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TOPOLOGICAL ASPECTS OF THE THREE DIMENSIONAL CRITICAL POINT EQUATION

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Abstract. Let (M^n, g) be a compact oriented Riemannian manifold. It has been conjectured that every solution of the equation $z_g = D_g df - \Delta_g f g - f r_g$ is an Einstein metric. In this article, we deal with the 3 dimensional case of the equation. In dimension 3, if the conjecture fails, there should be a stable minimal hypersurface in (M^3, g). We study some necessary conditions to guarantee that a stable minimal hypersurface exists in M^3.

1. Introduction

It is an important problem in differential geometry to find a canonical Riemannian metric on a given manifold. A metric of constant Ricci curvature has been considered as one of the canonical metrics by differential geometers. But it is still unsolved whether there exists a metric of constant Ricci curvature on a given manifold. One of the approaches to get a metric of constant Ricci curvature on a compact, oriented manifold M^n is the following.

Let $\mathcal{M}_1$ be the set of Riemannian metrics on M^n with volume 1. We look into the total scalar curvature functional $\mathcal{S} : \mathcal{M}_1 \to \mathbb{R}$ given by

$$\mathcal{S}(g) = \int_{M^n} s_g dvol_g,$$

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where s_g is scalar curvature function of g. Also, it is well known that any compact manifold carries many metrics with constant scalar curvature. Let $\Xi = \{g \in \mathcal{M}_1 | \, s_g \text{ is constant}\}$. A standard variational technique tells us that the metric g is critical of $\mathcal{S}$ restricted to Ξ if and only if, for some function f on M^n,

$$z_g = D_g df - \Delta_g f g - f r_g \tag{1.1}$$

holds, where r_g is the Ricci tensor and $z_g = r_g - (s_g/n)g$ is the traceless Ricci tensor. We call the equation (1.1) the critical point equation. Note that $f \equiv 0$ leads an Einstein metric trivially. Now the following conjecture arises naturally.

Conjecture 1.1. *If there is a non-zero function f satisfying (1.1), then the metric g is Einstein.*

Now we assume that (1.1) has a non-zero solution on M^3 throughout this article. We refer to [1] for details on Conjecture 1.1. As a remark, M. Obata showed that the metric g in Conjecture 1.1 should be isometric to the standard sphere $\mathbb{S}^n$(See [6]). But unfortunately, it is even not known whether the metric is homeomorphic to $\mathbb{S}^n$ or not. Related with this topological observation, we are interested in the stable minimal hypersurfaces in a three dimensional manifold M^3. Since the existence of stable minimal hypersurfaces in M^3 corresponds to the existence of non-zero elements of $H_2(M^3, \mathbb{Z})$, the existence of stable minimal hypersurfaces would be a topological obstruction for M^3 to have an Einstein metric. Now, let φ be a function on M^3 satisfying the following equation.

$$0 = D_g d\varphi - \Delta_g \varphi g - \varphi r_g \tag{1.2}$$

In [4], S. Hwang proved that every compact stable minimal hypersurface in M^3 should be contained in the set $\Gamma = \varphi^{-1}(0)$ for some non-zero function φ satisfying (1.2). Note that such a set Γ is totally geodesic submanifold of M^3(See [2]).

In this article, we study some properties of a compact stable minimal hypersurface Σ in M^3. To do this, we observe a function f satisfying (1.1) and a function φ satisfying (1.2) on Σ simultaneously. And we find $\tilde{f}$, another solution function of (1.1) obtained from f and φ. We hope that this analysis would help the efforts towards proving the nonexistence of possible compact stable minimal hypersurfaces in M^3.

2. Relation between solutions of (1.1) and (1.2)

The consideration of the two equations (1.1) and (1.2) at the same time gives us a lot of advantages to studying Conjecture 1.1. The following Theorem 2.1 is one of the examples. Let

$$\Sigma = \cup_{i=1}^{I} \Sigma_i,$$

$$\Gamma = \cup_{j=1}^{J} \Gamma_j,$$

where Σ_i and Γ_j are connected components of Σ and Γ respectively. Since $\Sigma \subset \Gamma$, we can arrange the indices satisfying $\Sigma_i \subset \Gamma_i$ for $1 \leq i \leq I \leq J$.

Theorem 2.1. $\Sigma \neq \Gamma$.

Proof. First, note that φ is an eigenfunction of Δ since $\Delta\varphi = -(s_g/2)\varphi$. Hence Γ is the nodal set(the boundary of the nodal domain) of φ, and it is well known that φ has no critical points in Γ(See [2]), i.e., $|d\varphi| \neq 0$. Furthermore, $|d\varphi|$ is constant on each component Γ_i of Γ because, for $\xi \in T\Gamma_i$, (1.2) gives

$$\xi\langle d\varphi, d\varphi\rangle = 2\langle D_\xi d\varphi, d\varphi\rangle = -s_g\varphi\langle\xi, d\varphi\rangle + 2r_g\varphi\langle\xi, d\varphi\rangle = 0.$$

So we can conclude that for some positive constant c_i,

$$|d\varphi|\Big|_{\Gamma_i} \equiv c_i.$$

Let $M_{0,\varphi} = \{x \in M^3 \,|\, \varphi(x) < 0\}$. Then the boundary of $M_{0,\varphi}$ is Γ. The following equation can be easily deduced from the facts $\Delta\varphi = -(s_g/2)\varphi$ and $\Delta f = -(s_g/2)f$.

$$\int_{M_{0,\varphi}} f\Delta\varphi = \int_{M_{0,\varphi}} \varphi\Delta f \tag{2.1}$$

And Green's formula with a unit normal vector field $d\varphi/|d\varphi|$ on Γ gives the following two equations.

$$\int_{M_{0,\varphi}} f\Delta\varphi = \int_\Gamma f\,|d\varphi| - \int_{M_{0,\varphi}} \langle df, d\varphi\rangle \tag{2.2}$$

$$\int_{M_{0,\varphi}} \varphi\Delta f = \int_\Gamma \varphi\,\langle df, \frac{d\varphi}{|d\varphi|}\rangle - \int_{M_{0,\varphi}} \langle d\varphi, df\rangle = -\int_{M_{0,\varphi}} \langle d\varphi, df\rangle \tag{2.3}$$

By combining (2.1), (2.2), and (2.3), we can get

$$\int_\Gamma f\,|d\varphi| = 0.$$

Now suppose $\Sigma = \Gamma$. Then $\Sigma_i = \Gamma_i$ and from the above calculation, we have

$$0 = \int_\Gamma f\,|d\varphi| = \sum_i \int_{\Gamma_i} f\,|d\varphi| = \sum_i c_i \int_{\Gamma_i} f = \sum_i c_i \int_{\Sigma_i} f < 0. \tag{2.4}$$

The last inequality comes from the fact that Σ should be contained in the set $\{x \in M^3 | f(x) < -1\}$(See [4]). So the inequality (2.4) is a contradiction, and it completes the proof. □

Just for simplicity, we assume that Γ is connected until the end of Corollary 3.1. Let f be a solution of (1.1) and φ be a solution of (1.2). It was proved that $\langle d\varphi, d\varphi\rangle$ and $\langle df, d\varphi\rangle$ are constant along Γ(See [4]). Hence we have the following.

$$\langle \frac{df}{|df|}, \frac{d\varphi}{|d\varphi|}\rangle = \frac{C}{|df|} \quad \text{on } \Gamma, \tag{2.5}$$

where C is a constant. From this equation, we know that $|df|$ times the cosine of the angle between df and $d\varphi$ is constant on Γ. Now let

$$\tilde{f} = f - \frac{\langle df, d\varphi\rangle\big|_\Gamma}{\langle d\varphi, d\varphi\rangle\big|_\Gamma}\,\varphi. \tag{2.6}$$

Since $\langle d\varphi, d\varphi\rangle$ and $\langle df, d\varphi\rangle$ are constant on Γ, it can be easily checked that $d\tilde{f}$ is tangent to Γ. So, we can get the following Lemma.

Lemma 2.1. *For a solution φ of (1.2), there exists a solution $\tilde{f}$ of (1.1) such that $\langle d\tilde{f}, d\varphi\rangle = 0$ on Γ.*

Though looking concise, Lemma 2.1 hides two cases which are geometrically opposite each other. Obviously, the equation $\langle d\tilde{f}, d\varphi\rangle = 0$ holds when (1) $d\tilde{f} = 0$, or (2) $d\tilde{f} \neq 0$ and $d\tilde{f} \perp d\varphi$. Now if we look at (2.6) again with the condition that df is parallel to $d\varphi$, then Lemma 2.1 imples $d\tilde{f} = 0$. Furthermore, in this case, the left hand side of (2.5) is constant, so $|df|$ is also constant. From this observation, though looking more complicated, it is useful that Lemma 2.1 is re-told geometrically as the following.

Lemma 2.2. *For a solution f of (1.1) and a solution φ of (1.2), on Γ,*

(1) df is parallel to $d\varphi$, or

(2) df is not parallel to $d\varphi$, and in this case we can get another solution $\tilde{f}$ of (1.1) such that $d\tilde{f} \not\equiv 0$ and $d\tilde{f} \perp d\varphi$.

Proof. Let f be any non-zero solution of (1.1) and suppose that df is not parallel to $d\varphi$ on Γ. Then df cannot be described as $k\,d\varphi$ for any constant k, hence

$$d\tilde{f} = df - \frac{\langle df, d\varphi\rangle\big|_\Gamma}{\langle d\varphi, d\varphi\rangle\big|_\Gamma}\,d\varphi \not\equiv 0.$$

And the property of $d\tilde{f} \perp d\varphi$ can be easily checked, it completes the proof. □

Remark 2.1. The condition that df and $d\varphi$ are parallel to each other actually means that

$$\frac{df}{|df|} = \pm \frac{d\varphi}{|d\varphi|}, \quad \text{if} \quad |df| \neq 0. \tag{2.7}$$

But it can be easily checked that if φ is a solution of (1.2), then so is $-\varphi$. So we can use the term 'parallel' only if two vectors have the same direction, i.e., we can take only the + sign in (2.7) without loss of generality. If $|df| \equiv 0$, we just regard f as $\tilde{f}$.

3. Properties of Σ

In this section, we deal with the only case of df is parallel to $d\varphi$ on Γ. We know that $|df|$ is constant on Γ as explained in the previous section. Furthermore, since $df/|df| = d\varphi/|d\varphi|$ on Γ from our assumption, we get $d\tilde{f} = 0$ on Γ from Lemma 2.1. Now we compute $Dd\tilde{f}$, the Hessian matrix of $\tilde{f}$, on Γ. Since $\tilde{f}$ is also a solution function of (1.1), $\tilde{f}$ satisfies the following equation.

$$z_g = D_g d\tilde{f} - \Delta_g \tilde{f} g - \tilde{f} r_g \tag{3.1}$$

Let $\{X_1, X_2, Y\}$ be an orthonormal frame on Γ such that X_i's are tangent to Γ. So Y should be a unit normal vector field on Γ. On the other hand, $df/|df| = d\varphi/|d\varphi|$ is also a unit normal vector field on Γ. Hence we can take $df/|df| = Y$.

Lemma 3.1. $\Delta\varphi \equiv 0$ *on* Γ.

Proof. On Γ, φ satisfies (1.2) with $\varphi \equiv 0$. By taking the trace of the both sides of (1.2), we have $0 = \Delta\varphi - 3\Delta\varphi$, i.e., $\Delta\varphi \equiv 0$. □

Theorem 3.1. *On* Γ,

$$Dd\tilde{f} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-s_g f}{2} \end{pmatrix}.$$

Proof. The following two equations are directly derived from the property of $|d\tilde{f}| = 0$ on Γ.

$$\begin{aligned} \langle D_{X_i} d\tilde{f}, X_j \rangle &= 0 \\ \langle D_{X_i} d\tilde{f}, Y \rangle &= 0 \end{aligned}$$

The following is just the equation (3.1) in the case of 3 dimension.

$$(1 + \tilde{f})z_g = Dd\tilde{f} + \frac{s_g \tilde{f}}{6} g \tag{3.2}$$

¿From (3.2), we have the following equations.

$$(1 + \tilde{f})z(X_i, X_i) = \langle D_{X_i} d\tilde{f}, X_i \rangle + \frac{s_g \tilde{f}}{6} \tag{3.3}$$

$$(1 + \tilde{f})z(Y, Y) = \langle D_Y d\tilde{f}, Y \rangle + \frac{s_g \tilde{f}}{6} \tag{3.4}$$

Since $\sum_i z(X_i, X_i) + z(Y, Y) = 0$, By adding the equations (3.3) and (3.4), we get

$$\langle D_Y d\tilde{f}, Y \rangle = -\frac{s_g \tilde{f}}{2}.$$

□

Corollary 3.1. *Σ consists of local minimum points of $\tilde{f}$.*

Proof. Since $\Sigma \subset \Gamma$, by Theorem 3.1, $Dd\tilde{f}$ is also given by the following on Σ.

$$Dd\tilde{f} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-s_g \tilde{f}}{2} \end{pmatrix}$$

But we know that Σ should be contained in the set $\{x \in M^3 | \tilde{f}(x) < -1\}$(See [4]). So, $\frac{-s_g \tilde{f}}{2} > 0$, hence $Dd\tilde{f}$ is non-negative. We already pointed that $d\tilde{f} = 0$ on Σ. So we can conclude that every point $p \in \Sigma$ is a local minimum point of $\tilde{f}$. □

Now we consider the general case that Σ consists of the connected components Σ_i's and Γ consists of the connected components Γ_j's. Suppose that df is parallel to $d\varphi$ on Γ, and let

$$\tilde{f}_j = f - \frac{\langle df, d\varphi\rangle|_{\Gamma_j}}{\langle d\varphi, d\varphi\rangle|_{\Gamma_j}}\,\varphi. \tag{3.5}$$

It is easily checked that $\tilde{f}_j$ is a solution of the equation (1.1) and $|d\tilde{f}_j| = 0$ on Γ_i for all $1 \leq j \leq J$. Since all of the calculations of this section are local arguments, we can easily formulated the followings. And the proofs of them are actually done above.

Theorem 3.2. *Let $\Gamma = \cup_j \Gamma_j$. On Γ_j,*

$$Dd\tilde{f}_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-s_g f}{2} \end{pmatrix},$$

where $\tilde{f}_j$ is obtained from the same manner as (3.5)

Corollary 3.2. *Let $\Sigma = \cup_i \Sigma_i$. For all $1 \leq i \leq I$, Σ_i consists of local minimum points of $\tilde{f}_i$.*

Remark 3.1 Finally we shortly remark that, for a component Σ_i of Σ, there exists a solution of the equation (1.1) such that Σ_i should be contained in a level set of the solution function. In fact, since Σ_i consists of local minimum points of $\tilde{f}_i$, it is easily checked that $\Sigma_i \subset \{x \in M^3 |\, \tilde{f}_i(x) = -a_i\}$ for some constant $a_i > 1$.

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