$$

Since $0 < q \leqq p$ it follows from (1.20) and Lemma 3.3 that the first term of the right side in (3.22) is dominated by its last term. Hence, the proof is completed by a direct application of Lemma 3.3 to (3.22).

We see from Theorem 3.4 that the rate of convergence is essentially given by

$$O((d(m) - p - q)^{-(\lambda-1)}q^{-(\sigma-1)}h^{\min(p,\sigma-1)} + q^{-(\sigma-1)}h^{\min(p,\sigma-1)} + (d(m) - p - q)^{-(\lambda-1)}). \tag{3.23}$$

If m is large enough with $q = p$, then the rate of convergence is asymptotically $O(p^{-(\sigma-1)}h^{min(p,\sigma-1)})$, which coincides with that of (1.20). Further, when λ is large enough (that is, a and $\widehat{a_{ij}}$ are sufficiently smooth), even when $d(m) \approx 2p+1$ with $q = p$ the second term in (3.23) may dominate, so that the rate of convergence is asymptotically $O(p^{-(\sigma-1)}h^{min(p,\sigma-1)})$. More precisely, in G-L quadrature rules, using I_m with $(p+1)$-point rules we would obtain an asymptotic rate $O(p^{-(\sigma-1)}h^{\min(p,\sigma-1)})$. But, when a and $\widehat{a_{ij}}$ are not smooth enough, the second term $q^{-(\sigma-1)}h^{\min(p,\sigma-1)}$ may be dominated by the other term of (3.23). In this situation, using an overintegration with a sufficiently large m we may reduce the error $\|u - \widetilde{u}_p^h\|_{1,\Omega}$ until the second term dominates again. In practice, when a and $\widehat{a_{ij}}$ are not smooth we may increase the value of $d(m)$ with $q \approx p$.

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