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On Lorentzian α-Sasakian Manifolds

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Abstract. The present paper deals with Lorentzian α-Sasakian manifolds with conformally flat and quasi conformally flat curvature tensor. It is shown that in both cases, the manifold is locally isometric with a sphere $S^{2^n+1}(c)$. Further it is shown that an Lorentzian α-Sasakian manifold with $R(X,Y).C = 0$ is locally isometric with a sphere $S^{2^n+1}(c)$, where $c = \alpha^2$.

0. Introduction

In [9], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes:

(1) homogeneous normal contact Riemannian manifolds with $c > 0$,

(2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and

(3) a warped product space $\mathbb{R} \times_f \mathbb{C}$ if $c < 0$.

It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu ([18]) characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian ([18]).

In the Gray-Hervella classification of almost Hermitian manifolds ([13]), there appears a class, W_4, of Hermitian manifolds which are closely related to locally

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conformal Kaehler manifolds ([20]). An almost contact metric structure on a manifold M is called a trans-Sasakian structure ([11]) if the product manifold $M \times \mathbb{R}$ belongs to the class W_4. The class $C_6 \oplus C_5$([16], [18]) coincides with the class of the trans-Sasakian structures of type (α, β). In fact, in [18], local nature of the two subclasses, namely, C_5 and C_6 structures, of trans-Sasakian structures are characterized completely.

We note that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic ([1]), β-Kenmotsu ([19]) and α-Sasakian ([19]) respectively. In ([12]) it is proved that trans-Sasakian structures are generalized quasi-Sasakian ([17]). Thus, trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures.

An almost contact metric structure (ϕ, ξ, η, g) on M is called a *trans-Sasakian structure* ([11]) if $(M \times \mathbb{R},\ J,\ G)$ belongs to the class W_4([13]), where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(X, fd/dt) = (\phi X - f\xi, \eta(X) fd/dt)$$

for all vector fields X on M and smooth functions f on $M \times \mathbb{R}$, and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition ([21])

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β).

Let (x, y, z) be Cartesian coordinates in $\mathbb{R}^3$, then (ϕ, ξ, η, g) given by

$$\xi = \partial/\partial z, \quad \eta = dz - ydx, \quad \phi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & -y & 0 \end{pmatrix}, \quad g = \begin{pmatrix} e^z + y^2 & 0 & -y \\ 0 & e^z & 0 \\ -y & 0 & 1 \end{pmatrix}$$

is a trans-Sasakian structure of type $(-1/(2e^z),\ 1/2)$ in $\mathbb{R}^3$([21]). In general, in a 3-dimensional K-contact manifold with structures tensors (ϕ, ξ, η, g) for a non-constant function f, if we define $g' = fg + (1 - f)\eta \otimes \eta$; then (ϕ, ξ, η, g') is a trans-Sasakian structure of type $(1/f,\ (1/2)\xi(\ln f))$([14], [15], [16]).

Corollary 1 ([7]). *A trans-Sasakian structure of type (α, β) with α a nonzero constant is always α-Sasakian.*

In this case α becomes a constant. If $\alpha = 1$, then α-Sasakian manifold is Sasakian.

In this paper, we investigate Lorentzian α-Sasakian manifolds in which

$$C = 0 \tag{1}$$

where C is the Weyl conformal curvature tensor. Then we study Lorentzian α-Sasakian manifolds in which

$$\widetilde{C} = 0 \tag{2}$$

where $\widetilde{C}$ is the quasi conformal curvature tensor. In the both cases, it is shown that an Lorentzian α-Sasakian manifold is isometric with a sphere $S^{2n+1}(c)$, where $c = \alpha^2$. Finally, an Lorentzian α-Sasakian manifold with

$$R(X,Y).C = 0 \tag{3}$$

has been considered, where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold of tangent vectors X, Y. It is easy to see that $R(X,Y).R = 0$ implies $R(X,Y).C = 0$. So it is meaningful to undertake the study of manifolds satisfying the condition (3).

1. Preliminaries

A differentiable manifold of dimension $(2n+1)$ is called Lorentzian α-Sasakian manifold if it admits a $(1,1)-$tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy ([2], [3], [4], [5], [6], [7])

$$\eta(\xi) = -1, \tag{4}$$
$$\phi^2 = I + \eta \otimes \xi, \tag{5}$$
$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{6}$$
$$g(X,\xi) = \eta(X), \tag{7}$$
$$\phi\xi = 0, \qquad \eta(\phi X) = 0 \tag{8}$$

for all $X, Y \in TM$.

Also an Lorentzian α-Sasakian manifold M is satisfying ([8])

$$\nabla_X \xi = -\alpha\phi X, \tag{9}$$
$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y). \tag{10}$$

where ∇ denotes the operator of covariant differentation with respect to the Lorentzian metric g.

An Lorentzian α-Sasakian manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{11}$$

for any vector fields X, Y, where a, b are functions on M.

Further, on an Lorentzian α-Sasakian manifold M the following relations hold ([7], [10])

$$R(\xi, X)Y = \alpha^2(g(X,Y)\xi - \eta(Y)X), \tag{12}$$
$$R(X,Y)\xi = \alpha^2(\eta(Y)X - \eta(X)Y), \tag{13}$$
$$R(\xi, X)\xi = \alpha^2(\eta(X)\xi + X), \tag{14}$$
$$S(X,\xi) = 2n\alpha^2\eta(X), \tag{15}$$
$$Q\xi = 2n\alpha^2\xi, \tag{16}$$
$$S(\xi,\xi) = -2n\alpha^2, \tag{17}$$

where S is the Ricci curvature and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$.

2. Lorentzian α-Sasakian manifolds with $C = 0$

The conformal curvature tensor C on M^{2n+1} is defined as

$$C(X,Y)Z = R(X,Y)Z + \frac{1}{2n-1}\begin{bmatrix} S(X,Z)Y - S(Y,Z)X \\ +g(X,Z)QY - g(Y,Z)QX \end{bmatrix} - \frac{\tau}{2n(2n-1)}\left[g(X,Z)Y - g(Y,Z)X\right] \tag{18}$$

where

$$S(X,Y) = g(QX,Y).$$

Using (1) we get from (18)

$$R(X,Y)Z = \frac{1}{2n-1}\begin{bmatrix} S(Y,Z)X - S(X,Z)Y \\ +g(Y,Z)QX - g(X,Z)QY \end{bmatrix} + \frac{\tau}{2n(2n-1)}\left[g(X,Z)Y - g(Y,Z)X\right]. \tag{19}$$

Taking $Z = \xi$ in (19) and using (7), (13) and (15), we find

$$\left(\alpha^2 + \frac{\tau}{2n(2n-1)} - \frac{2n\alpha^2}{(2n-1)}\right)(\eta(Y)X - \eta(X)Y) = \frac{1}{2n-1}(\eta(Y)QX - \eta(X)QY).$$

Taking $Y = \xi$ and using (4) we get

$$QX = \left(\frac{\tau}{2n} - \alpha^2\right)X + \left(\frac{\tau}{2n} - \alpha^2 - 2n\alpha^2\right)\eta(X)\xi. \tag{20}$$

Thus the manifold is η-Einstein.

Contracting (20) we get

$$\tau = 2n(2n+1)\alpha^2. \tag{21}$$

Using (21) in (20) we find

$$QX = 2n\alpha^2 X. \tag{22}$$

Putting (22) in (19) we get after a few steps

$$R(X,Y)Z = \alpha^2(g(Y,Z)X - g(X,Z)Y). \tag{23}$$

Thus a conformally flat Lorentzian α-Sasakian manifold is of constant curvature. The value of this constant is α^2. Hence we can state.

Theorem 2. *A conformally flat Lorentzian $\alpha-$Sasakian manifold is locally isometric to a sphere $S^{2n+1}(c)$, where $c = \alpha^2$.*

3. Lorentzian α-Sasakian manifolds with $\widetilde{C} = 0$

The quasi conformal curvature tensor $\widetilde{C}$ on M^{2n+1} is defined as

$$\begin{aligned}\widetilde{C}(X,Y)Z &= aR(X,Y)Z + b\{S(Y,Z)X - S(X,Z)Y \\ &\quad + g(Y,Z)QX - g(X,Z)QY\} \\ &\quad - \frac{\tau}{(2n+1)}(\frac{a}{2n} + 2b)\{g(Y,Z)X - g(X,Z)Y\}\end{aligned} \tag{24}$$

where a, b are constants such that $ab \neq 0$ and

$$S(Y,Z) = g(QY,Z).$$

Using (2), we find from (24)

$$\begin{aligned}R(X,Y)Z &= -\frac{b}{a}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ &\quad + \{\frac{\tau}{(2n+1)a}(\frac{a}{2n} + 2b)\}\{g(Y,Z)X - g(X,Z)Y\}.\end{aligned} \tag{25}$$

Taking $Z = \xi$ in (25) and using (7), (13) and (15), we get

$$\begin{aligned}\alpha^2(\eta(Y)X - \eta(X)Y) &= -\frac{b}{a}\{\eta(Y)QX - \eta(X)QY\} \\ &\quad + \{\frac{\tau}{(2n+1)a}(\frac{a}{2n} + 2b) - \frac{2n\alpha^2 b}{a}\}(\eta(Y)X - \eta(X)Y).\end{aligned} \tag{26}$$

Taking $Y = \xi$ and applying (4) we have

$$\begin{aligned}QX &= \{\frac{\tau}{(2n+1)b}(\frac{a}{2n} + 2b) - 2n\alpha^2 - \frac{a}{b}\alpha^2\}X \\ &\quad + \{\frac{\tau}{(2n+1)b}(\frac{a}{2n} + 2b) - 4n\alpha^2 - \frac{a}{b}\alpha^2\}\eta(X)\xi.\end{aligned} \tag{27}$$

Contracting (27), we get after a few steps

$$\tau = 2n(2n+1)\alpha^2. \tag{28}$$

Using (28) in (27), we get

$$QX = 2n\alpha^2 X. \tag{29}$$

Finally, using (29), we find from (25)

$$R(X,Y)Z = \alpha^2\{g(Y,Z)X - g(X,Z)Y\}.$$

Thus we can state

Theorem 3. *A quasi conformally flat Lorentzian α-Sasakian manifold is locally isometric to a sphere $S^{2n+1}(c)$, where $c = \alpha^2$.*

4. Lorentzian α-Sasakian manifolds satisfying $R(X,Y).C = 0$

Using (7), (12) and (15) we take $X = \xi$, we find from (18)

$$\begin{aligned} \eta(C(X,Y)Z) &= \frac{1}{2n-1}[(\frac{\tau}{2n} - \alpha^2)\{g(Y,Z)\eta(X) \\ &\quad - g(X,Z)\eta(Y)\} - \{S(Y,Z)\eta(X) - S(X,Z)\eta(Y)\}]. \end{aligned} \tag{30}$$

Putting $Z = \xi$ in (30) and using (7) and (15), we get

$$\eta(C(X,Y)\xi) = 0. \tag{31}$$

Again $X = \xi$ in (30)

$$\begin{aligned} \eta(C(\xi,Y)Z) &= (\alpha^2 - \frac{2n\alpha^2}{2n-1} + \frac{\tau}{2n(2n+1)})g(Y,Z) \\ &\quad - (\alpha^2 - \frac{4n\alpha^2}{2n-1} + \frac{\tau}{2n(2n+1)})\eta(Y)\eta(Z) \\ &\quad - \frac{1}{2n-1}S(Y,Z). \end{aligned} \tag{32}$$

Now

$$\begin{aligned} (R(\xi,Y)C)(U,V)W &= R(\xi,Y)C(U,V)W - C(R(\xi,Y)U,V)W \\ &\quad - C(U,R(\xi,Y)V)W - C(U,V)R(\xi,Y)W. \end{aligned} \tag{33}$$

Using (3), we find from above

$$\begin{aligned} &g[R(\xi,Y)C(U,V)W,\xi] - g[C(R(\xi,Y)U,V)W,\xi] \\ &\quad - g[C(R(\xi,Y)UV)W,\xi] - g[C(U,V)R(\xi,Y)W,\xi] = 0. \end{aligned}$$

Using (7) and (12) we get

$$(34)\qquad \begin{aligned} &-\alpha^2 g(C(U,V)W,Y)-\alpha^2\eta(Y)\eta(C(U,V)W)-\alpha^2 g(Y,U)\eta(C(\xi,V)W)\\ &+\alpha^2\eta(U)\eta(C(Y,V)W)-\alpha^2 g(Y,V)\eta(C(U,\xi)W)+\alpha^2\eta(V)\eta(C(U,Y)W)\\ &-\alpha^2 g(Y,W)\eta(C(U,V)\xi)+\alpha^2\eta(W)\eta(C(U,V)Y) \;=\; 0. \end{aligned}$$

Putting $U=Y$ in (34)

$$(35)\qquad \begin{aligned} &-\alpha^2 g(C(U,V)W,U)-\alpha^2\eta(U)\eta(C(U,V)W)-\alpha^2 g(U,U)\eta(C(\xi,V)W)\\ &+\alpha^2\eta(U)\eta(C(U,V)W)-\alpha^2 g(U,V)\eta(C(U,\xi)W)+\alpha^2\eta(V)\eta(C(U,U)W)\\ &-\alpha^2 g(U,W)\eta(C(U,V)\xi)+\alpha^2\eta(W)\eta(C(U,V)U) \;=\; 0. \end{aligned}$$

Let $\{\widetilde{e_i} : i=1,\cdots,2n+1\}$ be an orthonormal basis of the tangent space at any point, then the sum for $1\le i\le n$ of the relations (35) for $U=\widetilde{e_i}$ gives

$$\begin{aligned} -2n\alpha^2\eta(C(\xi,V)W) &= 0,\\ (36)\qquad \eta(C(\xi,V)W) &= 0 \quad \text{as } n>1. \end{aligned}$$

Using (31) and (36), (34) takes the form

$$(37)\qquad \begin{aligned} &-\alpha^2 g(C(U,V)W,Y)-\alpha^2\eta(Y)\eta(C(U,V)W)+\alpha^2\eta(U)\eta(C(Y,V)W)\\ &+\alpha^2\eta(V)\eta(C(U,Y)W)+\alpha^2\eta(W)\eta(C(U,V)Y)\\ =\;& 0. \end{aligned}$$

Using (30) in (37), we get

$$(38)\qquad \begin{aligned} &-\alpha^2 g(C(U,V)W,Y)\\ &+\alpha^2\eta(W)\frac{1}{2n-1}[(\frac{\tau}{2n}-1)\{\eta(U)g(Y,V)-\eta(V)g(U,Y)\}\\ &-\{\eta(U)S(Y,V)-\eta(V)S(U,Y)\}]\\ =\;& 0. \end{aligned}$$

In virtue of (36), (32) reduces to

$$(39)\qquad S(Y,Z) \;=\; (\frac{\tau}{2n}-\alpha^2)g(Y,Z)+(\frac{\tau}{2n+1}-(2n+1)\alpha^2)\eta(Y)\eta(Z).$$

Using (39), (37) reduces to

$$(40)\qquad -\alpha^2 g(C(U,V)W,Y) \;=\; 0,$$

i.e.,

$$(41)\qquad C(U,V)W \;=\; 0.$$

Hence the manifold is conformally flat. Using the Theorem 1, we state

Theorem 4. *If in an Lorentzian α-Sasakian manifold $M^{2n+1}(n \geq 1)$ the relation $R(X,Y).C = 0$ holds, then it is locally isometric with a sphere $S^{2n+1}(c)$, where $c = \alpha^2$.*

For a conformally symmetric Riemannian manifold [1], we have $\nabla C = 0$. Hence for such a manifold $R(X,Y).C = 0$ holds. Thus we have the following corollary of the above theorem:

Corollary 5. *A conformally symmetric Lorentzian α-Sasakian manifold M^{2n+1} $(n \geq 1)$ is locally isometric with a sphere $S^{2n+1}(c)$, where $c = \alpha^2$.*

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