Honam Mathematical J. **27** (2005), No. 3, pp. 353–367

INTUITIONISTIC FUZZY $(1,2)$-IDEALS OF SEMIGROUPS

Young Bae Jun, Eun Hwan Roh* and Seok Zun Song

Abstract. Some properties of the intuitionistic fuzzy $(1,2)$-ideal is considered. Characterizations of an intuitionistic fuzzy $(1,2)$-ideal are given. We show that every intuitionistic fuzzy $(1,2)$-ideal in a group is constant. Using a chain of $(1,2)$-ideals of a semigroup S, an intuitionistic fuzzy $(1,2)$-ideal of S is established.

1. Introduction

After the introduction of fuzzy sets by Zadeh [7], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [1] is one among them. In [4], Kuroki gave some properties of fuzzy ideals and fuzzy bi-ideals in semigroups. The concept of $(1,2)$-ideals in semigroups was introduced by S. Lajos [5]. Jun and Lajos [2, 6] considered the fuzzification of $(1,2)$-ideals in semigroups. In this paper, we investigate further properties of intuitionistic fuzzy $(1,2)$-ideal. We give characterizations of an intuitionistic fuzzy $(1,2)$-ideal of a semigroup S. Using a chain of $(1,2)$-ideals of a semigroup S, we establish an intuitionistic fuzzy $(1,2)$-ideal of S. We show that every intuitionistic fuzzy $(1,2)$-ideal in a group is constant.

Received May 24, 2005. Accepted August 2, 2005.

2000 Mathematics Subject Classification : 20M12, 03F55, 03E72.

Key words and phrases : $(1,2)$-ideal, intuitionistic fuzzy $(1,2)$-ideal.

*Corresponding author. Tel.: +82 55 740 1232.

2. Preliminaries

Let S be a semigroup. By a *subsemigroup* of S we mean a non-empty subset A of S such that $A^2 \subseteq A$, and by a *left* (*right*) *ideal* of S we mean a non-empty subset A of S such that $SA \subseteq A$ ($AS \subseteq A$). By *two-sided ideal* or simply *ideal*, we mean a non-empty subset of S which is both a left and a right ideal of S. A subsemigroup A of a semigroup S is called a *bi-ideal* of S if $ASA \subseteq A$. A subsemigroup A of S is called a $(1,2)$-*ideal* of S if $ASA^2 \subseteq A$. A semigroup S is said to be $(2,2)$-*regular* if $x \in x^2Sx^2$ for any $x \in S$. A semigroup S is said to be *regular* if, for each $x \in S$, there exists $y \in S$ such that $x = xyx$. A semigroup S is said to be *completely regular* if, for each $x \in S$, there exists $y \in S$ such that $x = xyx$ and $xy = yx$. For a semigroup S, note that S is completely regular if and only if S is a union of groups if and only if S is $(2,2)$-regular. A semigroup S is said to be *left* (resp. *right*) *duo* if every left (resp. right) ideal of S is a two-sided ideal of S

A function $\mu : X \to [0,1]$ is called *a fuzzy set* in a set X, and the complement of μ, denoted by $\bar{\mu}$, is the fuzzy set in X given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$. An *intuitionistic fuzzy set* (IFS for short) A in X (see [1]) is an object having the form

$$A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A, respectively, and

$$0 \le \mu_A(x) + \gamma_A(x) \le 1$$

for each $x \in X$. For the sake of simplicity, we shall use the notation $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ instead of $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle \mid x \in X\}$.

Definition 2.1. [3] An IFS $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ in a semigroup S is called an *intuitionistic fuzzy subsemigroup* of S if

(i) $(\forall x, y \in S)\ (\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\})$,

(ii) $(\forall x, y \in S)\ (\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\})$.

Definition 2.2. [3] An IFS $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ in a semigroup S is called an *intuitionistic fuzzy left ideal* of S if $\mu_A(xy) \geq \mu_A(y)$ and $\gamma_A(xy) \leq \gamma_A(y)$ for all $x, y \in S$. An *intuitionistic fuzzy right ideal* of a semigroup S is defined in an analogous way. An IFS $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ in a semigroup S is called an *intuitionistic fuzzy ideal* of S if it is both an intuitionistic fuzzy right and an intuitionistic fuzzy left ideal of S.

Note that every intuitionistic fuzzy left (right) ideal of a semigroup S is an intuitionistic fuzzy subsemigroup of S.

Definition 2.3. [3] An intuitionistic fuzzy subsemigroup $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ of a semigroup S is called an *intuitionistic fuzzy bi-ideal* of S if

(i) $(\forall w, x, y \in S)\ (\mu_A(xwy) \geq \min\{\mu_A(x), \mu_A(y)\})$,

(ii) $(\forall w, x, y \in S)\ (\gamma_A(xwy) \leq \max\{\gamma_A(x), \gamma_A(y)\})$.

Note that every intuitionistic fuzzy left (right) ideal of a semigroup S is an intuitionistic fuzzy bi-ideal of S.

3. Intuitionistic fuzzy $(1, 2)$-ideals

In what follows, let S denote a semigroup unless otherwise specified.

Definition 3.1. [3] An intuitionistic fuzzy subsemigroup $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ of S is called an *intuitionistic fuzzy* $(1, 2)$*-ideal* of S if

(i) $(\forall w, x, y, z \in S)\ (\mu_A(xw(yz)) \geq \min\{\mu_A(x), \mu_A(y), \mu_A(z)\})$,

(ii) $(\forall w, x, y, z \in S)\ (\gamma_A(xw(yz)) \leq \max\{\gamma_A(x), \gamma_A(y), \gamma_A(z)\})$.

Example 3.2. (1) Let $S = \{a, b, c, d, e\}$ be a semigroup with the following Cayley table

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	e
e	a	d	e	a	a

An IFS $A = \langle x, \mu_A(x), \gamma_A(x) \rangle$ in S given by

$$A = \left\langle x, \left(\tfrac{a}{1}, \tfrac{b}{1}, \tfrac{c}{1}, \tfrac{d}{0}, \tfrac{e}{0}\right), \left(\tfrac{a}{0}, \tfrac{b}{0}, \tfrac{c}{0}, \tfrac{d}{1}, \tfrac{e}{1}\right) \right\rangle$$

is an intuitionistic fuzzy $(1, 2)$-ideal of S (see [3]).

(2) Let $S = \{a, b, c, d, e\}$ be a semigroup with the following Cayley table

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	c	c	e

An IFS $B = \langle x, \mu_B(x), \gamma_B(x) \rangle$ in S given by

$$B = \left\langle x, \left(\tfrac{a}{0.6}, \tfrac{b}{0.5}, \tfrac{c}{0.4}, \tfrac{d}{0.3}, \tfrac{e}{0.3}\right), \left(\tfrac{a}{0.3}, \tfrac{b}{0.3}, \tfrac{c}{0.4}, \tfrac{d}{0.5}, \tfrac{e}{0.6}\right) \right\rangle$$

is an intuitionistic fuzzy $(1, 2)$-ideal of S (see [3]).

Note that every intuitionistic fuzzy left (resp. right, bi-) ideal is an intuitionistic fuzzy $(1, 2)$-ideal (see [3]). The following example shows that an intuitionistic fuzzy $(1, 2)$-ideal is not an intuitionistic fuzzy bi-ideal in general.

Example 3.3. Let $S := \{a, b, c, x, y, z\}$ be the semigroup with Cayley table as follows:

$\cdot$	a	b	c	x	y	z
a	a	a	a	a	a	a
b	a	a	a	a	a	a
c	a	a	a	a	a	a
x	a	a	a	a	a	b
y	a	a	a	a	b	c
z	a	a	b	a	x	a

Note that $M := \{a, z\}$ is a $(1,2)$-ideal of S. Let $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ be an IFS in S given by

$$A = \left\langle x, \left(\tfrac{a}{m}, \tfrac{b}{0}, \tfrac{c}{0}, \tfrac{x}{0}, \tfrac{y}{0}, \tfrac{z}{m}\right), \left(\tfrac{a}{n}, \tfrac{b}{1}, \tfrac{c}{1}, \tfrac{x}{1}, \tfrac{y}{1}, \tfrac{z}{n}\right)\right\rangle,$$

where $m, n \in (0,1)$ with $m + n \leq 1$. Then $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is an intuitionistic fuzzy $(1,2)$-ideal of S but it is not an intuitionistic fuzzy bi-ideal of S since

$$\mu_A(zyz) = \mu_A(b) = 0 < m = \min\{\mu_A(z), \mu_A(z)\},$$

$$\gamma_A(zyz) = \gamma_A(b) = 1 > n = \max\{\gamma_A(z), \gamma_A(z)\}.$$

Theorem 3.4. *If $\{A_i : i \in \Lambda\}$ is an arbitrary family of intuitionistic fuzzy $(1,2)$-ideals of S, then $\cap A_i$ is an intuitionistic fuzzy $(1,2)$-ideal of S, where $\cap A_i = \{(x, \wedge\mu_{A_i}(x), \vee\gamma_{A_i}(x)) \mid x \in S\}$.*

Proof. Let $w, x, y, z \in S$. Then

$$\wedge\mu_{A_i}(xy) \geq \bigwedge\{\min\{\mu_{A_i}(x), \mu_{A_i}(y)\}\} = \min\{\wedge\mu_{A_i}(x), \wedge\mu_{A_i}(y)\},$$

$$\vee\gamma_{A_i}(xy) \leq \bigvee\{\max\{\gamma_{A_i}(x), \gamma_{A_i}(y)\}\} = \max\{\vee\gamma_{A_i}(x), \vee\gamma_{A_i}(y)\},$$

$$\begin{aligned}\wedge\mu_{A_i}(xw(yz)) &\geq \wedge\{\min\{\mu_{A_i}(x), \mu_{A_i}(y), \mu_{A_i}(z)\}\} \\ &= \min\{\wedge\mu_{A_i}(x), \wedge\mu_{A_i}(y), \wedge\mu_{A_i}(y)\},\end{aligned}$$

$$\begin{aligned}\vee\gamma_{A_i}(xw(yz)) &\leq \vee\{\max\{\gamma_{A_i}(x), \gamma_{A_i}(y), \gamma_{A_i}(z)\}\} \\ &= \max\{\vee\gamma_{A_i}(x), \vee\gamma_{A_i}(y), \vee\gamma_{A_i}(z)\}.\end{aligned}$$

Hence $\cap A_i$ is an intuitionistic fuzzy $(1,2)$-ideal of S. $\square$

Let $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ be an IFS in S and let $\alpha, \beta \in [0,1]$. Then the sets

$$U(\mu_A;\alpha) := \{x \in S \mid \mu_A(x) \geq \alpha\},\ L(\gamma_A;\beta) := \{x \in S \mid \gamma_A(x) \leq \beta\}$$

are called a *μ-level set* and a *γ-level set* of A, respectively.

Theorem 3.5. *An IFS $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ in S is an intuitionistic fuzzy $(1,2)$-ideal of S if and only if μ_A and $\bar{\gamma}_A$ are fuzzy $(1,2)$-ideals of S.*

Proof. If $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is an intuitionistic fuzzy $(1,2)$-ideal of S, then clearly μ_A is a fuzzy $(1,2)$-ideal of S and for any $w, x, y, z \in S$, we have

$$\begin{aligned}\bar{\gamma}_A(xy) &= 1 - \gamma_A(xy) \geq 1 - \max\{\gamma_A(x), \gamma_A(y)\}\\ &= \min\{1 - \gamma_A(x), 1 - \gamma_A(y)\} = \min\{\bar{\gamma}_A(x), \bar{\gamma}_A(y)\},\end{aligned}$$

and

$$\begin{aligned}\bar{\gamma}_A(xw(yz)) &= 1 - \gamma_A(xw(yz)) \geq 1 - \max\{\gamma_A(x), \gamma_A(y), \gamma_A(z)\}\\ &= \min\{1 - \gamma_A(x), 1 - \gamma_A(y), 1 - \gamma_A(z)\} = \min\{\bar{\gamma}_A(x), \bar{\gamma}_A(y), \bar{\gamma}_A(z)\}.\end{aligned}$$

Hence $\bar{\gamma}_A$ is a fuzzy $(1,2)$-ideal of S. The converse is straightforward. □

Theorem 3.6. *An IFS $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ in S is an intuitionistic fuzzy $(1,2)$-ideal of S if and only if $\Box A$ and $\Diamond A$ are intuitionistic fuzzy $(1,2)$-ideals of S, where*

$$\Box A = \{\langle x, \mu_A(x), \bar{\mu}_A(x)\rangle \mid x \in S\},\ \Diamond A = \{\langle x, \bar{\gamma}_A(x), \gamma_A(x)\rangle \mid x \in S\}.$$

Proof. Assume that $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is an intuitionistic fuzzy $(1,2)$-ideal of S. Since $\mu_A = \bar{\bar{\mu}}_A$ and $\bar{\gamma}_A$ are fuzzy $(1,2)$-ideal of S, it follows from Theorem 3.5 that $\Box A$ and $\Diamond A$ are intuitionistic fuzzy $(1,2)$-ideals of S. The converse is straightforward. □

Theorem 3.7. *If S is a group, then every intuitionistic fuzzy $(1,2)$-ideal of S is a constant function.*

Proof. Let $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ be an intuitionistic fuzzy $(1,2)$-ideal of S and denote by e the identity of S. Then for any $x \in S$, we have

$$\begin{aligned}\mu_A(x) &= \mu_A(ex(ee)) \geq \min\{\mu_A(e), \mu_A(e), \mu_A(e)\}\\ &= \mu_A(e) = \mu_A(ee) = \mu_A((xx^{-1})((x^2)^{-1}x^2))\\ &= \mu_A(x(x^{-1}(x^2)^{-1})(xx))\\ &\geq \min\{\mu_A(x), \mu_A(x), \mu_A(x)\} = \mu_A(x),\end{aligned}$$

$$\begin{aligned}\gamma_A(x) &= \gamma_A(ex(ee)) \leq \max\{\gamma_A(e), \gamma_A(e), \gamma_A(e)\}\\ &= \gamma_A(e) = \gamma_A(ee) = \gamma_A((xx^{-1})((x^2)^{-1}x^2))\\ &= \gamma_A(x(x^{-1}(x^2)^{-1})(xx))\\ &\leq \max\{\gamma_A(x), \gamma_A(x), \gamma_A(x)\} = \gamma_A(x).\end{aligned}$$

These show that $\mu_A(e) = \mu_A(x)$ and $\gamma_A(e) = \gamma_A(x)$ for all $x \in S$. Hence $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is constant. □

Theorem 3.8. *If an IFS $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ in S is an intuitionistic fuzzy $(1,2)$-ideal of S, then the μ-level set $U(\mu_A;\alpha)$ and γ-level set $L(\gamma_A;\beta)$ of A are $(1,2)$-ideals of S for every $\alpha, \beta \in \mathrm{Im}(\mu_A) \cap \mathrm{Im}(\gamma_A) \subseteq [0,1]$ with $\alpha + \beta \leq 1$.*

Proof. Let $\alpha, \beta \in \mathrm{Im}(\mu_A) \cap \mathrm{Im}(\gamma_A) \subseteq [0,1]$ with $\alpha + \beta \leq 1$. Let $x, y \in U(\mu_A;\alpha)$. Then $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$. It follows that

$$\mu_A(xy) \geq \min\{\mu_A(x),\ \mu_A(y)\} \geq \alpha$$

so that $xy \in U(\mu_A;\alpha)$. If $x, y \in L(\gamma_A;\beta)$, then $\gamma_A(x) \leq \beta$ and $\gamma_A(y) \leq \beta$, and so

$$\gamma_A(xy) \leq \max\{\gamma_A(x),\ \gamma_A(y)\} \leq \beta, \text{ i.e., } xy \in L(\gamma_A;\beta).$$

Hence $U(\mu_A;\alpha)$ and $L(\gamma_A;\beta)$ are subsemigroups of S. Let $w \in S$ and $x, y, z \in U(\mu_A;\alpha)$. Then $\mu_A(x) \geq \alpha$, $\mu_A(y) \geq \alpha$, and $\mu_A(z) \geq \alpha$. Using Definition 3.1(i), we get

$$\mu_A(xw(yz)) \geq \min\{\mu_A(x), \mu_A(y), \mu_A(z)\},$$

and thus $xw(yz) \in U(\mu_A;\alpha)$, i.e., $U(\mu_A;\alpha)SU(\mu_A;\alpha)^2 \subseteq U(\mu_A;\alpha)$. Finally let $w \in S$ and $a,b,c \in L(\gamma_A;\beta)$. Then $\gamma_A(a) \leq \beta$, $\gamma_A(b) \leq \beta$, and $\gamma_A(c) \leq \beta$, and so

$$\gamma_A(aw(bc)) \leq \max\{\gamma_A(a), \gamma_A(b), \gamma_A(c)\} \leq \beta$$

by Definition 3.1(ii). It follows that $aw(bc) \in L(\gamma_A;\beta)$, i.e.,

$$L(\gamma_A;\beta)SL(\gamma_A;\beta)^2 \subseteq L(\gamma_A;\beta).$$

This completes the proof. □

We call $U(\mu_A;\alpha)$ (resp. $L(\gamma_A;\beta)$) is called the *μ-level $(1,2)$-ideal* (resp. *γ-level $(1,2)$-ideal*).

Corollary 3.9. *Let a be a fixed element of S. If an IFS $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is an intuitionistic fuzzy $(1,2)$-ideal of S, then the sets*

$$[\mu > a] := \{x \in S \mid \mu_A(x) \geq \mu_A(a)\}$$

and

$$[\gamma < a] := \{x \in S \mid \gamma_A(x) \leq \gamma_A(a)\}$$

are $(1,2)$-ideals of S.

Proof. Straightforward. □

Theorem 3.10. *Let $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ be an IFS in S such that the nonempty sets $U(\mu_A;\alpha)$ and $L(\gamma_A;\beta)$ are $(1,2)$-ideals of S for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Then $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is an intuitionistic fuzzy $(1,2)$-ideal of S.*

Proof. Assume that Definition 2.1(i) is false. Then there exists $x_0, y_0 \in S$ such that $\mu_A(x_0y_0) < \min\{\mu_A(x_0), \mu_A(y_0)\}$. Taking

$$\alpha_0 := \frac{1}{2}(\mu_A(x_0y_0) + \min\{\mu_A(x_0), \mu_A(y_0)\}),$$

we get $\mu_A(x_0y_0) < \alpha_0 < \min\{\mu_A(x_0), \mu_A(y_0)\}$. Hence $x_0, y_0 \in U(\mu_A; \alpha_0)$ and $x_0y_0 \notin U(\mu_A; \alpha_0)$, which is a contradiction. Therefore Definition 2.1(i) is valid. Now suppose that there exists $a_0, b_0 \in S$ such that

$$\gamma_A(a_0b_0) > \max\{\gamma_A(a_0), \gamma_A(b_0)\}.$$

If we put $\beta_0 := \frac{1}{2}(\gamma_A(a_0b_0)+\max\{\gamma_A(a_0), \gamma_A(b_0)\})$, then $a_0, b_0 \in L(\gamma_A; \beta_0)$ but $a_0b_0 \notin L(\gamma_A; \beta_0)$, a contradiction. Thus Definition 2.1(ii) holds. Therefore $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is an intuitionistic fuzzy subsemigroup of S. Suppose that Definition 3.1(i) is not valid. Then

$$\mu_A(x_0w_0(y_0z_0)) < \min\{\mu_A(x_0), \mu_A(y_0), \mu_A(z_0)\}$$

for some $x_0, y_0, z_0, w_0 \in S$, and so $x_0, y_0, z_0 \in U(\mu_A; \alpha)$ and $x_0w_0(y_0z_0)) \notin U(\mu_A; \alpha)$ for every $\alpha \in [0, 1]$ with

$$\mu_A(x_0w_0(y_0z_0)) < \alpha < \min\{\mu_A(x - 0), \mu_A(y_0), \mu_A(z_0)\}.$$

This is a contradiction. Finally assume that Definition 3.1(ii) is false. Then there exists $a_0, b_0, c_0, u_0 \in S$ such that

$$\gamma_A(a_0u_0(b_0c_0)) > \max\{\gamma_A(a_0), \gamma_A(b_0), \gamma_A(c_0)\}.$$

Taking $\beta := \frac{1}{2}(\gamma_A(a_0u_0(b_0c_0)) + \max\{\gamma_A(a_0), \gamma_A(b_0), \gamma_A(c_0)\})$, we have a_0, b_0, $c_0 \in L(\gamma_A; \beta)$ but $a_0u_0(b_0c_0) \notin L(\gamma_A; \beta)$. This is a contradiction. Consequently, $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is an intuitionistic fuzzy $(1, 2)$-ideal of S. □

Theorem 3.11. *Let M be a $(1, 2)$-ideal of S and let $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ be an IFS in S defined by*

$$\mu_A(x) := \begin{cases} \alpha_0 & \text{if } x \in M, \\ \alpha_1 & \text{otherwise,} \end{cases} \qquad \gamma_A(x) := \begin{cases} \beta_0 & \text{if } x \in M, \\ \beta_1 & \text{otherwise,} \end{cases}$$

for all $x \in S$ and $\alpha_i, \beta_i \in [0, 1]$ such that $\alpha_0 > \alpha_1$, $\beta_0 < \beta_1$ and $\alpha_i+\beta_i \leq 1$ for $i = 0, 1$. Then $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is an intuitionistic fuzzy $(1, 2)$-ideal of S and $U(\mu_A; \alpha_0) = M = L(\gamma_A; \beta_0)$.

Proof. Let $x, y \in S$. If anyone of x and y does not belong to M, then

$$\mu_A(xy) \geq \alpha_1 = \min\{\mu_A(x), \mu_A(y)\},$$

$$\gamma_A(xy) \leq \beta_1 = \max\{\gamma_A(x), \gamma_A(y)\}.$$

Other cases are trivial, and we omit the proof. Hence $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is an intuitionistic fuzzy subsemigroup of S. Now let $w, x, y, z \in S$. If anyone of x, y and z does not belong to M, then anyone of $\mu_A(x)$, $\mu_A(y)$ and $\mu_A(z)$ is equal to α_1; and anyone of $\gamma_A(x)$, $\gamma_A(y)$ and $\gamma_A(z)$ is equal to β_1. It follows that

$$\mu_A(xw(yz)) \geq \alpha_1 = \min\{\mu_A(x), \mu_A(y), \mu_A(z)\},$$

$$\gamma_A(xw(yz)) \leq \beta_1 = \max\{\gamma_A(x), \gamma_A(y), \gamma_A(z)\}.$$

Assume that $x, y, z \in M$. Then $xw(yz) \in M$ because M is a $(1,2)$-ideal of S. Hence

$$\mu_A(xw(yz)) = \min\{\mu_A(x), \mu_A(y), \mu_A(z)\},$$

$$\gamma_A(xw(yz)) = \max\{\gamma_A(x), \gamma_A(y), \gamma_A(z)\}.$$

Therefore $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is an intuitionistic fuzzy $(1,2)$-ideal of S and obviously $U(\mu_A; \alpha_0) = M = L(\gamma_A; \beta_0)$. □

Theorem 3.11 suggest that any $(1,2)$-ideal of S can be realized as μ- and γ-level $(1,2)$-ideals of some intuitionistic fuzzy $(1,2)$-ideal of S. We now consider the converse of Theorem 3.11.

Theorem 3.12. *Let M be a nonempty subset of S. If an IFS $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ in S which is given in Theorem 3.11 is an intuitionistic fuzzy $(1,2)$-ideal of S, then M is a $(1,2)$-ideal of S.*

Proof. Let $x, y \in M$. Then $\mu_A(x) = \mu_A(y) = \alpha_0$ and $\gamma_A(x) = \gamma_A(y) = \beta_0$. Thus

$$\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y)\} = \alpha_0,$$

$$\gamma_A(xy) \leq \max\{\gamma_A(x), \gamma_A(y)\} = \beta_0,$$

and so $xy \in U(\mu_A; \alpha_0) \cap L(\gamma_A; \beta_0) = M$. Hence M is a subsemigroup of S. Now let $w \in S$ and $x, y, z \in M$. Then $\mu_A(x) = \mu_A(y) = \mu_A(z) = \alpha_0$ and $\gamma_A(x) = \gamma_A(y) = \gamma_A(z) = \beta_0$. It follows from Definition 3.1 that

$$\mu_A(xw(yz)) \geq \min\{\mu_A(x), \mu_A(y), \mu_A(z)\} = \alpha_0,$$

$$\gamma_A(xw(yz)) \leq \max\{\gamma_A(x), \gamma_A(y), \gamma_A(z)\} = \beta_0$$

so that $\mu_A(xw(yz)) = \alpha_0$ and $\gamma_A(xw(yz)) = \beta_0$. Therefore

$$xw(yz) \in U(\mu_A; \alpha_0) \cap L(\gamma_A; \beta_0) = M,$$

and consequently M is a $(1, 2)$-ideal of S. □

Theorem 3.13. *Consider a chain of $(1, 2)$-ideals of S*

$$P_0 \subset P_1 \subset \cdots \subset P_n = S,$$

where $\subset$ denotes proper inclusion. Then there exists an intuitionistic fuzzy $(1, 2)$-ideal of S whose μ- and γ-level $(1, 2)$-ideals are exactly the $(1, 2)$-ideals in the above chain.

Proof. Let $\{\alpha_k \mid k = 0, 1, \cdots, n\}$ (resp. $\{\beta_k \mid k = 0, 1, \cdots, n\}$) be a finite decreasing (resp. increasing) sequence in $[0, 1]$ such that $\alpha_i + \beta_i \leq 1$ for $i = 0, 1, \cdots, n$. Let $A = \langle x, \mu_A(x), \gamma_A(x) \rangle$ be an IFS in S defined by $\mu_A(P_0) = \alpha_0$, $\gamma_A(P_0) = \beta_0$, $\mu_A(P_k \setminus P_{k-1}) = \alpha_k$ and $\gamma_A(P_k \setminus P_{k-1}) = \beta_k$ for $0 < k \leq n$. Let $x, y \in S$. If $x, y \in P_k \setminus P_{k-1}$, then $xy \in P_k$, $\mu_A(x) = \alpha_k = \mu_A(y)$ and $\gamma_A(x) = \beta_k = \gamma_A(y)$. It follows that

$$\mu_A(xy) \geq \alpha_k = \min\{\mu_A(x),\ \mu_A(y)\}$$

and

$$\gamma_A(xy) \leq \beta_k = \max\{\gamma_A(x),\ \gamma_A(y)\}.$$

For $i > j$, if $x \in P_i \setminus P_{i-1}$ and $y \in P_j \setminus P_{j-1}$, then $\mu_A(x) = \alpha_i < \alpha_j = \mu_A(y)$, $\gamma_A(x) = \beta_i > \beta_j = \gamma_A(y)$ and $xy \in P_i$. Hence

$$\mu_A(xy) \geq \alpha_i = \min\{\mu_A(x),\ \mu_A(y)\}$$

and

$$\gamma_A(xy) \leq \beta_i = \max\{\gamma_A(x),\, \gamma_A(y)\}.$$

Thus $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is an intuitionistic fuzzy subsemigroup of S. Now let $w, x, y, z \in S$. If $x, y, z \in P_k \setminus P_{k-1}$, then $xw(yz) \in P_k$, $\mu_A(x) = \mu_A(y) = \mu_A(z) = \alpha_k$ and $\gamma_A(x) = \gamma_A(y) = \gamma_A(z) = \beta_k$. Hence

$$\mu_A(xw(yz)) \geq \alpha_k = \min\{\mu_A(x),\, \mu_A(y),\, \mu_A(z)\}$$

and

$$\gamma_A(xw(yz)) \leq \beta_k = \max\{\gamma_A(x),\, \gamma_A(y),\, \gamma_A(z)\}.$$

For $i > j > k$, if $x \in P_i \setminus P_{i-1}$, $y \in P_j \setminus P_{j-1}$, and $z \in P_k \setminus P_{k-1}$, then $\mu_A(x) = \alpha_i < \alpha_j = \mu_A(y) < \alpha_k = \mu_A(z)$, $\gamma_A(x) = \beta_i > \beta_j = \gamma_A(y) > \beta_k = \gamma_A(z)$, and $xw(yz) \in P_i$. Hence

$$\mu_A(xw(yz)) \geq \alpha_i = \min\{\mu_A(x),\, \mu_A(y),\, \mu_A(z)\}$$

and

$$\gamma_A(xw(yz)) \leq \beta_i = \max\{\gamma_A(x),\, \gamma_A(y),\, \gamma_A(z)\}.$$

Therefore $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ is an intuitionistic fuzzy $(1,2)$-ideal of S. Note that $\mathrm{Im}(\mu_A) = \{\alpha_0, \alpha_1, \cdots, \alpha_n\}$ and $\mathrm{Im}(\gamma_A) = \{\beta_0, \beta_1, \cdots, \beta_n\}$. It follows that the μ-level $(1,2)$-ideals and the γ-level $(1,2)$-ideals of $A = \langle x, \mu_A(x), \gamma_A(x)\rangle$ are given by the chain of $(1,2)$-ideals

$$U(\mu_A; \alpha_0) \subset U(\mu_A; \alpha_1) \subset \cdots \subset U(\mu_A; \alpha_n) = S$$

and

$$L(\gamma_A; \beta_0) \subset L(\gamma_A; \beta_1) \subset \cdots \subset L(\gamma_A; \beta_n) = S,$$

respectively. Obviously, we have

$$U(\mu_A; \alpha_0) = \{x \in S \mid \mu_A(x) \geq \alpha_0\} = P_0,$$

$$L(\gamma_A; \beta_0) = \{x \in S \mid \gamma_A(x) \leq \beta_0\} = P_0.$$

We now prove that $U(\mu_A; \alpha_k) = P_k = L(\gamma_A; \beta_k)$ for $0 < k \leq n$. Clearly $P_k \subseteq U(\mu_A; \alpha_k)$ and $P_k \subseteq L(\gamma_A; \beta_k)$. If $x \in U(\mu_A; \alpha_k)$, then $\mu_A(x) \geq \alpha_k$ and so $x \notin P_i$ for $i > k$. Hence $\mu_A(x) \in \{\alpha_1, \alpha_2, \cdots, \alpha_k\}$, which implies $x \in P_j$ for some $j \leq k$. Since $P_j \subseteq P_k$, it follows that $x \in P_k$. Consequently, $U(\mu_A; \alpha_k) = P_k$ for $0 \leq k \leq n$. Now if $y \in L(\gamma_A; \beta_k)$, then $\gamma_A(y) \leq \beta_k$ and thus $y \notin P_i$ for $i > k$. Hence $\gamma_A(y) \in \{\beta_1, \beta_2, \cdots, \beta_k\}$ and so $y \in P_j$ for some $j \leq k$. Since $P_j \subseteq P_k$, we have $y \in P_k$. Therefore $L(\gamma_A; \beta_k) = P_k$ for $0 \leq k \leq n$. This completes the proof. □

Theorem 3.14. *Let $\{M_\alpha \mid \alpha \in \Lambda \subseteq [0, \frac{1}{2}]\}$ be a collection of $(1,2)$-ideals of S such that $X = \bigcup_{\alpha \in \Lambda} M_\alpha$, and for every $\alpha, \beta \in \Lambda$, $\alpha < \beta$ if and only if $M_\beta \subset M_\alpha$. Then an IFS $A = \langle x, \mu_A(x), \gamma_A(x) \rangle$ in S defined by*

$$\mu_A(x) = \bigvee \{\alpha \in \Lambda \mid x \in M_\alpha\} \text{ and } \gamma_A(x) = \bigwedge \{\alpha \in \Lambda \mid x \in M_\alpha\}$$

for all $x \in S$ is an intuitionistic fuzzy $(1,2)$-ideal of S.

Proof. According to Theorem 3.10, it is sufficient to show that the nonempty sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \beta)$ are $(1,2)$-ideals of S for every $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. In order to show that $U(\mu_A; \alpha)$ is a $(1,2)$-ideal, we divide into the following two cases:

$$\text{(i) } \alpha = \bigvee \{\delta \in \Lambda \mid \delta < \alpha\} \text{ and (ii) } \alpha \neq \bigvee \{\delta \in \Lambda \mid \delta < \alpha\}.$$

Case (i) implies that

$$\begin{aligned} x \in U(\mu_A; \alpha) &\Leftrightarrow x \in M_\delta \text{ for all } \delta < \alpha \\ &\Leftrightarrow x \in \bigcap_{\delta < \alpha} M_\delta, \end{aligned}$$

so that $U(\mu_A; \alpha) = \bigcap_{\delta < \alpha} M_\delta$, which is a $(1,2)$-ideal of S. For the case (ii), we claim that $U(\mu_A; \alpha) = \bigcup_{\delta \geq \alpha} M_\delta$. If $x \in \bigcup_{\delta \geq \alpha} M_\delta$, then $x \in M_\delta$ for some $\delta \geq \alpha$. It follows that $\mu_A(x) \geq \delta \geq \alpha$ so that $x \in U(\mu_A; \alpha)$. This proves that $\bigcup_{\delta \geq \alpha} M_\delta \subset U(\mu_A; \alpha)$. Now assume that $x \notin \bigcup_{\delta \geq \alpha} M_\delta$. Then $x \notin M_\delta$ for all $\delta \geq \alpha$. Since $\alpha \neq \bigvee \{\delta \in \Lambda \mid \delta < \alpha\}$, there exists $\varepsilon > 0$ such that $(\alpha - \varepsilon, \alpha) \cap \Lambda = \emptyset$. Hence $x \notin M_\delta$ for all $\delta > \alpha - \varepsilon$, which means that if $x \in M_\delta$ then $\delta \leq \alpha - \varepsilon$. Thus $\mu_A(x) \leq \alpha - \varepsilon < \alpha$, and so $x \notin U(\mu_A; \alpha)$. Therefore $U(\mu_A; \alpha) = \bigcup_{\delta \geq \alpha} M_\delta$. Next we show that

$L(\gamma_A;\beta)$ is a $(1,2)$-ideal of S for all $\beta \in [\gamma_A(0),1]$. We consider the following two cases:

$$\text{(iii) } \beta = \bigwedge\{\delta \in \Lambda \mid \beta < \delta\} \text{ and (iv) } \beta \neq \bigwedge\{\delta \in \Lambda \mid \beta < \delta\}.$$

For the case (iii) we have

$$\begin{aligned} x \in L(\gamma_A;\beta) &\Leftrightarrow x \in M_\delta \text{ for all } \beta < \delta \\ &\Leftrightarrow x \in \bigcap_{\beta<\delta} M_\delta, \end{aligned}$$

and hence $L(\gamma_A;\beta) = \bigcap_{\beta<\delta} M_\delta$, which is a $(1,2)$-ideal of S. For the case (iv), we will show that $L(\gamma_A;\beta) = \bigcup_{\beta\geq\delta} M_\delta$. If $x \in \bigcup_{\beta\geq\delta} M_\delta$, then $x \in M_\delta$ for some $\beta \geq \delta$. It follows that $\gamma_A(x) \leq \delta \leq \beta$ so that $x \in L(\gamma_A;\beta)$. Hence $\bigcup_{\beta\geq\delta} M_\delta \subset L(\gamma_A;\beta)$. Conversely, if $x \notin \bigcup_{\beta\geq\delta} M_\delta$ then $x \notin M_\delta$ for all $\delta \leq \beta$. Since $\beta \neq \bigwedge\{\delta \in \Lambda \mid \beta < \delta\}$, there exists $\varepsilon > 0$ such that $(\beta, \beta+\varepsilon) \cap \Lambda = \emptyset$, which implies that $x \notin M_\delta$ for all $\delta < \beta+\varepsilon$, that is, if $x \in M_\delta$ then $\delta \geq \beta+\varepsilon$. Thus $\gamma_A(x) \geq \beta+\varepsilon > \beta$, that is, $x \notin L(\gamma_A;\beta)$. Therefore $L(\gamma_A;\beta) \subset \bigcup_{\beta\geq\delta} M_\delta$ and consequently $L(\gamma_A;\beta) = \bigcup_{\beta\geq\delta} M_\delta$. This completes the proof. □

References

[1] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Syst. **20** (1986), 87–96.

[2] Y. B. Jun and S. Lajos, *On fuzzy $(1,2)$-ideals in semigroups,* PU. M. A. **8** (1997), no. 1, 67–74.

[3] K. H. Kim and Y. B. Jun, *Intuitionistic fuzzy ideals of semigroups,* Indian J. Pure Appl. Math. **33** (2002), no. 4, 443–449.

[4] N. Kuroki, *On fuzzy ideals and fuzzy bi-ideals in semigroups,* Fuzzy Sets Syst. **5** (1981), 203–215.

[5] S. Lajos, *$(1,2)$-ideal characterizations of unions of groups,* Math. Seminar Notes (presently, Kobe J. Math.) **5** (1977), 447–450.

[6] S. Lajos and Y. B. Jun, *On fuzzy $(1,2)$-ideals in semigroups II,* PU. M. A. **8** (1994), no. 2-3-4, 335–338.

[7] L. A. Zadeh, *Fuzzy sets,* Inform. Control **8** (1965), 338–353.

Young Bae Jun
Department of Mathematics Education (and RINS)
Gyeongsang National University
Chinju 660-701, Korea
Email : ybjun@gsnu.ac.kr jamjana@korea.com

Eun Hwan Roh
Department of Mathematics Education
Chinju National University of Education
Chinju 660-756, Korea
Email : ehroh@cue.ac.kr

Seok Zun Song
Department of Mathematics
Cheju National University
Cheju 690-756, Korea
Email : szsong@cheju.ac.kr