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THE STABILITY OF HILBERT SPACE FRAMELETS AND RIESZ FRAMES

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Abstract. We consider the stability of Hilbert space framelets and related Riesz frames. Our results are in spirit close to classical results for orthonormal bases, due to Mazur and Schauder.

1. Introduction

The classical Banach's Basic Sequence Theorem states the following : Let $(f_i)_{i\in I}$ be in the Banach space $\mathbf{B}$. Then in order that $(f_i)_{i\in I}$ be a basic sequence, it is both necessary and sufficient that there exists a constant $A>0$ such that for all scalars $c_i(i=1,2,\cdots,m)$ we have

$$||\sum_{i=1}^{n} c_i f_i|| \le A||\sum_{i=1}^{m} c_i f_i||,\ (n\le m).$$

The conditions imply that there exist bounded expansion operators P_n, from $\overline{\mathrm{Span}}(f_m)$ to itself such that $P_n(\sum c_i f_i)=\sum_{j=1}^{n} c_j f_j$, each of whose operator norms is $\le A$; it follows immediately that each P_n has a continuous extension.

The above formulation is due to Diestel(cf. [17]). The Banach's Basic Sequence Theorem is useful in order to apply that a frame $(f_i)_{i\in I}$ admits of a Riesz frame for a Hilbert space, so the result is sometimes used in wavelet analysis([1,3,6,7,11,15]).

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Our aim here is to show that a similar criterional problem holds for unconditionally framelets Cauchy frames. Our approach is motivated by the results in [2,10] about stabilities of basic sequences concerning abstract Banach spaces.

2. Preliminaries and Notation

Let H be a separable Hilbert space, with the inner product $< \cdot, \cdot >$ linear in the first entry. $L(H)$ denotes the set of bounded linear operators on H. I and J will be countable index sets and sometimes the natural numbers N used as the index set. c_0 denotes the space of scalar null sequences with the property that each of the coordinate maps $p_i : a \mapsto a_i$ is continuous and possess the finite section property.

The needed facts about frames and framelets can be found in the paper ([4,8,12,14]).

The framelets of MRA version as a specifically modified frame system introduced by Petukov et. al. [16] was concentrated essentially on wavelet frames and their construction via multiresolution analysis (MRA). The redundant representation offered by framelets using associated with given refinable functions has already been put to good use in wide range of applications of wavelet theory [8,9]. We give here, by obvious modifications of [16], some functional intrinsic characterizations of coefficient framelets which are associated with coherent frames, i.e., frames of the form $(g_i(f_i))_{i \in I}$.

Recall that if $g_i := < \bar{f}_i, \cdot >$; $\bar{f}_i$ is the conjugate element [13] to f_i, is a uniquely determined, bounded framelets-linear form in the topology of uniform convergence on H associated with weighted norm bounded frame $(f_i)_{i \in I}$ for H such that $\sum_{i \in I} |g_i(f_i)| < \infty$, then defining the weighted framelet mapping $S \in L(H)$; $Sf = \sum \lambda_i g_{i,f}(f_i), (\lambda_i) \in c_0$, which converges unconditionally for all $f \in H$; equivalently by tensoring $S = \sum_{i \in I} \lambda_i g_i \otimes f_i$, we obtain an isomorphism on H, and S is invertible, self-adjoint. This leads to the following; from

$||Sf|| = \sup_{||g_i||\le 1} |g_i(\sum_i \lambda_i g_{i,f}(f_i)| \le (\sup_i |\lambda_i g_{i,f}|) \sup_{||g_i||\le 1} \sum_i |g_i(f_i)|$

follows $||S|| \le ||\lambda||_\infty \sup_{||g_i||\le 1} \sum_i |g_i(f_i)|$, which implies that every linear mapping of finite rank is weighted framelet mapping and that every weighted framelet mapping associated with framelets is compact as the norm-limit of mappings of finite rank.

We define the weighted framelet norm associated with framelets by

$$||S||_w = \inf ||\lambda||_\infty \sup_{||g_i||\le 1} \sum_i |g_i(f_i)|,$$

where the infimum is taken over all weighted framelet representations described above.

The preceding framelets result allows us to set up the following :

Theorem 2.1. $S \in L(H)$ is a weighted framelet mapping associated with coefficient framelets (g_i) if and only if S satisfies an inequality of the form

$$||Sf|| \le ||\lambda||_\infty \sup_{||g_i||\le 1} \sum_i |g_{i,f}(f_i)|, \quad \lim_i ||g_i|| = 0. \tag{1}$$

Moreover, $||S||_w = \inf ||\lambda||_\infty \sup_{||g_i||\le 1} \sum_i |g_i(f_i)|$, where the infimum is taken over all framelets $g_i \in H$ satisfying (1).

3. Stability of framelets related Riesz frames

Let $\hat{N}$ denote the family of all finite subsets of the natural numbers N directed under inclusion. We recall that a weighted norm-bounded frame $(f_i)_{i\in I}$ for H is said to be "unconditionally framelets Cauchy" if $\sup_{||g_i||\le 1} \sum_i |g_i(f_i)| < \infty$; and has a finite section property if and only if $\sup_{F\in\hat{N}} ||\sum_{n\in F} f_n|| < \infty$.

We say that $(f_i)_{i\in I}$ is a Riesz frame if every subfamily $(f_i)_{i\in J}$ is a frame for its closed linear span, with the same frame bounds A, B for each subfamily. So if $(f_i)_{i\in I}$ is a frame for H and $J \subset I$ is finite, then

$(f_i)_{i\in I-J}$ is a Riesz frame. The following theorem in [5] gives one of the characterizations of Riesz frames.

Theorem 3.1. Every Riesz frame for H contains a Riesz basis.

Our first application concerns the factorization of weighted framelet mapping through the classical space $\ell^p(1 < p < \infty)$.

Theorem 3.2. Let $S \in L(H)$ be a weighted framelet mapping associated with coefficient framelets $(g_i)_{i\in I}$ of unconditionally framelets Cauchy $(f_i)_{i\in I}$ for H with finite section property. Let $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1, \epsilon > 0$. Then S factors through ℓ^p and $||T|| \leq k(||S||_\omega + \epsilon)^{\frac{1}{p}}, ||R|| \leq \frac{1}{k}(||S||_\omega + \epsilon)^{\frac{1}{q}}, k > 0$, where $T \in L(H, \ell^p), R \in L(\ell^p, H)$.

Proof. S has a representation

$$Sf = \sum_{i=1}^{\infty} \lambda_i g_{i,f}(f_i), \ \sum_{i=1}^{\infty} |\lambda_i| \leq ||S||_\omega + \epsilon$$

since (f_i) is unconditionally framelets Cauchy and ℓ_1 is c_0-invariant. We define T by

$$Tf = (|\frac{1}{M_1^p}\lambda_i|^{\frac{1}{p}} g_i(f))_{i\in I}, \ f \in H, \ M_1 > 0.$$

¿From $||Tf|| \leq \frac{1}{M_1}(\sum |\lambda_i|)^{\frac{1}{p}}||f|| \leq (||S||_\omega + \epsilon)^{\frac{1}{p}} \cdot \frac{1}{M_1}||f||$ follows the assertion for T.

Let the operator R be defined by

$$R\rho = \sum_{i=1}^{\infty} \rho_i |\lambda_i|^{\frac{1}{q}} f_i, \ \ \rho = (\rho_i) \in \ell^p.$$

Then by the finite section property for norm-bounded frame (f_i) and Hölder's inequality, we have

$$\begin{aligned} ||R(\rho_i)|| &\leq \sum M_2 |\rho_i||\lambda_i|^{\frac{1}{q}} \leq (\sum |\rho_i|^p)^{\frac{1}{p}} (\sum |\lambda_i|)^{\frac{1}{q}} M_2 \\ &\leq ||(\rho_i)||_p (||S||_\omega + \epsilon)^{\frac{1}{q}} M_2, \ M_2 > 0. \end{aligned}$$

Hence, it follows that $R \in L(\ell^p, H)$, $Sf = RTf$, and

$$||S||_\omega \leq ||R||||T|| \leq \frac{1}{k}(||S||_\omega + \epsilon)^{\frac{1}{q}}(||S||_\omega + \epsilon)^{\frac{1}{p}}k,$$
$$\frac{1}{M} = k,\ M = max(M_1, M_2).$$

□

Using the projection method in connection with coherent framelets we have, in analogy with [10], the following characterization of unconditionally framelets Cauchy:

Theorem 3.3. Let $(f_i)_{i\in I}$ be an unconditionally framelets Cauchy for H with weighted framelet mapping S. Then $(f_i)_{i\in I}$ is an uncoditional Riesz frame if and only if there exists a number $A > 0$ such that for any choice of associated framelets $(g_i)_{i\in I}$ on H with $\sum|\lambda_i g_i(f_i)| < \infty$ we have

$$||\sum_{i=1}^{n} \lambda_i g_i \otimes S^{-1} f_i|| \leq A||\sum_{i=1}^{m} \lambda_i g_i \otimes S^{-1} f_i||,\ (n \leq m).$$

Proof. Let first $(f_i)_{i\in J}$ be a weighted unconditional frame for $\overline{\text{Span}}(f_i)_{i\in J}$ with corresponding coherent framelets $(g_i)_{i\in J}$ and using orthogonal projection method, consider $P_k : \overline{\text{Span}}(f_i)_{i\in J} \longrightarrow \overline{\text{Span}}(f_i)_{i\in J}$ by

$$P_k(\sum_i \lambda_i g_{i,f}(S^{-1}f_i)) = \sum_{i=1}^{k} \lambda_i g_{i,f}(S^{-1}f_i),\ f \in \overline{\text{Span}}(f_i)_{i\in J}.$$

Since each of the coefficient framelets $g_i (1 \leq i \leq k)$ is continuous, we have $P_k f \longrightarrow f = \sum_{i=1}^{\infty} \lambda_i g_{i,f}(S^{-1}f_i)$ for $k \to \infty$. Thus, a standard uniform boundedness argument shows that $\sup_i ||P_i|| < \infty$.

Now, let $n \leq m$ and $\sum_k \lambda_k g_{k,f}(S^{-1}f_k) \in H$, then

$$\begin{aligned}
||\sum_{k=1}^{n} \lambda_k g_{k,f}(S^{-1}f_k)|| &= ||P_n \sum_{k} \lambda_k g_{k,f}(S^{-1}f_k)|| \\
&= ||P_n P_m \sum_{k} \lambda_k g_{k,f}(S^{-1}f_k)|| = ||P_n \sum_{k=1}^{m} \lambda_k g_{k,f}(S^{-1}f_k)|| \\
&\leq ||P_n|| || \sum_{k=1}^{m} \lambda_k g_{k,f}(S^{-1}f_k)|| \leq \sup_m ||P_m|| || \sum_{k=1}^{m} \lambda_k g_{k,f}(S^{-1}f_k)||
\end{aligned}$$

Hence, the "only if " part follows by setting $A = \sup_m ||P_m||$.

Suppose conversely that the above inequality holds as described whenever $n \leq m$. Since for any $j, k \in I$, $||\lambda_j g_j \otimes S^{-1}f_j|| = |\lambda_j g_{j,f}| ||S^{-1}f_j|| \leq A|| \sum_{i=j}^{j+k} \lambda_i g_i \otimes S^{-1}f_i||$ implies that $|\lambda_j g_{j,f}| \leq \frac{A}{||S^{-1}f_j||} || \sum_{j \leq i} \lambda_i g_i \otimes S^{-1}f_i||$, hence an $f \in H$ can be represented as the form ;

$$f = \sum_n \lambda_n g_n \otimes S^{-1}f_n = \lim_n \sum_{i=1}^{n} \lambda_i g_i \otimes S^{-1}f_i.$$

Now we show that every $f \in \overline{\text{Span}}(f_i)_{i \in J}$ has a representation in the form

$$f = \lim_n \sum_{i=1}^{n} \lambda_i g_i \otimes S^{-1}f_i = \sum_n \lambda_n g_n \otimes S^{-1}f_n.$$

Let $f \in \overline{\text{Span}}(f_i)_{i \in J}$ and $\epsilon > 0$ be given. Then for some $i(\epsilon)$, there is a $g \in \text{Span}\{f_1, \cdots, f_{i(\epsilon)}\}_{i \in J}$ for which $||f - g|| < \epsilon$.
Thus, let $m \geq i_\epsilon$. Then

$$||P_m f - f|| \leq ||f - g|| + ||g - P_m g|| + ||P_m g - P_m f|| \leq \epsilon + ||P_m||\epsilon \leq (1 + A)\epsilon.$$

So that, it follows that

$$P_m f \to f = \lim_m \sum_{j=1}^{m} \lambda_j g_{j,f}(S^{-1}f_j) = \lim_m \sum_{j=1}^{m} \lambda_j S^{-1} g_j \otimes f_j$$

and the proof is complete. □

Remark. As corresponding counterparts for unconditionally framelets Cauchy, our results for coefficient framelets version can be used to modify the classical basic sequences [10, P.42], which leads to the following :

Corollary 3.4. Let $(f_i)_{i\in I}$ be an unconditionally framelets Cauchy for H. Then $(f_i)_{i\in I}$ admits of a Riesz frame.

Proof. We proceed, following a suggestion of Bessaga-Pelczynski Selection Principle [10]. Let $0 < \epsilon_i \leq 1 (i \geq 0)(i = 1, 2, \cdots)$ be given such that $(1 - \epsilon_i)^i > 1 - \epsilon_0$ and suppose we may choose $f_{i_1}, \cdots, f_{i_k}$ with $i_1 < i_2 < \cdots < i_k$.

Letting $h_1, \cdots, h_n$ be in the closed unit ball $B_{\overline{\text{Span}}\{f_1,\cdots,f_{i_k}\}}$ so that each $g \in B_{\overline{\text{Span}}\{f_1,\cdots,f_{i_k}\}}$ lies within $\frac{\epsilon_k}{4}$ of any of $\{h_1, \cdots, h_n\}$. Since, there are correspondingly, associated coherent framelets $\bar{h_1}, \cdots, \bar{h_n}$ in $(g_i)_{i\in I}$ for which $\bar{h}_i h_i > 1 - \frac{\epsilon_k}{4}$ for $i = 1, 2, \cdots, n$, so that an $f_{i_{k+1}} (i_k < i_{k+1})$ can be chosen such that for each $j; j = 1, \cdots, n$, $0 < |\bar{h}_j f_{i_{k+1}}| \leq \frac{\epsilon_k}{4}$. Again, since either $\overline{\text{Span}}(f_i)_{i\in I} = H$ or $(f_i)_{i\in I}$ can still be a frame for the subspace $\overline{\text{Span}}(f_i)_{i\in I}$, so by applying of Mazur's construction of basic sequences [10], we are done if we can show that for a $g \in B_{\overline{\text{Span}}\{f_i,\cdots,f_{i_k}\}}$ and any choice of associated framelet $\bar{h} \in (\bar{h}_i)_{1\leq i\leq n}$, $(1 - \epsilon_k)||g|| \leq ||g + \bar{h}_{i,f} f_{i_{k+1}}||$.

First we assume that $|\bar{h}_{i,f}| < 2$. Then by choosing $h_1, \cdots, h_n$ such that $||g - h_i|| < \frac{\epsilon_k}{4}(1 \leq i \leq n)$, we have

$$\begin{aligned} ||g + \bar{h}_{i,f} f_{i_{k+1}}|| &\geq |\bar{h}_i(g + \bar{h}_{i,f} f_{i_{k+1}})| \\ &\geq |\bar{h}_i h_i| - |\bar{h}_i(g - h_i)| - |\bar{h}_i(\bar{h}_{i,f} f_{i_{k+1}})| \\ &\geq (1 - \frac{\epsilon_k}{4}) - ||g - h_i|| - 2|\bar{h}_i f_{i_{k+1}}| \geq (1 - \epsilon_k)||g||. \end{aligned}$$

Now assume that $|\bar{h}_{i,f}| \geq 2$.
Then it follows that

$$||g + \bar{h}_{i,f} f_{i_{k+1}}|| \geq |\bar{h}_{i,f}|||f_{i_{k+1}}|| - ||g|| \geq (1 - \epsilon_k)||g||.$$

□

Remark. It is a classical result, cf.[1, Theorem 3.2], that if X is a Banach space, then each series $\sum_i a_i x_i$ converges uncoditionally for some coefficient scalars (a_i) so that (x_i) is an unconditional basis for its closed span if and only if X does not contain a subspace isomorphic to c_0. Therefore, using such terminology, the associated coherent framelets version states as follows : if a normalized frame $((f_i)_{i \in I}$ for H is unconditionally framelets Cauchy if and only if H contains no copy of c_0.

Consequently, we have :

Corollary 3.5. Every Riesz frame with action of unconditionally framelets Cauchy is equivalent to the unit vector basis of c_0.

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