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CONFORMAL CHANGE OF THE CONNECTION IN 8-DIMENSIONAL g-UFT

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Abstract. We investigate change of the connection induced by the conformal change in 8-dimensional g-unified field theory. These topics will be studied for the second class with the first category in 8-dimensional case.

1. Introduction

The conformal change in a generalized 4-dimensional Riemannian space conneted by an Einstein's connection was primarily studied by HLAVAT Ý[10]. CHUNG[8] also investigated the same topic in 4-dimens ional $^*g$-unified field theory.

The Einstein's connection induced by the conformal change for the second class with the first category of the torsion tensor $S_{\omega\mu}{}^{\nu}$ and $U_{\omega\mu}{}^{\nu}$ in 7-dimensional case were investigated by CHO[3,4].

In the present paper, we investigate change of conncection $\Gamma^{\nu}_{\omega\mu}$ induced by the conformal change in 8-dimensional g-unified field theory. These topics will be studied for the second class with the first category in 8-dimensional case.

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2. Preliminaries

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may be referred to CHUNG[5-7], CHO[1-4].

2.1. n-dimensional g-unified field theory

The n-dimensional g-unified field theory (n-g-UFT hereafter) was originally suggested by HLAVATÝ[10] and systematically introduced by CHUNG[8].

Let $X_n (n \geq 2)$ be an n-dimensional generalized Riemannian manifold, referred to a real coordinate system x^ν obeying coordinate transformations $x^\nu \to x^{\nu'}$, for which

$$Det\left(\left(\frac{\partial x}{\partial x'}\right)\right) \neq 0. \tag{2.1}$$

In the usual Einstein's n-dimensional unified field theory, the manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$Det((g_{\lambda\mu})) \neq 0 \qquad Det((h_{\lambda\mu})) \neq 0. \tag{2.3}$$

Therefore we may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.4}$$

In our n-g-UFT, the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of the tensors in X_n in the usual manner.

The manifold X_n is connected by a general real connection $\Gamma^\nu_{\omega\mu}$ with the following transformation rule :

$$\Gamma^{\nu'}_{\omega'\mu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\omega'}} \cdot \frac{\partial x^\gamma}{\partial x^{\mu'}} \Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial x^{\omega'} \partial x^{\mu'}}\right) \tag{2.5}$$

and satisfies the system of Einstein's equations

$$D_\omega g_{\lambda\mu} = 2S_{\omega\mu}{}^{\alpha} g_{\lambda\alpha} \tag{2.6}$$

where D_ω denotes the covariant derivative with respect to $\Gamma^\nu_{\lambda\mu}$ and

$$S_{\omega\mu}{}^{\nu} = \Gamma^\nu_{[\omega\mu]} \tag{2.7}$$

is the *torsion tensor* of $\Gamma^\nu_{\omega\mu}$. The connection $\Gamma^\nu_{\omega\mu}$ satisfying (2.6) is called the *Einstein's connection.*

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \cdots$ are frequently used :

$$\mathfrak{g} = Det((g_{\lambda\mu})) \neq 0, \quad \mathfrak{h} = Det((h_{\lambda\mu})) \neq 0, \qquad \mathfrak{k} = Det((k_{\lambda\mu})), \tag{2.8a}$$

$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}}, \tag{2.8b}$$

$$K_p = k_{[\alpha_1}{}^{\alpha^1} \cdots k_{\alpha_p]}{}^{\alpha^p}, \quad (p = 0, 1, 2, \cdots) \tag{2.8c}$$

$$^{(0)}k_\lambda{}^\nu = \delta_\lambda{}^\nu, \quad ^{(1)}k_\lambda{}^\nu = k_\lambda{}^\nu, \quad ^{(p)}k_\lambda{}^\alpha = {}^{(p-1)}k_\lambda{}^\alpha k_\alpha{}^\nu, \tag{2.8d}$$

$$K_{\omega\mu\nu} = \nabla_\nu k_{\omega\mu} + \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu}, \tag{2.8e}$$

$$\sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \tag{2.8f}$$

where ∇_ω is the symbolic vector of the convariant derivative with respect to the Christoffel symbols $\{{}^{\nu}_{\lambda\mu}\}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

$$K_0 = 1; K_n = k \quad \text{if } n \text{ is even}; \quad K_p = 0 \quad \text{if } p \text{ is odd}, \tag{2.9a}$$

$$g = 1 + K_2 + \cdots + K_{n-\sigma}, \tag{2.9b}$$

$$^{(p)}k_{\lambda\mu} = (-1)^{p(p)}k_{\mu\lambda}, \quad ^{(p)}k^{\lambda\mu} = (-1)^{p(p)}k^{\nu\lambda}. \tag{2.9c}$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by T:

$$\overset{pqr}{T} = \overset{pqr}{T}_{\omega\mu\nu} = T_{\alpha\beta\gamma}{}^{(p)}k_\omega{}^{\alpha(q)}k_\mu{}^{\beta(r)}k_\nu{}^\gamma, \tag{2.10a}$$

$$T = T_{\omega\mu\nu} = \overset{000}{T}, \tag{2.10b}$$

$$2\overset{pqr}{T}_{\omega[\lambda\mu]} = \overset{pqr}{T}_{\omega\lambda\mu} - \overset{pqr}{T}_{\omega\mu\lambda}, \tag{2.10c}$$

$$2\overset{(pq)r}{T}_{\omega\lambda\mu} = \overset{pqr}{T}_{\omega\lambda\mu} + \overset{qpr}{T}_{\omega\lambda\mu}. \tag{2.10d}$$

We then have

$$\overset{pqr}{T}_{\omega\lambda\mu} = -\overset{qpr}{T}_{\lambda\omega\mu}. \tag{2.11}$$

If the system (2.6) admits $\Gamma^{\nu}_{\lambda\mu}$, using the above abbreviations it was shown that the connection is of the form

$$\Gamma^{\nu}_{\omega\mu} = \{{}^{\nu}_{\omega\mu}\} + S_{\omega\mu}{}^{\nu} + U^{\nu}_{\omega\mu} \tag{2.12}$$

where

$$U_{\nu\omega\mu} = 2\overset{001}{S}_{\nu(\omega\mu)}. \tag{2.13}$$

The above two relations show that our problem of determining $\Gamma^{\nu}_{\omega\mu}$ in terms of $g_{\lambda\mu}$ is reduced to that of studying the tensor $S_{\omega\mu}{}^{\nu}$. On the other hand, it has also been shown that the tensor $S_{\omega\mu}{}^{\nu}$ satisfies

$$S = B - 3\overset{(110)}{S} \tag{2.14}$$

where

$$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}k^{\alpha}_{\omega]}k^{\beta}_{\nu}. \tag{2.15}$$

2.2. Some results for the second class in 8-g-UFT

In this section, we introduce some results of 8-g-UFT without proof, which are needed in our subsequent considerations.
They may be referred to CHUNG[5-7].

DEFINITION 2.1. *In 8-g-UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be the second class with the first category, if $K_2 \neq 0$, $K_4 = K_6 = K_8 = 0$.*

THEOREM 2.2. *(Main recurrence relations). For the second class with the first category in 8-g-UFT, the following recurrence relation hold*

$$^{(p+2)}k_{\lambda}{}^{\nu} = -K_2^{(p)}k_{\lambda}{}^{\nu}, \quad (p = 0, 1, 2, \cdots). \tag{2.16}$$

THEOREM 2.3. *(For the second class with the first category in 8-g-UFT). A necessary and sufficient condition for the existence and uniqueness of the solution of (2.5) is*

$$1-(K_2)^2 \neq 0. \tag{2.17}$$

If the condition (2.17) is satisfied, the unique solution of (2.14) is given by

$$(1-K_2^2)(B-S)=K_2(1-K_2)B+2\overset{(10)1}{B}. \tag{2.18}$$

3. Conformal change of the 8-dimensional conection for the second class with first category

In this final chapter we investigate the change $\Gamma^{\nu}_{\omega\mu} \to \overline{\Gamma}^{\nu}_{\omega\mu}$ of the connection for the second class with first category induced by the conformal change of the tensor $g_{\lambda\mu}$, using the recurrence relations and theorems introduced in the preceding chapter.

We say that X_n and $\overline{X}_n$ are conformal if and only if

$$\overline{g}_{\lambda\mu}(x)=e^{\Omega}g_{\lambda\mu}(x) \tag{3.1}$$

where $\Omega=\Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the connection $\Gamma^{\nu}_{\omega\mu}$. An explicit representation of the change of 8-dimensional vector $\Gamma^{\nu}_{\omega\mu}$ for the second class will be exhibited in this chapter.

AGREEMENT 3.1. *Throughout this section, we agree that, if T is a funtion of $g_{\lambda\mu}$, then we denote $\overline{T}$ the same function of $\overline{g}_{\lambda\mu}$. In particular, if T is a tensor, so is $\overline{T}$. Furthermore, the indices of $T(\overline{T})$ will be raised and/or lowered by means of $h^{\lambda\nu}(\overline{h}^{\lambda\nu})$ and/or $h_{\lambda\nu}(\overline{h}_{\lambda\nu})$.*

The results in the following theorems are needed in our further considerations. They may be referred to CHUNG[5-7], CHO[1-2].

THEOREM 3.2. *In n-g-UFT, the conformal change (3.1) induces the following changes:*

$$ {}^{(p)}\overline{k}_{\lambda\mu} = e^{\Omega(p)} k_{\lambda\mu}, \quad {}^{(p)}\overline{k}_{\lambda} = {}^{(p)}k^{\nu}_{\lambda}, \quad {}^{(p)}\overline{k}^{\lambda\mu} = e^{-\Omega(p)} k^{\lambda\mu}, \tag{3.2a} $$

$$ \overline{g} = g, \quad \overline{K}_p = K_p, \qquad (p = 1, 2, \cdots). \tag{3.2b} $$

THEOREM 3.3. *In n-g-UFT, the conformal change (3.1) induces the following changes:*

$$ \overline{\left\{ \begin{matrix} \mu \\ \omega\mu \end{matrix} \right\}} = \left\{ \begin{matrix} \mu \\ \omega\mu \end{matrix} \right\} + \delta^{\nu}{}_{(\omega}\Omega_{\mu)} - \frac{1}{2} h_{\omega\mu}\Omega^{\nu}. \tag{3.3} $$

where $\Omega_\alpha = \partial_\alpha \Omega$

THEOREM 3.4. *(For the second class with the first category in 8-g-UFT). The change of the tensor* $B_{\omega\mu\nu}$ *induced by the conformal change (3.1) may be given by*

$$ \begin{aligned} \overline{B}_{\omega\mu\nu} &= e^{\Omega}(B_{\omega\mu\nu} + k_{\nu[\omega}\Omega_{\mu]} - k_{\omega\mu}\Omega_{\nu} \\ &\quad - h_{\nu[\omega}k_{\mu]}{}^{\alpha}\Omega_{\alpha} - 2K_2\delta_{\nu[\omega}k_{\mu]}{}^{\alpha}\Omega_{\alpha} - K_2 k_{\omega\mu}\delta^{\alpha}_{\nu}\Omega_{\alpha}.) \end{aligned} \tag{3.4} $$

Now, we are ready to derive representations of the changes $S_{\omega\mu}{}^{\nu} \to \overline{S}_{\omega\mu}{}^{\nu}$ and $U_{\omega\mu}{}^{\nu} \to \overline{U}_{\omega\mu}{}^{\nu}$ in 8-g-UFT for the second class with the first category induced by the conformal change (3.1).

THEOREM 3.5. *(For the second class with first category in 8-g-UFT). The conformal change (3.1) induces the following change:*

$$ \begin{aligned} 2\,\overline{\overset{(10)1}{B}}_{\omega\mu\nu} &= e^{\Omega}(2\,\overset{(10)1}{B}_{\omega\mu\nu} + (-2(K_2)^2\delta_{\nu[\omega}k_{\mu]}{}^{\alpha} - 2K_2\delta_{\nu[\omega}k_{\mu]}{}^{\alpha} \\ &\quad + K_2 k_{\nu[\omega}k_{\mu]}{}^{\alpha})\Omega_{\alpha} + K_2 k_{\nu[\omega}\Omega_{\mu]}). \end{aligned} \tag{3.5} $$

THEOREM 3.6. *The conformal change (3.1) may be represented by*

$$\overset{\overline{ppq}}{B}{}_{\omega\mu\nu} = e^{\Omega}[\overset{ppq}{B}{}_{\omega\mu\nu} + (-1)^p\{2^{(p+q+2)}k_{\nu[\omega}{}^{(p+1)}k_{\mu]}{}^{\delta}$$

$$+{}^{(2p+1)}k_{\omega\mu}{}^{(2+q)}k_{\nu}{}^{\delta} - {}^{(2p+1)}k_{\omega\mu}{}^{(q)}k_{\nu}{}^{\delta} \tag{3.6}$$

$$+{}^{(p+q+1)}k_{\nu[\omega}{}^{(p)}k_{\mu]}{}^{\delta} - {}^{(p+q)}k_{\nu[\omega}{}^{(p+1)}k_{\mu]}{}^{\delta}\}\Omega_{\delta}].$$

$$\begin{pmatrix} p = 0,1,2,3,4,\cdots \\ q = 0,1,2,3,4,\cdots \end{pmatrix}$$

By the above relation (3.6), we obtain $\overset{\overline{001}}{B}$.

THEOREM 3.7. *The change* $S_{\omega\mu}{}^{\nu} \to \overline{S}_{\omega\mu}{}^{\nu}$ *induced by conformal change (3.1) may be represented by*

$$\overline{S}_{\omega\mu}{}^{\nu} = S_{\omega\mu}{}^{\nu} + \underset{1}{S}{}_{\omega\mu}{}^{\nu}, \tag{3.7}$$

where

$$\underset{1}{S}{}_{\omega\mu}{}^{\nu} = -k_{\omega\mu}\Omega^{\nu} + \frac{1}{1-K_2{}^2}[(1-2K_2)k^{\nu}{}_{[\omega}\Omega_{\mu]}$$

$$+((K_2-1)h^{\nu}{}_{[\omega}k_{\mu]}{}^{\alpha} + 4K_2{}^2\delta^{\nu}{}_{[\omega}k_{\mu]}{}^{\alpha}$$

$$-2K_2k^{\nu}{}_{[\omega}\delta_{\mu]}{}^{\alpha})\Omega_{\alpha}].$$

Proof. In virtue of (2.18) and Agrement (3.1), we have

$$(1-\overline{K_2}{}^2)(\overline{B}-\overline{S}) = \overline{K}_2(1-\overline{K}_2)\overline{B} + 2\overset{\overline{(10)1}}{B} \tag{3.8}$$

The relation (3.7) follows by substituiting (3.2), (3.4), (3.5), (2.10), (2.16), Definition (2.1), into (3.8). ∎

THEOREM 3.8. *The change $U_{\omega\mu}{}^{\nu} \to \overline{U}_{\omega\mu}{}^{\nu}$ induced by conformal change (3.1) may be represented by*

$$\begin{aligned}\overline{U}^{\nu}_{\omega\mu} &= U^{\nu}_{\omega\mu} + \tfrac{1}{1-K_2{}^2}[(-K_2 + 3K_2{}^2)\,\delta^{\nu}{}_{(\omega}\Omega_{\mu)} \\ &\quad + (K_2 - 4K_2{}^2)\,\delta_{\omega\mu}\Omega^{\nu} - (1 + K_2 + 2K_2{}^2)\,k^{\nu}{}_{(\omega}k_{\mu)}{}^{\alpha}\Omega_{\alpha}].\end{aligned} \tag{3.9}$$

Proof. *In virtue of (2.13) and Agreement (3.1), we have*

$$\overline{U}_{\nu\omega\mu} = 2\,\overline{\overset{001}{S}}_{\nu(\omega\mu)}. \tag{3.10}$$

The relation (3.9) follows by substituting (3.7), (2.10), Definition (2.1), (2.16), Theorem 3.6, (3.2) into (3.10). ∎

THEOREM 3.9. *The change of the connection $\Gamma_{\omega\mu}{}^{\nu} \to \overline{\Gamma}_{\omega\mu}{}^{\nu}$ induced by conformal change (3.1) may be represented by*

$$\overline{\Gamma}_{\omega\mu}{}^{\nu} = \Gamma_{\omega\mu}{}^{\nu} + \underset{1}{S}_{\omega\mu}{}^{\nu} + \underset{1}{U}_{\omega\mu}{}^{\nu}, \tag{3.11}$$

where

$$\begin{aligned}\underset{1}{U}_{\omega\mu}{}^{\nu} = &\; -\tfrac{1}{2}h_{\omega\mu}\Omega^{\nu} + \tfrac{1}{1-K_2{}^2}[(1 - K_2 + 2K_2{}^2)\,\delta^{\nu}{}_{(\omega}\Omega_{\mu)} \\ &+ (K_2 - 4K_2{}^2)\,\delta_{\omega\mu}\Omega^{\nu} - (1 + K_2 + 2K_2{}^2)\,k^{\nu}{}_{(\omega}k_{\mu)}{}^{\alpha}\Omega_{\alpha}].\end{aligned}$$

Proof. *In virtue of (2.12) and Agreement (3.1), we have*

$$\overline{\Gamma}_{\omega\mu}{}^{\nu} = \overline{\left\{ \begin{array}{c} \mu \\ \omega\mu \end{array} \right\}} + \overline{S}_{\omega\mu}{}^{\nu} + \overline{U}_{\omega\mu}{}^{\nu}. \tag{3.12}$$

The relation (3.11) follows by substituting (3.3), (3.7), (3.9) into (3.12).
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