KYUNGPOOK Math. J. 45(2005), 161-166

Generalized Transformation Semigroups Whose Sets of Quasi-ideals and Bi-ideals Coincide

Ronnason Chinram
Department of Mathematics, Faculty of Science, Prince of Songkla University, Songkhla 90110, Thailand
e-mail : ronnason.c@psu.ac.th

Abstract. Let $\boldsymbol{BQ}$ be the class of all semigroups whose bi-ideals are quasi-ideals. It is known that regular semigroups, right [left] simple semigroups and right [left] 0-simple semigroups belong to $\boldsymbol{BQ}$. Every zero semigroup is clearly a member of this class. In this paper, we characterize when generalized full transformation semigroups and generalized Baer-Levi semigroups are in $\boldsymbol{BQ}$ in terms of the cardinalities of sets.

1. Introduction and preliminaries

A subsemigroup Q of a semigroup S is called a *quasi-ideal* of S if $SQ \cap QS \subseteq Q$, and by a *bi-ideal* of S we mean a subsemigroup B of S such that $BSB \subseteq B$. Quasi-ideals are a generalization of left ideals and right ideals and bi-ideals are a generalization of quasi-ideals. Moreover, the intersection of a left ideal and a right ideal of S is a quasi-ideal of S and every quasi-ideal of S can be obtained in this way. The notion of quasi-ideal was first introduced by O. Steinfeld in [7]. In fact, the notion of bi-ideal was given earlier by R. A. Good and D. R. Hughes [3]. It was actually introduced in 1952.

For a nonempty subset A of a semigroup S, $(A)_q$ and $(A)_b$ denote the quasi-ideal and the bi-ideal of S generated by A, respectively, that is $(A)_q$ is the intersection of all quasi-ideals of S containing A and $(A)_b$ is the intersection of all bi-ideals of S containing A ([8], page 10 and 12).

Proposition 1.1 ([2], page 84-85)**.** *For any nonempty subset A of S,*

$$(A)_q = S^1A \cap AS^1 \text{ and } (A)_b = AS^1A \cup A.$$

Let $\boldsymbol{BQ}$ denote the class of all semigroups whose sets of bi-ideals and quasi-ideals coincide. It is known that the following semigroups belong to $\boldsymbol{BQ}$: regular semigroups ([6]), left [right] simple semigroups ([5]) and left [right] 0-simple semigroups ([5]). Not only these semigroups are in $\boldsymbol{BQ}$. A nontrivial zero semigroup is an obvious example. In fact, J. Calais [1] has characterized the semigroups in $\boldsymbol{BQ}$ as follows: A semigroup S is in $\boldsymbol{BQ}$ if and only if $(x, y)_q = (x, y)_b$ for all $x, y \in S$.

Received August 13, 2003, and, in revised form, February 14, 2005.
2000 Mathematics Subject Classification: 20M20.
Key words and phrases: generalized transformation semigroups, bi-ideals, quasi-ideals.

It is not easy to see from this characterization whether a given semigroup belongs to $\boldsymbol{BQ}$. The purpose of this paper is to find necessary and sufficient conditions for generalized full transformation semigroups and generalized Baer-Levi semigroups to belong to $\boldsymbol{BQ}$.

2. Generalized full transformation semigroups

Let X be a nonempty set and $T(X)$ the full transformation semigroup on X. It is well-known that $T(X)$ is regular ([4], page 14), so $T(X) \in \boldsymbol{BQ}$. To generalize this, let X and Y be nonempty sets, k a cardinal number such that $k > 0$,

$$T(X,Y) \;=\; \text{the set of all full transformations from } X \text{ to } Y$$

and

$$T_k(X,Y) = \{\alpha \in T(X,Y) \mid |\operatorname{ran}\alpha| \leqslant k\}.$$

For a cardinal number k such that $k > 0$ and $\theta \in T(Y,X)$, let $(T_k(X,Y),\theta)$ be a semigroup $(T_k(X,Y),*)$ where $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in T_k(X,Y)$. It is well-defined because $\operatorname{ran}\alpha\theta\beta \subseteq \operatorname{ran}\beta$ and $|\operatorname{ran}\beta| \leqslant k$ for all $\alpha, \beta \in T_k(X,Y)$. The following theorem characterizes when the semigroups $(T_k(X,Y),\theta)$ are regular.

Theorem 2.1. *For a cardinal number k such that $k > 0$ and $\theta \in T(Y,X)$, $(T_k(X,Y),\theta)$ is regular if and only if one of the following statements holds.*

(i) $|X| = 1$. (ii) $|Y| = 1$. (iii) $k = 1$. (iv) θ is bijective.

Proof. Assume that $|X| = 1, |Y| = 1$ or $k = 1$. Let $\alpha,\ \beta \in T_k(X,Y)$. Since $\operatorname{ran}\alpha\theta\beta\theta\alpha \subseteq \operatorname{ran}\alpha$ and $|\operatorname{ran}\alpha| = 1$, $\operatorname{ran}\alpha\theta\beta\theta\alpha = \operatorname{ran}\alpha$, and hence $\alpha\theta\beta\theta\alpha = \alpha$.

Next, assume that θ is bijective. Let $\alpha \in T_k(X,Y)$. For all $y \in \operatorname{ran}\alpha$, let $x_y \in X$ be such that $x_y\alpha = y$. Let $y_0 \in \operatorname{ran}\alpha$. Define $\beta \in T(X,Y)$ by

$$x\beta = \begin{cases} x_y\theta^{-1} & \text{if } x = y\theta \ \text{ for some } y \in \operatorname{ran}\alpha, \\ x_{y_0}\theta^{-1} & \text{if } x \in X \setminus (\operatorname{ran}\alpha)\theta. \end{cases}$$

Then $|\operatorname{ran}\beta| = |\{x_y\theta^{-1} \mid y \in \operatorname{ran}\alpha\}| = |\operatorname{ran}\alpha| \leqslant k$. Therefore $\beta \in T_k(X,Y)$. For $y \in \operatorname{ran}\alpha$, we have

$$y\theta\beta\theta\alpha = x_y\theta^{-1}\theta\alpha = x_y\alpha = y.$$

Then $\theta\beta\theta\alpha|_{\operatorname{ran}\alpha} = id_{\operatorname{ran}\alpha}$, and hence $\alpha\theta\beta\theta\alpha = \alpha$.

To prove the converse, assume that $k,\ |X|,\ |Y| > 1$ and θ is not bijective.

Case 1: θ is not one-to-one. Then there exist $y_1, y_2 \in Y$ such that $y_1 \neq y_2$ but $y_1\theta = y_2\theta$. Since $|X| > 1$, let $x_1, x_2 \in X$ be such that $x_1 \neq x_2$. Define $\alpha \in T_k(X,Y)$ by

$$x\alpha = \begin{cases} y_1 & \text{if } x = x_1, \\ y_2 & \text{if } x \neq x_1. \end{cases}$$

Let $\beta \in T_k(X,Y)$. We have

$$x_1\alpha = y_1 \neq y_2 = x_2\alpha$$

but

$$x_1\alpha\theta\beta\theta\alpha = y_1\theta\beta\theta\alpha = y_2\theta\beta\theta\alpha = x_2\alpha\theta\beta\theta\alpha.$$

Therefore $\alpha \neq \alpha\theta\beta\theta\alpha$ for all $\beta \in T_k(X,Y)$.

Case 2: θ is not onto. Let $y_1, y_2 \in Y$ be distinct. Let $\alpha \in T_k(X,Y)$ be such that

$$x\alpha = \begin{cases} y_1 & \text{if } x \in \operatorname{ran}\theta, \\ y_2 & \text{otherwise.} \end{cases}$$

Then $\operatorname{ran}\alpha = \{y_1, y_2\}$ and $\operatorname{ran}\theta\alpha = \{y_1\}$. Then for all $\beta \in T_k(X,Y)$, since $\operatorname{ran}\alpha\theta\beta\theta\alpha \subseteq \operatorname{ran}\theta\alpha = \{y_1\}$, $\alpha\theta\beta\theta\alpha \neq \alpha$.

From both two cases, $(T_k(X,Y),\ \theta)$ is not regular. □

Proposition 2.2. *If $|X|,\ |Y|,\ k > 1$, then $(T_k(X,Y),\theta)$ is neither left simple nor right simple.*

Proof. Let $\alpha \in T_1(X,Y)$ and $\beta \in T_k(X,Y)$. Since $\operatorname{ran}\beta\theta\alpha \subseteq \operatorname{ran}\alpha$, $|\operatorname{ran}\beta\theta\alpha| = 1$, so $\beta\theta\alpha \in T_1(X,Y)$. We have $\operatorname{ran}\alpha\theta\beta = (\operatorname{ran}\alpha)\theta\beta$. So $|\operatorname{ran}\alpha\theta\beta| = |(\operatorname{ran}\alpha)\theta\beta| \leqslant |\operatorname{ran}\alpha| = 1$, it follows $|\operatorname{ran}\alpha\theta\beta| = 1$. Thus $\alpha\theta\beta \in T_1(X,Y)$. Hence $T_1(X,Y)$ is a proper ideal of $(T_k(X,Y),\theta)$. Therefore $(T_k(X,Y),\theta)$ is neither left simple nor right simple, as required. □

Theorem 2.3. *For a cardinal number k such that $k > 0$ and $\theta \in T(Y,X)$, $(T_k(X,Y),\theta)$ belongs to $\boldsymbol{BQ}$.*

Proof. Since every quasi-ideal of any semigroup S is a bi-ideal of S, it suffices to show that every bi-ideal of $(T_k(X,Y),\theta)$ is a quasi-ideal of $(T_k(X,Y),\theta)$.

Let B be a bi-ideal of $(T_k(X,Y),\theta)$. Let $\alpha \in B\theta T_k(X,Y) \cap T_k(X,Y)\theta B$. Then $\alpha = \beta\theta\gamma = \lambda\theta\eta$ for some $\beta,\eta \in T_k(X,Y)$ and $\gamma,\lambda \in B$. Since $\lambda\theta \in T(X)$ and $T(X)$ is regular, $\lambda\theta\mu\lambda\theta$ for some $\mu \in T(X)$. Then we have

$$\alpha = \lambda\theta\eta = \lambda\theta\mu\lambda\theta\eta = \lambda\theta\mu\beta\theta\gamma.$$

Since $\operatorname{ran}\mu\beta \subseteq \operatorname{ran}\beta, \mu\beta \in T_k(X,Y)$. Therefore $\alpha = \lambda\theta\mu\beta\theta\gamma \in B\theta T_k(X,Y)\theta B \subseteq B$, and hence B is a quasi-ideal of $(T_k(X,Y),\theta)$, as required. □

Remark 2.4. From Theorem 2.1 and Proposition 2.2, we have known that if $|X|,|Y|,k > 1$ and θ is not a bijection, then $(T_k(X,Y),\theta)$ is not regular, not left simple and not right simple. However, we have shown in Theorem 2.3, it belongs to $\boldsymbol{BQ}$. Then $(T_k(X,Y),\theta)$ where $|X|,|Y|,k > 1$ and θ is not a bijection becomes a nontrivial example of semigroups without zero in $\boldsymbol{BQ}$ which are not regular, not left simple and not right simple.

3. Generalized Baer-Levi semigroups

Let X be a countably infinite set and $BL(X) = \{\alpha \in T(X) \mid \alpha$ is one-to-one and $X \setminus \operatorname{ran}\alpha$ is infinite$\}$. $BL(X)$ is called *Baer-Levi semigroup* on X([4], page 14). We have known that $BL(X)$ is right simple ([4], page 14). Then $BL(X)$ belongs to $\boldsymbol{BQ}$. To generalize this, let X and Y be infinite sets, k an infinite cardinal number such that $k \leqslant |Y|$,

$$M(Y,X) \;=\; \{\alpha \in T(Y,X) \mid \alpha \text{ is one-to-one}\} \text{ and}$$

$$BL_k(X,Y) = \{\alpha \in T(X,Y) \mid \alpha \text{ is one-to-one and } |Y \setminus \operatorname{ran}\alpha| \geqslant k\}.$$

It is clear that $BL_k(X,Y) \neq \emptyset$ and $M(Y,X) \neq \emptyset$ if and only if $|X| = |Y|$.

Proposition 3.1. *Let X and Y be infinite sets such that $|X| = |Y|$ and k an infinite cardinal number such that $k \leqslant |Y|$. If $\theta \in M(Y,X)$, then $\alpha\theta\beta \in BL_k(X,Y)$ for all $\alpha,\ \beta \in BL_k(X,Y)$.*

Proof. Since $\alpha,\ \theta$ and β are one-to-one, $\alpha\theta\beta$ is one-to-one. Since $\operatorname{ran}\alpha\theta\beta \subseteq \operatorname{ran}\beta$, $|Y \setminus \operatorname{ran}\alpha\theta\beta| \geqslant |Y \setminus \operatorname{ran}\beta| \geqslant k$. Therefore $\alpha\theta\beta \in BL_k(X,Y)$, as required. □

In the reminder, let X and Y be infinite sets such that $|X| = |Y|$ and k an infinite cardinal number such that $k \leqslant |Y|$. For $\theta \in M(Y,X)$, let $(BL_k(X,Y),\ \theta)$ be a semigroup $(BL_k(X,Y), *)$ where $\alpha * \beta \;=\; \alpha\theta\beta$ for all $\alpha,\ \beta \in BL_k(X,Y)$.

Proposition 3.2. *$(BL_k(X,Y),\theta)$ is not regular for all $\theta \in M(Y,X)$.*

Proof. Let $\alpha,\ \beta \in BL_k(X,Y)$ be such that $\alpha \;=\; \alpha\theta\beta\theta\alpha$. Since α is one-to-one, $\alpha\theta\beta\theta = id_X$. Since $\theta\beta\theta$ is one-to-one, α must be onto, a contradiction. Therefore $(BL_k(X,Y),\ \theta)$ is not regular. □

Lemma 3.3. *Let $\theta \in M(Y,X)$. If $|X| = |Y| = k$, then $(BL_k(X,Y),\ \theta)$ is right simple.*

Proof. Let R be a right ideal of $(BL_k(X,Y),\theta), \alpha \in R$ and $\beta \in BL_k(X,Y)$. By infiniteness of $Y \setminus \operatorname{ran}\beta$, there exist Y_1 and Y_2 such that $|Y \setminus \operatorname{ran}\beta| = |Y_1| = |Y_2|, Y \setminus \operatorname{ran}\beta = Y_1 \cup Y_2$ and $Y_1 \cap Y_2 = \emptyset$. Since $|X \setminus X\alpha\theta| \leqslant |X| = k = |Y \setminus \operatorname{ran}\beta| = |Y_1|$, there exists $\phi : (X \setminus X\alpha\theta) \to Y_1$ is injective. For all $x \in X\alpha\theta$, let $a_x \in X$ be such that $x = a_x\alpha\theta$. Define $\gamma \in T(X,Y)$ by

$$x\gamma = \begin{cases} a_x\beta & \text{if } x \in X\alpha\theta, \\ x\phi & \text{if } x \in X \setminus X\alpha\theta. \end{cases}$$

Then γ is one-to-one and $|Y \setminus \operatorname{ran}\gamma| \geqslant |Y_2| = k$. Therefore $\gamma \in BL_k(X,Y)$. Since $\alpha\theta$ is one-to-one, $x\alpha\theta\gamma = x\beta$ for all $x \in X$. Hence $\alpha\theta\gamma = \beta$. Therefore $R = BL_k(X,Y)$, and hence $(BL_k(X,Y),\theta)$ is right simple. □

The following theorem gives necessary and sufficient condition for $(BL_k(X,Y),\ \theta)$ to belong to $\boldsymbol{BQ}$.

Theorem 3.4. $(BL_k(X,Y),\ \theta) \in \boldsymbol{BQ}$ *if and only if* $|X| = |Y| = k$.

Proof. If $|X| = |Y| = k$, by Lemma 3.3, $(BL_k(X,Y),\theta)$ is right simple, so $(BL_k(X,Y),\theta) \in \boldsymbol{BQ}$.

For the converse, assume that $k < |X| = |Y|$. By infiniteness of Y, there exist subsets Y_1, Y_2 and Y_3 of Y such that

$$|Y| = |Y_1| = |Y_2|, \quad Y = Y_1 \cup Y_2, Y_1 \cap Y_2 = \emptyset,$$
$$Y_3 \subseteq Y_2 \text{ and } |Y_2 \smallsetminus Y_3| = |Y_2| = |Y_3|.$$

Let Y_4 be a subset of Y such that $|Y_4| = k$. Then $|Y \smallsetminus Y_4| = |Y|$. Since $|X| = |Y_1|$ and $|X| = |Y \smallsetminus Y_4|$, there are bijections $\alpha : X \to Y_1$ and $\beta : X \to Y \smallsetminus Y_4$. It follows that $\alpha, \beta \in BL_k(X,Y)$. Then $\alpha\theta : X \to Y_1\theta$. Moreover, $(\alpha\theta)^{-1}\beta\theta\alpha : Y_1\theta \to Y_1$ is one-to-one. Since $Y_2\theta \subseteq X \smallsetminus Y_1\theta$ and θ is one-to-one, $|X \smallsetminus Y_1\theta| = |Y_2\theta| = |Y_2| = |Y_3|$. Thus there is a bijection $\phi : X \smallsetminus Y_1\theta \to Y_3$. Define $\gamma : X \to Y$ by

$$x\gamma = \begin{cases} x(\alpha\theta)^{-1}\beta\theta\alpha & \text{if } x \in Y_1\theta, \\ x\phi & \text{if } x \in X \smallsetminus Y_1\theta. \end{cases}$$

Since $Y_1 \cap Y_3 = \emptyset$ and θ is one-to-one, γ is one-to-one. Also we have

$$\operatorname{ran}\gamma \subseteq Y_1 \cup Y_3$$

which implies that

$$|Y \smallsetminus \operatorname{ran}\gamma| \geqslant |Y_2 \smallsetminus Y_3| = |Y_2| = |Y| \geqslant k.$$

Therefore $\gamma \in BL_k(X,Y)$. By definition of γ and $\operatorname{ran}\alpha\theta = Y_1\theta$, $\gamma|_{\operatorname{ran}\alpha\theta} = (\alpha\theta)^{-1}\beta\theta\alpha|_{\operatorname{ran}\alpha\theta}$. Thus $\beta\theta\alpha = \alpha\theta\gamma \in BL_k(X,Y)\theta\alpha \cap \alpha\theta BL_k(X,Y) \subseteq (\alpha)_q$. Suppose that $\beta\theta\alpha \in (\alpha)_b$. By Proposition 1.1,

$$(\alpha)_b = \alpha\theta BL_k(X,Y)\theta\alpha \cup \{\alpha, \alpha\theta\alpha\}.$$

Case 1: $\beta\theta\alpha = \alpha$. Since α is one-to-one, $\beta\theta = id_X$. Since θ is one-to-one, β is onto, a contradiction.

Case 2: $\beta\theta\alpha = \alpha\theta\alpha$. Then $\beta = \alpha$ because α and θ are one-to-one. This is a contradiction because $|Y \smallsetminus \operatorname{ran}\beta| = k < |Y| = |Y_2| = |Y \smallsetminus \operatorname{ran}\alpha|$.

Case 3: $\beta\theta\alpha \in \alpha\theta BL_k(X,Y)\theta\alpha$. Then $\beta\theta\alpha = \alpha\theta\lambda\theta\alpha$ for some $\lambda \in BL_k(X,Y)$. This implies that $\beta = \alpha\theta\lambda$ since α and θ are one-to-one. Hence

$$\begin{aligned} |Y_4| = |Y \smallsetminus \operatorname{ran}\beta| &= |Y \smallsetminus \operatorname{ran}\alpha\theta\lambda| \geqslant |X\lambda \smallsetminus (\operatorname{ran}\alpha\theta)\lambda| \\ &= |(X \smallsetminus \operatorname{ran}\alpha\theta)\lambda| \quad \text{since } \lambda \text{ is one-to-one} \\ &= |(X \smallsetminus Y_1\theta)\lambda| \geqslant |Y_2\theta\lambda| = |Y_2| \quad \text{since } \theta\lambda \text{ is one-to-one,} \end{aligned}$$

a contradiction.

Hence $(BL_k(X,Y),\theta) \notin \boldsymbol{BQ}$. □

Corollary 3.5. *Let* $\theta \in M(Y,X)$. *Then the following statements are equivalent*:

(i) $|X| = |Y| = k$.

(ii) $(BL_k(X, Y),\ \theta)$ *is right simple.*

(iii) $(BL_k(X, Y),\ \theta) \in \boldsymbol{BQ}$.

Proof. By Lemma 3.3, Theorem 3.4 and the fact that every right simple semigroup belongs to $\boldsymbol{BQ}$. □

References

[1] J. Calais, *Demi-groups quasi-inversif*, C. R. Acad. Paris, **252**(1961), 2357-2359.

[2] A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Vol. I., Amer. Math. Soc., Providence I(1961).

[3] R. A. Good and D. R. Hughes, *Associated groups for a semigroup*, Bull. Amer. Math. Soc., **58**(1952), 624-625(Abstract).

[4] P. Higgins, Techniques of Semigroup Theory, Oxford University Press: Oxford-New York-Tokyo, (1992).

[5] K. M. Kapp, *On bi-ideals and quasi-ideals in semigroups*, Publ. Math. Debrecen, **16**(1969), 179-185.

[6] S. Lajos, *Generalized ideals in semigroups*, Acta. Sci. Math., **22**(1961), 217-222.

[7] O. Steinfeld, *Über die quasiideale von halbgruppen*, Publ. Math. Debrecen, **4**(1956), 262-275.

[8] O. Steinfeld, Quasi-ideals in Rings and Semigroups, Akadémiai Kiadó Budapest, (1978).