Honam Mathematical J. **27** (2005), No. 3, pp. 445–463

GENERALIZED FUZZY WEAK VECTOR QUASIVARIATIONAL-LIKE INEQUALITIES*

BYUNG-SOO LEE

Abstract. In this paper, we introduce a Stampacchia type of generalized weak vector quasivariational-like inequalities for fuzzy mappings and consider the existence of solutions to them under non-compact assumption.

1. Introduction

The vector variational inequality has grown to be the central part of nonlinear functional analysis in the academic and professional communities since the path-breaking paper [14] introduced it firstly. And then Chen and Cheng [9] published the first primary existence result for the following Stampacchia-type of weak vector variational inequality for single-valued mappings:
Find a vector $\overline{x} \in K$ such that

$$\langle T(\overline{x}), y - \overline{x} \rangle \notin -int\, C, \quad y \in K,$$

where K and C are a subset and a closed convex cone of Hausdorff topological vector spaces X and Y, respectively, and $T : K \to L(X,Y)$ is a mapping from K to the set $L(X,Y)$ of all linear continuous mappings from X to Y.

Received July 8, 2005. Revised August 22, 2005.

2000 Mathematics Subject Classification : 49J40.

Key words and phrases : Fuzzy mapping, weak vector quasivariational-like inequality, genealized Ky Fan's section theorem, topologically open fuzzy set-valued, weakly open fuzzy set-valued.

* This research was supported by Kyungsung University Research Grants in 2004.

Konnov and Yao [19] considered the following Stampacchia-type of weak vector variational-like inequality for set-valued mappings:
Find $\overline{x} \in K$ and $s \in T(\overline{x})$ such that

$$\langle s, \eta(\overline{x}, y)\rangle \notin -int\, C, \quad y \in K,$$

where $\eta : K \times K \to X$ is a mapping, $T : K \to 2^{L(X,Y)}$ is a set-valued mapping and $C : K \to 2^Y$ is a set-valued mapping such that $C(x)$ is a closed and convex cone with $C(x) \neq Y$ and $int\, C(x) \neq \emptyset$.

Chen and Li [11] considered the following weak vector quasivariational inequality for set-valued mappings:
Find $\overline{x} \in K(\overline{x})$ and $s \in T(\overline{x})$ such that

$$\langle s, y - \overline{x}\rangle \notin -int\, C, \quad y \in K(\overline{x}),$$

where $K : X_0(\subset X) \to 2^{X_0}$ and $T : K_0 \to 2^{L(X,Y)}$ are set-valued mappings.

Lee et al. [27] considered the following weak vector quasivariational inequality for set-valued mappings:
Find $\overline{x} \in S(\overline{x})$ and $\overline{y} \in T(\overline{x})$ such that

$$\psi(\overline{x}, \overline{y}, x) - \psi(\overline{x}, \overline{y}, \overline{x}) \notin -int\, K, \quad x \in S(\overline{x}),$$

where C, D are subsets of a locally convex Hausdorff topological space X, K is a cone of a Hausdorff topological vector space Y, and $T : C \to 2^X$, $S : C \to 2^C$ are set-valued mappings and $\psi : C \times D \times C \to Y$ is a mapping. Also they considered the following weak vector quasivariational-like inequality for set-valued mapping:
Find $\overline{x} \in S(\overline{x})$ and $\overline{y} \in T(\overline{x})$ such that

$$\langle M(\overline{x}, \overline{y}), \eta(x, \overline{x})\rangle \notin -int\, K \quad \text{for} \quad x \in S(\overline{x}),$$

where $M : C \times D \to L(X,Y)$ and $\eta : C \times C \to X$ are mappings.

Besides, many other vector variational-like inequalities and vector quasivariational inequalities were considered on topological vector spaces in [1, 4, 8, 11, 27, 32-33] and [16, 18, 27, 34], respectively.

Very recently, Qun [31] obtained existence theorems for the following vector variational-like inequalities for set-valued mappings:
Find $\overline{x} \in K$ such that there exists $\overline{s} \in T(\overline{x})$ satisfying

$$\langle \overline{s}, \eta(y, \overline{x}) \rangle \notin -int\, C(\overline{x}), \quad y \in K,$$

under both the compact and non-compact assumption of K, where X, Y are Hausdorff topological vector spaces, K is a subset of X and $\eta : K \times K \to X$ is a mapping. Especially, in the non-compact case, by using the concept of escaping sequence, he obtained the existence theorem of solutions.

On the other hand, since Chang and Zhu [7] introduced a variational inequality problem for fuzzy mappings, some authors [4, 5, 17, 20, 22, 24-25] considered vector variational inequality problems for fuzzy mappings. In particular, Chang et al. studied vector quasivariational inequalities [4-5] and vector variational-like inequalities [3] for fuzzy mappings and Lee et al. [20] obtained a fuzzy extension of Siddiqi et al.'s results [32] for vector variational-like inequalities. In [17], the authors considered the existence of solutions to generalized fuzzy vector quasivariational-like inequalities **(F-VQVLI)** under the compact assumption.

In this paper, we introduce a Stampacchia type of generalized weak vector quasivariational-like inequalities for fuzzy mappings **(F-WVQVL I)** and consider the existence of solutions for them under the non-compact assumption. Our results generalize the existences of solutions for **(F-VQVLI)** studied in [17].

2. Preliminaries

Let X, Y be topological spaces and $T : X \to 2^Y$ be a multivalued mapping. Let $T^- : Y \to 2^X$ be a multivalued mapping defined by

$$x \in T^-(y) \quad \text{if and only if} \quad y \in T(x).$$

DEFINITION 2.1 [9].

(1) T is said to be upper semicontinuous (in short, u.s.c.) at $x \in X$ if, for every open set V in Y containing $T(x)$, there is an open set U containing x such that $T(u) \subseteq V$ for all $u \in U$;
(2) T is said to be u.s.c. on X if T is u.s.c. at every point of X.
(3) T is said to be lower semicontinuous (in short, l.s.c.) at $x \in X$ if, for every open set V in Y with $T(x) \cap V \neq \emptyset$, there is an open set U containing x such that $T(u) \cap V \neq \emptyset$ for all $u \in U$;
(4) T is said to be l.s.c. on X if T is l.s.c. at every point of X.
(5) T is said to be continuous at x if T is both u.s.c. and l.s.c. at x.
(6) T is said to be compact if $T(X)$ is contained in some compact subset of Y,
(7) T is said to be closed if the graph of T, $G_rT = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.
(8) T is said to have open lower sections if for any $y \in Y$, $T^-(y)$ is open in X.

LEMMA 2.1. *T is l.s.c. at $x \in X$ if, and only if, for any $y \in T(x)$ and for any net $\{x_\alpha\}$ in X converging to x, there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ for each α, and y_α converging to y.*

Let X, Y be sets and $F : X \to \Im(Y)$ be a fuzzy mapping. We denote a fuzzy set $F(x)$ by F_x in Y for $x \in X$, where $\Im(Y)$ is the collection of all fuzzy sets in Y.

DEFINITION 2.2 [3-5, 7].

(1) F is said to be convex on a set X if Y is a convex subset of a topological vector space and for any $x \in X$, y, $z \in Y$ and $\lambda \in [0, 1]$,

$$F_x(\lambda y + (1 - \lambda)z) \geq \min\{F_x(y), F_x(z)\}.$$

(2) F is said to be closed fuzzy set-valued if for each $y \in Y$, $F_x(y)$ is u.s.c. on $X \times Y$ as a real ordinary function.
(3) F is said to be topologically open fuzzy set-valued if, for each $x_0 \in X$ and for each open subset V of Y with $F_{x_0}(y) \geq \gamma$ for some

$y \in V$ $(\gamma \in (0,1])$, there is a neighborhood U of x_0 in X such that if $x \in U$, then $F_x(y) \geq \gamma$ for some $y \in V$.

(4) F is said to be weakly open fuzzy set-valued if for each $y \in Y$, $F_x(y)$ is l.s.c. on $X \times Y$ as a real ordinary function.

LEMMA 2.2 [3]. *Let K be a nonempty closed convex subset of a real Hausdorff topological vector space X, D a nonempty closed convex subset of a real Hausdorff topological vector space Y and $\beta : K \to (0,1]$ a l.s.c. function. Let $F : K \to \Im(D)$ be a fuzzy mapping with the cut set $(F_x)_{\beta(x)} := \{d \in D : F_x(d) \geq \beta(x)\} \neq \emptyset$ for any $x \in K$. Let $\overline{F} : K \to 2^D$ be a multivalued mapping defined by $\overline{F}(x) = (F_x)_{\beta(x)}$. If F is a convex fuzzy mapping with closed fuzzy set-values, then $\overline{F}$ is a closed mapping with nonempty convex-values.*

LEMMA 2.3 [4]. *Let X and Y be topological spaces, and $F : X \to \Im(Y)$ be a fuzzy mapping such that for any $x \in X$, the cut set $(F_x)_\gamma := \{y \in Y : F_x(y) \geq \gamma\}$ is nonempty for $\gamma \in (0,1]$. Let $\overline{F} : X \to 2^Y$ be a multivalued mapping defined by $\overline{F}(x) = (F_x)_\gamma$. If F is convex and topologically open fuzzy set-valued, then $\overline{F}$ is a l.s.c. mapping with nonempty convex-values.*

LEMMA 2.4 [5]. *Let K be a nonempty closed convex subset of a real Hausdorff topological vector space X, D a nonempty closed convex subset of a Hausdorff topological vector space Y and $\beta : X \to (0,1]$ a u.s.c. function. Let $F : K \to \Im(D)$ be a fuzzy mapping such that for any $x \in K$, the strong cut set $[F_x]_{\beta(x)} := \{d \in D : F_x(d) > \beta(x)\}$ is nonempty. Let $\overline{F} : K \to 2^D$ be a multivalued mapping defined by $\overline{F}(x) = [F_x]_{\beta(x)}$.*

(1) If F is convex, then $\overline{F}$ has nonempty convex-values.

(2) If F is weakly open fuzzy set-valued, then $\overline{F}$ has open lower sections.

DEFINITION 2.3. Let X, Z be two vector spaces, $K \subset X$ be a nonempty convex set and $P \subset Z$ a pointed, closed convex cone with apex at the origin and nonempty interior *int* P. A multivalued mapping

$H : K \times K \to 2^Z$ is said to be P-convex with respect to the first variable if, for x_1, x_2, $y \in K$, $u_1 \in H(x_1, y)$, $u_2 \in H(x_2, y)$ and $\lambda \in [0, 1]$, there exists $u \in H(\lambda x_1 + (1-\lambda)x_2, y)$ such that

$$\lambda u_1 + (1-\lambda)u_2 \in u + P.$$

DEFINITION 2.4. Let X, Z be vector spaces. A mapping $\eta : X \times X \to Z$ is said to be linear if

$$\eta(\lambda(x_1, y_1) + (x_2, y_2)) = \lambda\eta(x_1, y_1) + \eta(x_2, y_2)$$

for all (x_1, x_2), $(y_1, y_2) \in X \times X$ and $\lambda \in \mathbb{R}$.

DEFINITION 2.5 [36]. A point z_0 in a nonempty subset C of Z is called a vector maximal point of C if the set $\{z \in C : z_0 \leq_P z, z \neq z_0\} = \emptyset$, which is equivalent to

$$C \cap (z_0 + P) = \{z_0\}.$$

LEMMA 2.5 [28]. *Let C be a nonempty compact subset of an ordered Banach space Z. Then* $\max C \neq \emptyset$, *where* $\max C$ *denotes the set of all vector maximal points of C.*

3. Main results

Throughout this section, X denotes a Hausdorff topological vector space, Y is a topological vector space and Z is an ordered topological vector space. Let K be a nonempty convex subset of X, D be a nonempty subset of Y and $\{C(x) : x \in K\}$ be a family of solid convex cones in Z, that is, for any $x \in K$, $int\, C(x)$ is nonempty and $C(x) \neq Z$. Let $F : K \to \Im(D)$ and $G : K \to \Im(K)$ be two fuzzy mappings, $M : K \times D \to 2^{L(X,Z)}$ and $H : K \times K \to 2^Z$ be two multi-valued mappings, $\eta : X \times X \to X$ be a mapping, $\beta : X \to (0, 1]$ be a function and γ be a constant in $(0, 1]$.

An ordering $\leq$ with respect to the cone C in Z is defined as $y \nleq_{int\, C} x$ if and only if $x - y \notin -int\, C$ for $x, y \in Z$.

We consider the existence of solutions to the following Stampacchia type of generalized weak vector quasivariational-like inequalities for fuzzy mappings:

(F-WVQVLI) Find $\bar{x} \in K$ such that there exists $\bar{s} \in (F_{\bar{x}})_{\beta(\bar{x})}$ satisfying the following inequality:

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z)\rangle + u \notin -int\, C(\bar{x})$$

for any $x \in K$, $z \in (G_{\bar{x}})_{\gamma}$ and $u \in H(x, \bar{x})$, where

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z)\rangle = \max_{s \in M(\bar{x}, \bar{s})} \langle s, \eta(x, z)\rangle$$

and $\langle s, \eta(x, z)\rangle$ denotes the evaluation of continuous linear operator s from X into Z at $\eta(x, z)$.

In addition, we obtain the existence of solutions to the following Stampacchia type of generalized vector quasivariational-like inequalities for fuzzy mappings:

(F-VQVLI) Find $\bar{x} \in K$ such that, for any $x \in K$, there exists $\bar{s} \in (F_{\bar{x}})_{\beta(\bar{x})}$ satisfying the following inequality:

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z)\rangle + u \notin -int\, C(\bar{x})$$

for any $z \in (G_{\bar{x}})_{\gamma}$ and $u \in H(x, \bar{x})$.

Replacing $\Im(K)$ and $\Im(D)$ with 2^K and 2^D, respectively, in **(F-WVQVLI)** and **(F-VQVLI)**, we obtain the following Stampacchia type of generalized weak vector quasivariational-like inequalities and vector quasivariational-like inequalities for multivalued mappings:

(WVQVLI) Find $\bar{x} \in K$ and $\bar{s} \in F(\bar{x})$ such that

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z)\rangle + u \notin -int\, C(\bar{x})$$

for any $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$.

(VQVLI) Find $\bar{x} \in K$ such that, for any $x \in K$, that exists $\bar{s} \in F(\bar{x})$ satisfying the following inequality:

$$\max\langle M(\bar{x}, \bar{s}), \eta(x, z)\rangle + u \notin -int\, C(\bar{x})$$

for any $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$.

Deleting a topological vector space Y and a fuzzy mapping F, first, and then replacing Z with an ordered topological vector space Y, $H : K \times K \to 2^Z$ with $H : K \times K \to Y$, and $M : K \times D \to 2^{L(X,Z)}$ with $S : K \to 2^{L(X,Y)}$ in **(F-WVQVLI)** or **(F-VQVLI)**, respectively, we obtain the following vector variation-like inequality for fuzzy mappings:

(F-VVLI) Find $\bar{x} \in K$ satisfying the following inequality:

$$\max\langle S(\bar{x}), \eta(x, y)\rangle + H(x, \bar{x}) \notin -int\, C(\bar{x})$$

for any $x \in K$ and any $y \in (G_{\bar{x}})_{\beta(\bar{x})}$, where $\{C(x) : x \in K\}$ is a family of closed convex cones in Y.

Replacing a fuzzy mapping $G : K \to \Im(K)$ with a multivalued mapping $G : K \to 2^K$ defined by $G(x) = K$ for $x \in K$ and putting $H \equiv 0$ in **(F-VVLI)**, we obtain the following vector variational-like inequalities for multivalued mappings:

(VVLI) Find $\bar{x} \in K$ such that

$$\max\langle S(\bar{x}), \eta(x, y)\rangle \notin -int\, C(\bar{x}), \quad x, y \in K,$$

which is a generalized form of the following vector variational-like inequalities for multivalued mappings introduced and studied by Chang, Thompson and Yuan [6]:

(VVLI)′ Find $\bar{x} \in K$ satisfying the following inequality:

$$\max\langle S(\bar{x}), \eta(x,\bar{x})\rangle \notin -int\, C(\bar{x}) \quad x \in K.$$

Putting $Z = Y$, $\eta(x,z) = x - z$, $H = \bar{0}$ and replacing $M : K \times D \to 2^{L(X,Z)}$ with $S : K \to L(X,Y)$ in **(WVQVLI)** or **(VQVLI)**, respectively, we have the following variational inequality:

(VVI) Find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - z\rangle \notin -int\, C(\bar{x}), \quad x \in K,\, z \in G(\bar{x}).$$

Putting $C(x) \equiv C$ for $x \in K$ and $\eta(x,y) = x - y$ in **(VVLI)′**, we obtain the following vector-valued variational inequality considered by Lee et al. [26]:
Find $\bar{x} \in K$ such that there exists $\bar{s} \in S(\bar{x})$ satisfying

$$\langle \bar{s}, x - \bar{x}\rangle \notin -int\, C, \quad x \in K.$$

Putting $Z = \mathbb{R}$, $L(X,Z) = X^*$, the dual of X and $C(x) \equiv \mathbb{R}^+$, the positive orchant for $x \in K$ in **(VVLI)′**, we obtain the following scalar-valued variational inequality considered by Cottle and Yao [12], Isac [15], and Noor [29]:
Find $\bar{x} \in K$ such that

$$\sup_{u \in S(\bar{x})} \langle u, \eta(x,\bar{x})\rangle \geq 0, \quad x \in K.$$

Replacing $S : K \to 2^{L(X,Z)}$ with $S : X \to L(X,Z)$ and putting $\eta(x,z) = x - g(z)$, respectively, where $g : K \to K$ is a mapping, then **(VVLI)′** reduces to the following vector variational inequality **(VVI)′** considered by Siddiqi et al. [33]:

(VVI)′ Find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - g(\bar{x})\rangle \notin -int\, C(\bar{x}), \quad x \in K.$$

Putting $G(x) = \{x\}$ for $x \in K$ in **(VVI)** or $g(x) = x$ for $x \in K$ in **(VVI)**$'$, we obtain the following vector-valued variational inequality considered by Chen [8]:
Find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - \bar{x} \rangle \notin -int\, C(\bar{x}), \quad \text{for } x \in K.$$

Putting $C(x) \equiv C$ and $g(x) = x$ for $x \in K$ in **(VVI)**$'$, we obtain the following vector-valued variational inequality considered by Chen et al. [8-10]:
Find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - \bar{x} \rangle \notin -int\, C, \quad x \in K.$$

The following particular form of the generalized Ky Fan's section theorem which is due to Park [30] will be used in dealing with **(F-WVQVLI)** for the noncompact set case.

THEOREM 3.1 [30]. *Let K be a nonempty convex subset of X and $A \subset K \times K$ satisfy the following conditions:*

(i) $(x, x) \in A$, $x \in K$;
(ii) $A_x = \{y \in K : (x, y) \in A\}$, $x \in K$, *is closed;*
(iii) $A^y = \{x \in K : (x, y) \notin A\}$, $y \in K$, *is convex or empty;*
(iv) there exists a nonempty compact subset B of K such that for each finite subset N of K there exists a nonempty compact convex subset L_N of K containing N such that

$$L_N \cap \{y \in K : (x, y) \in A \text{ for any } x \in L_N\} \subset B.$$

Then there exists a $y_0 \in B$ such that $K \times \{y_0\} \subset A$.

In particular, if $K = B$, that is, K is a compact convex subset of X, then the condition (iv) is obviously true and thus we obtain Ky Fan's section theorem [13], which will be used in considering **(F-VQVLI)** under compact assumption.

Now, we consider the existence of solutions to the Stampacchia type of **(F-WVQVLI)** for non-compact set case.

LEMMA 3.2 [35]. *Let X be a paracompact Hausdorff topological space and Y be a topological vector space. Let $F : X \to 2^Y$ be a multivalued mapping with nonempty convex-values. If F has open lower sections, then there exists a continuous function $f : X \to Y$ such that $f(x) \in F(x)$ for any $x \in X$.*

PROPOSITION 3.3. *Let K be a nonempty convex subset of X and D be a nonemtpy subset of Y. Let $f : K \to D$ be a continuous function and $G : K \to 2^K$ be a l.s.c. multivalued mapping with convex-values. Let $M : K \times D \to 2^{L(X,Z)}$ be a multivalued mapping, and a multivalued mapping $W : K \to 2^Z$ defined by $W(x) = Z \backslash \{-int\, C(x)\}$, $x \in K$, be closed. Let $\eta : X \times X \to X$ be linear, $y \mapsto \eta(\cdot, y)$ be continuous and $H : K \times K \to 2^Z$ be P-convex with respect to the first variable and l.s.c. with respect to the second, where $P = \bigcap_{x \in K} C(x)$. Suppose further that*

(i) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha)\rangle$ converges to $\max\langle M(y, s), \eta(x, z)\rangle$ provided that $y_\alpha \to y$, $s_\alpha \to s$ and $z_\alpha \to z$;

(ii) $\langle M(x, \cdot), \eta(x, \cdot)\rangle = 0$ and $H(x, x) = \{0\}$ for all $x \in K$,

(iii) there is a nonempty compact subset B of K such that for each nonempty finite subset N of K, there is a nonempty compact convex subset L_N of K containing N such that, for $y \in L_N \backslash B$, there exist $x \in L_N$, $z \in G(y)$ and $u \in H(x, y)$ such that

$$\max\langle M(y, f(y)), \eta(x, z)\rangle + u \in -int\, C(y).$$

Then there exists $\bar{x} \in K$ such that

$$\max\langle M(\bar{x}, f(\bar{x})), \eta(x, z)\rangle + u \notin -int\, C(\bar{x})$$

for any $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$.

Proof. Let $A = \{(x, y) \in K \times K : \max\langle M(y, f(y)), \eta(x, z)\rangle + u \notin -int\, C(y)$ for any $z \in G(y)$ and $u \in H(x, y)\}$. It is easily shown that $(x, x) \in A$ for $x \in K$ from the condition (ii). And $A_x = \{y \in K : (x, y) \in A\}$, $x \in K$, is closed. In fact, for any net $\{y_\alpha\}$ in A_x converging to y, we have $\max\langle M(y_\alpha, f(y_\alpha)), \eta(x, z_\alpha)\rangle + u_\alpha \notin -int\, C(y_\alpha)$ for any $z_\alpha \in G(y_\alpha)$ and $u_\alpha \in H(x, y_\alpha)$. From Lemma 2.1 and the condition (i), $\max\langle M(y, f(y)), \eta(x, z)\rangle + u \notin -int\, C(y)$ for any $z \in G(y)$ and $u \in H(x, y)$, so that we have $y \in A_x$, which shows that A_x is closed for $x \in K$. And $A^y = \{x \in K : (x, y) \notin A\}$, $y \in K$, is convex. Indeed, let x_1, $x_2 \in A^y$ and $\lambda \in [0, 1]$. Then, from the fact that $(x_1, y) \notin A$ for any $s \in \overline{F}(y)$, there exist $z_1 \in \overline{G}(y)$ and $u_1 \in H(x_1, y)$ such that

$$\max\langle M(y, s), \eta(x_1, z_1)\rangle + u_1 \in -int\, C(y)$$

and from the fact that $(x_2, y) \notin A$ for any $s \in \overline{F}(y)$, there exist $z_2 \in \overline{G}(y)$ and $u_2 \in H(x_2, y)$ such that

$$\max\langle M(y, s), \eta(x_2, z_2)\rangle + u_2 \in -int\, C(y).$$

On the other hand, from the convexity of G, $\overline{G}$ is convex-valued due to Lemma 2.4(1). Hence, for any $s \in \overline{F}(y)$, there exist $u \in H(\lambda x_1 + (1 - \lambda)x_2, y)$ and $z := \lambda z_1 + (1 - \lambda)z_2 \in \overline{G}(y)$ for $\lambda \in [0, 1]$ such that

$$\begin{aligned}
&\max\langle M(y, s), \eta(\lambda x_1 + (1 - \lambda)x_2, z)\rangle + u \\
&= \max\langle M(y, s), \eta(\lambda x_1 + (1 - \lambda)x_2, \lambda z_1 + (1 - \lambda)z_2)\rangle + u \\
&= \max\langle M(y, s), (\lambda\eta(x_1, z_1) + (1 - \lambda)\eta(x_2, z_2))\rangle + u \\
&\leq \lambda \max\langle M(y, s), \eta(x_1, z_1)\rangle + (1 - \lambda) \max\langle M(y, s), \eta(x_2, z_2)\rangle + u \\
&\in \lambda \max\langle M(y, s), \eta(x_1, z_1)\rangle + (1 - \lambda) \max\langle M(y, s), \eta(x_2, z_2)\rangle + \lambda u_1 + \\
&\qquad\qquad (1 - \lambda)u_2 - P \\
&= \lambda(\max\langle M(y, s), \eta(x_1, z_1)\rangle + u_1) + \\
&\qquad\qquad (1 - \lambda)(\max\langle M(y, s), \eta(x_2, z_2)\rangle + u_2) - P \\
&\subseteq -\, int\, C(y) - int\, C(y) - C(y) \\
&= -\, int\, C(y).
\end{aligned}$$

Thus $\lambda x_1 + (1 - \lambda)x_2 \in A^y$, which shows that A^y is convex. Further, note that the condition (iii) implies that, for $y \in L_N \backslash B$, there exists

$x \in L_N$ such that $y \notin A_x$. Hence the condition (iv) of Theorem 3.1 is satisfied. Thus there exists $\bar{x} \in K$ such that

$$\max\langle M(\bar{x}, f(\bar{x})), \eta(x,z)\rangle + u \notin -int\, C(\bar{x})$$

for any $x \in K$, $z \in G(\bar{x})$ and $u \in H(x, \bar{x})$. This completes the proof. □

Now, we show the existence of solution for the problem **(F-WVQVLI)**.

THEOREM 3.4. *Let K be a nonempty paracompact convex subset of X and D be a nonempty closed convex subset of Y. Let $F : K \to \Im(D)$ be a convex fuzzy mapping with weakly open fuzzy set-values and nonempty strong cut set $[F_x]_{\beta(x)}$ for a u.s.c. function $\beta : X \to (0,1]$, $G : K \to \Im(K)$ be a convex fuzzy mapping with topologically open fuzzy set-values and nonempty cut set $(G_x)_\gamma$ for $\gamma \in (0,1]$, a multivalued mapping $W : K \to 2^Z$ defined by $W(x) = Z \backslash \{-int\, C(x)\}$, $x \in K$, be closed, and $M : K \times D \to 2^{L(X,Z)}$ be a multivalued mapping. Let $\eta : X \times X \to X$ be linear, $y \mapsto \eta(\cdot, y)$ continuous and $H : K \times K \to 2^Z$ be P-convex with respect to the first variable and l.s.c. with respect to the second, where $P = \bigcap_{x \in K} C(x)$.*

Suppose further that

(i) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha)\rangle \to \max\langle M(y,s), \eta(x,z)\rangle$ provided that $y_\alpha \to y$, $s_\alpha \to s$ and $z_\alpha \to z$,

(ii) $\langle M(x,\cdot), \eta(x,\cdot)\rangle = 0$ and $H(x,x) = \{0\}$ for all $x \in K$,

(iii) there is a nonempty compact subset B of K such that for any nonempty finite subset N of K, there is a nonempty compact convex subset L_N of K containing N such that for any $y \in L_N \backslash B$, there exist $x \in L_N$, $z \in (G_y)_\gamma$ and $u \in H(x,y)$ such that

$$\max\langle M(y,s), \eta(x,z)\rangle + u \in -int\, C(y)$$

for any $s \in (F_y)_{\beta(y)}$.

Then the problem **(F-WVQVLI)** *is solvable, i.e., there exist $\bar{x} \in K$ and $\bar{s} \in (F_{\bar{x}})_{\beta(\bar{x})}$ such that*

$$\max\langle M(\bar{x}, \bar{s}), \eta(x,z)\rangle + u \notin -int\, C(\bar{x})$$

for any $x \in K$, $z \in (G_{\bar{x}})_\gamma$ *and* $u \in H(x, \bar{x})$.

Proof. Define two multivalued mappings $\overline{F} : K \to 2^D$ and $\overline{G} : K \to 2^K$ by $\overline{F}(x) = [F_x]_{\beta(x)}$ and $\overline{G}(x) = (G_x)_\gamma$, for $x \in K$, respectively. It follows from Lemma 2.3 that $\overline{G}$ is l.s.c. and has nonempty convex-values and, from Lemma 2.4, that $\overline{F}$ has nonempty convex-values such that $\overline{F}^-(y)$ is open in X for $y \in D$. Thus, by Lemma 3.2, there exists a continuous function $f : K \to D$ such that $f(x) \in \overline{F}(x)$ for $x \in K$. So, by Proposition 3.3, there exists $\bar{x} \in K$ such that

$$\max\langle M(\bar{x}, f(\bar{x})), \eta(x, z)\rangle + u \notin -int\, C(\bar{x})$$

for any $x \in K$, $z \in (G_{\bar{x}})_\gamma$ and $u \in H(x, \bar{x})$. Letting $\bar{s} = f(\bar{x})$, we obtain the desired conclusion of Theorem 3.4. This completes the proof. □

From Theorem 3.4, we obtain the following theorem for Stampacchia type of the generalized weak vector quasivariational-like inequalities **(WVQVLI)** for multivalued mappings.

THEOREM 3.5. *Let* K *be a nonempty paracompact convex subset of* X *and* D *be a nonemtpy closed convex subset of* Y. *Let* $F : K \to 2^D$ *be a multivalued mapping with nonempty convex-values and open lower sections,* $G : K \to 2^K$ *be a multivalued l.s.c. mapping with nonempty convex-values, a multivalued mapping* $W : K \to 2^Z$ *defined by* $W(x) = Z\backslash\{-int\, C(x)\}$, $x \in K$, *closed, and* $M : K \times D \to 2^{L(X,Z)}$ *be a multivalued mapping. Let* $\eta : X \times X \to X$ *be linear,* $y \mapsto \eta(\cdot, y)$ *be continuous and* $H : K \times K \to 2^Z$ *be* P*-convex with respect to the first variable and l.s.c. with respect to the second, where* $P = \bigcap_{x\in K} C(x)$.

Suppose further that

(i) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha)\rangle \to \max\langle M(y, s), \eta(x, z)\rangle$ *provided that* $y_\alpha \to y$, $s_\alpha \to s$ *and* $z_\alpha \to z$,

(ii) $\langle M(x, \cdot), \eta(x, \cdot)\rangle = 0$ *and* $H(x, x) = \{0\}$ *for all* $x \in K$,

(iii) *there is a nonempty compact subset* B *of* K *such that, for any nonempty finite subset* N *of* K, *there is a nonempty compact convex subset* L_N *of* K *containing* N *such that, for any* $y \in L_N\backslash B$, *there exist* $x \in L_N$, $z \in G(y)$ *and* $u \in H(x, y)$ *such that*

$$\max\langle M(y,s), \eta(x,z)\rangle + u \in -int\, C(y), \quad s \in F(y).$$

Then the problem **(WVQVLI)** *is solvable.*

For the compact set case, by using Ky Fan's section theorem [13], we obtain the following existence of solutions for the vector variational inequalities **(F-VQVLI)**, **(F-VVI)**, **(VQVLI)** as special cases of **(WVQ VLI)**.

COROLLARY 3.6 [17]. *Let K be a nonempty compact convex subset of X and D be a nonemtpy closed convex subset of Y. Let $F : K \to \Im(D)$ be a convex fuzzy mapping with closed fuzzy set-values, $G : K \to \Im(K)$ be a convex fuzzy mapping with topologically open fuzzy set-values, $M : K \times D \to 2^{L(X,Z)}$ be a multivalued mapping and a multivalued mapping $W : K \to 2^Z$ defined by $W(x) = Z \backslash \{-int\, C(x)\}$, $x \in K$, be closed. Let $\eta : X \times X \to X$ be linear, $y \mapsto \eta(\cdot, y)$ be continuous, and $H : K \times K \to 2^Z$ be P-convex with respect to the first variable and l.s.c. with respect to the second, where $P := \bigcap_{x \in K} C(x)$.*

Suppose further that

(i) *there exist a l.s.c. function $\beta : X \to (0,1]$ and a constant $\gamma \in (0,1]$ such that for any $x \in K$, the cut sets $(F_x)_{\beta(x)}$ and $(G_x)_\gamma$ are nonempty;*

(ii) *$\bigcup_{x \in K} (F_x)_{\beta(x)}$ is contained in some compact subset of D;*

(iii) *$\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha)\rangle$ converges to $\max\langle M(y,s), \eta(x,z)\rangle$ provided that $y_\alpha \to y$, $s_\alpha \to s$ and $z_\alpha \to z$;*

(iv) *$\langle M(x,\cdot), \eta(x,\cdot)\rangle = 0$ and $H(x,x) = \{0\}$ for all $x \in K$.*

Then the problem **(F-VQVLI)** *is solvable from Theorem 3.1.*

COROLLARY 3.7 [25]. *Let K be a nonempty compact convex subset of X. Let $F : K \to \Im(L(X,Y))$ be a fuzzy mapping with closed fuzzy set-values, $G : K \to \Im(K)$ be a convex fuzzy mapping with topologically open fuzzy set-values and a multivalued mapping $W : K \to 2^Y$ defined* *by* $W(x) = Y \backslash [-int\, C(x)]$, $x \in K$, *be closed, where* $\{C(x) : x \in K\}$ *is a family of solid convex cones in* Y. *Let* $P = \bigcap_{x \in K} C(x)$ *and* $h : K \to Y$ *be a continuous* P*-convex function. Suppose further that*

(i) *there exist a l.s.c. function* $\beta : X \to (0, 1]$ *and a constant* $\gamma \in (0, 1]$ *such that for any* $x \in K$ *the cut sets* $(F_x)_{\beta(x)}$ *and* $(G_x)_\gamma$ *are nonempty;*

(ii) $\bigcup_{x \in K} (F_x)_{\beta(x)}$ *is contained in some compact subset of* $L(X, Y)$*;*

(iii) *for any* $x \in K$, *there exists* $s \in (F_x)_{\beta(x)}$ *such that* $\langle s, x - z \rangle \notin -int\, C(x)$ *for any* $z \in (G_x)_\gamma$.

Then the following variational inequality:

(F-VVI) *Find* $\bar{x} \in K$ *such that, for any* $x \in K$, *there exists* $\bar{s} \in (F_{\bar{x}})_{\beta(\bar{x})}$ *such that*

$$\langle \bar{s}, x - z \rangle + h(x) - h(\bar{x}) \notin -int\, C(\bar{x}), \quad z \in (G_{\bar{x}})_\gamma,$$

is solvable.

The following theorem for the existence of solutions for **(VQVLI)** is a special case of Corollary 3.6.

COROLLARY 3.8 [17]. *Let* K *be a nonempty compact convex subset of* X *and* D *be a nonempty subset of* Y. *Let* $F : K \to 2^D$ *be closed,* $G : K \to 2^K$ *be l.s.c. and nonempty convex-valued,* $M : K \times D \to 2^{L(X,Z)}$ *be nonempty compact-valued and a multivalued mapping* $W : K \to 2^Z$ *defined by* $W(x) = Z \backslash \{-int\, C(x)\}$, $x \in K$, *be closed. Let* $\eta : X \times X \to X$ *be linear and* $H : K \times K \to 2^Z$ *be* P*-convex with respect to the first variable and l.s.c. with respect to the second, where* $P := \bigcap_{x \in K} C(x)$.

Suppose further that

(i) $\langle M(x, \cdot), \eta(x, \cdot) \rangle = 0$ *and* $H(x, x) = \{0\}$ *for all* $x \in K$*;*

(ii) F *is compact;*

(iii) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha) \rangle$ *converges to* $\max\langle M(y, s), \eta(x, z) \rangle$ *provided that* $y_\alpha \to y$, $s_\alpha \to s$ *and* $z_\alpha \to z$.

Then the problem **(VQVLI)** *is solvable.*

References

[1] Q. H. Ansari, *A note on generalized vector variational-like inequalities*, Optimization **41** (1997), 197–205.

[2] G. H. Ansari, A. H. Siddiqu and J. C. Yao, *Generalized vector variational-like inequalities and their scalarizations*, Vector Variational Inequalities and Vector Equilibria (F. Giannessi, eds.), Kluwer Publishers, Dordrecht, 2000.

[3] S. S Chang, *Coincidence theorems and variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **61** (1994), 359–368.

[4] S. S. Chang, G. M. Lee and B. S. Lee, *Vector quasivariational inequalities for fuzzy mappings (I)*, Fuzzy Sets and Systems **87** (1997), 307–315.

[5] S. S. Chang, G. M. Lee and B. S. Lee, *Vector quasivariational inequalities for fuzzy mappings (II)*, Fuzzy Sets and Systems **102** (1999), 333–344.

[6] S. S. Chang, H. B. Thompson and G. X. Z. Yuan, *The existence theorems of solutions for generalized vector-valued variational-like inequalities*, Comput. Math. Appl. **37** (1999), 1–9.

[7] S. S. Chang and Y. G. Zhu, *On variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **32** (1989), 359–367.

[8] G. Y. Chen, *Existence of solutions for a vector variational inequality: An extension of the Hartmann-Stampacchia theorem*, J. Opti. Th. Appl. **74(3)** (1992), 445–456.

[9] G. Y. Chen and G. M. Cheng, *Vector Variational Inequalities and Vector Optimization*, Lecture Notes in Econ. and Math. Systems **285**, Springer Verlag, Berlin, 1987.

[10] G. Y. Chen and B. D. Craven, *A vector variational inequality and optimization over an efficient set*, Zeitscrift für Operations Research **3** (1990), 1–12.

[11] G. Y. Chen and S. J. Li, *Existence of solutions for a generalized quasi-vector variational inequality*, J. Opti. Th. Appl. Vol. 90 (1996), 321–334.

[12] R. W. Cottle and J. C. Yao, *Pseudo-monotone complementarity problems in Hilbert spaces*, J. Opti. Th. Appl. **75(2)** (1992), 281–295.

[13] Ky Fan, *A generalization of Tychonoff's fixed-point theorem*, Mathematische Annalen **142** (1961), 305–310.

[14] F. Giannessi, *Theorems of alternative, quadratic programs, and complementarity problems*, Variational Inequalities and Complementarity Problems (Cottle, Giannessi and Lions, eds.), John Wiley and Sons, New York, 1980.

[15] G. Isac, *A special variational inequality and the implicit complementarity problem*, J. of the Faculty of Sciences, Univ. of Tokyo, Section IA, Mathematics **37**

(1990), 107–127.

[16] M. K. Kang and B. S. Lee, *Generalized vector quasivariational-like inequalities*, Honam Math. J. **26** (4) (2004).

[17] M. K. Kang and B. S. Lee, *Generalized fuzzy vector quasivariational-like inequalities*, to appear.

[18] W. K. Kim and K. K. Tan, *On generalized vector quasi-variational inequalities*, Optimization **46** (1999), 185–198.

[19] V. Konnov and J. C. Yao, *On the generalized vector variational inequality problem*, J. Math. Anal. Vol. 206 (1997), 42–58.

[20] B. S. Lee and D. Y. Jung, *A fuzzy extension of Siddiqi et al.'s results for vector variational-like inequalities*, Indian J. pure appl. Math. **34** (10) (2003), 1495–1502.

[21] B. S. Lee and G. M. Lee, *A vector version of Minty's lemma and application*, Appl. Math. Lett. **12** (1999), 43–50.

[22] B. S. Lee, G. M. Lee and D. S. Kim, *Generalized vector-valued variational inequalities and fuzzy extensions*, J. Korean Math. Soc. **33** (1996), 609–624.

[23] B. S. Lee, G. M. Lee and D. S. Kim, *Generalized vector variational-like inequalities on locally convex Hausdorff topological vector spaces*, Indian J. pure appl. Math. **28** (1997), 33–41.

[24] G. M. Lee, D. S. Kim and B. S. Lee, *Strongly quasivariational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **78** (1996), 381–386.

[25] G. M. Lee, D. S. Kim and B. S. Lee, *Vector variational inequalities for fuzzy mappings*, Nonlinear Analysis Forum **4** (1999), 119–129.

[26] G. M. Lee, D. S. Kim, B. S. Lee and S. J. Cho, *Generalized vector variational inequality and fuzzy extension*, Appl. Math. Lett. **6** (1993), 47–51.

[27] G. M. Lee, B. S. Lee and S. S. Chang, *On vector quasivariational inequalities*, J. Math. Anal. Appl. **203** (1996), 626–638.

[28] D. C. Luc, *Theory of Vector Optimization*, Lectures Notes in Econ. and Math. Systems, Vol. 319, Springer Verlag, Berlin, 1989.

[29] M. A. Noor, *Generalized variational inequality*, Appl. Math. Lett. **1** (1988), 119–122.

[30] Sehie Park, *A unified approach to generalizations of the KKM-type theorems related to acyclic maps*, Numer. Funct. Anal. & Optimiz. **15** (1994), 105–119.

[31] L. Qun, *Generalized vector variational-like inequalities*, Vector Variational Inequalities and Vector Equilibria (F. Giannessi, eds.), Kluwer Publishers, Dordrecht, 2000.

[32] A. H. Siddiqi, Q. H. Ansari and R. Ahmad, *On vector variational-like inequalities*, Indian J. pure appl. Math. **28(8)** (1997), 1009–1016.

[33] A. H. Siddiqi, Q. H. Ansari and A. Khaliq, *On vector variational inequalities*, J. Opti. Th. Appl. **84** (1995), 171–180.

[34] N. X. Tan, *Quasi-variational inequalities in topological linear locally convex Hausdorff spaces*, Mathematische Nachrichten, **122** (1985), 231–245.

[35] N. C. Yannelis and N. D. Prabhakar, *Existence of maximal elements and equilibria in linear topological spaces*, J. Math. Econ. **12** (1983), 233–245.

[36] P. L. Yu, *Cone convexity, cone extreme points and nondominated solutions in decision problems with multiobjectives*, J. Opti. Th. Appl. **14** (1974), 319–377.

Byung-Soo Lee
Department of Mathematics
Kyungsung University
Busan 608-736, Korea
E-mail : bslee@ks.ac.kr