KYUNGPOOK Math. J. 45(2005), 115-121

Quasi BCC-algebras

YOUNG BAE JUN
Department of Mathematics Education, Gyeongsang National University, Chinju 660-701, Korea
e-mail : ybjun@gsnu.ac.kr

SEOK ZUN SONG
Department of Mathematics, Cheju National University, Cheju 690-756, Korea
e-mail : szsong@cheju.ac.kr

ABSTRACT. Using the concept of fuzzy points, the notions of fuzzy point BCC-(sub)algebras, quasi BCC-(sub)algebras and quasi BCC-ideals are introduced. Some characterizations of them are discussed.

1. Introduction

In 1966, Y. Imai and K. Iséki [8] defined a class of algebras of type (2,0) called *BCK-algebras* which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental non-nullary operation, on the other hand the notion of implication algebra. The class of all BCK-algebras is a quasivariety. K. Iséki posed an interesting problem (solved by A. Wroński [10]) whether the class of BCK-algebras is a variety. In connection with this problem, Y. Komori [9] introduced a notion of BCC-algebras, and W. A. Dudek [1], [2] redefined the notion of BCC-algebras by using a dual form of the ordinary definition in the sense of Y. Komori. In [6], W. A. Dudek and X. H. Zhang introduced a notion of BCC-ideals in BCC-algebras, and W. A. Dudek and Y. B. Jun [4] established the fuzzification of BCC-ideals in BCC-algebras (see also [5]). In this paper, we use the notion of fuzzy points to establish the notion of fuzzy point BCC-(sub)algebras, quasi BCC-(sub)algebras and quasi BCC-ideals. We investigate some related properties.

2. Preliminaries

Recall that a BCC-*algebra* is an algebra $(X, *, 0)$ of type (2,0) satisfying the following axioms:

Received January 9, 2004, and, in revised form, March 19, 2004.
2000 Mathematics Subject Classification: 06F35, 03G25, 03B52.
Key words and phrases: Quasi BCC-(sub)algebra, fuzzy point BCC-(sub)algebra, quasi BCC-ideal.
This work was supported by grant R01-2001-000-00004-0 from the Korea Science and Engineering Foundation.

(C1) $((x * y) * (z * y)) * (x * z) = 0$,

(C2) $0 * x = 0$,

(C3) $x * 0 = x$,

(C4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

for every $x, y, z \in X$. For any BCC-algebra X, the relation $\leq$ defined by $x \leq y$ if and only if $x * y = 0$ is a partial order on X. In a BCC-algebra X, the following holds (see [7]).

- $x * y \leq x$,
- $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$

for all $x, y \in X$. A nonempty subset S of a BCC-algebra X is said to be a BCC-*subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A nonempty subset A of a BCC-algebra X is called a BCC-*ideal* of X if it satisfies

- $0 \in A$,
- $(x * y) * z \in A$ and $y \in A$ imply $x * z \in A$ for all $x, y, z \in X$.

Note that every BCC-ideal of a BCC-algebra X is a BCC-subalgebra of X. A fuzzy set μ in a set X is called a *fuzzy point* if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is $\alpha \in (0, 1]$ we denote this fuzzy point by x_α, where the point x is called its *support*. For a fuzzy set μ in X and $\alpha \in [0, 1]$, the set $U(\mu; \alpha) := \{x \in X \mid \mu(x) \geq \alpha\}$ is called a *upper level set* of μ. A fuzzy set μ in a BCC-algebra X is called a *fuzzy* BCC-*subalgebra* of X if $\mu(x * y) \geq \min\{\mu(x),\, \mu(y)\}$ for all $x, y \in X$. A fuzzy set μ in a BCC-algebra X is called a *fuzzy* BCC-*ideal* of X it it satisfies

- $\mu(0) \geq \mu(x)$ for all $x \in X$,
- $\mu(x * z) \geq \min\{\mu((x * y) * z),\, \mu(y)\}$ for all $x, y, z \in X$.

3. Quasi BCC-algebras

In what follows, let X denote a BCC-algebra unless otherwise specified. Let $\mathbb{F}(X)$ denote the set of all fuzzy points in X and define a binary operation "$\odot$" on $\mathbb{F}(X)$ by $x_\alpha \odot y_\beta = (x * y)_{\min\{\alpha,\beta\}}$ for all $x_\alpha, y_\beta \in \mathbb{F}(X)$. We can easily check the following results:

(q1) $((x_\alpha \odot y_\beta) \odot (z_\gamma \odot y_\beta)) \odot (x_\alpha \odot z_\gamma) = 0_{\min\{\alpha,\beta,\gamma\}}$,

(q2) $0_\delta \odot x_\alpha = 0_{\min\{\alpha,\delta\}}$,

(q3) $x_\alpha \odot 0_\delta = x_{\min\{\alpha,\delta\}}$

for all $x_\alpha, y_\beta, z_\gamma, 0_\delta \in \mathbb{F}(X)$. But the following implication (q4) $x_\alpha \odot y_\beta = 0_{\min\{\alpha,\beta\}},\ y_\beta \odot x_\alpha = 0_{\min\{\alpha,\beta\}} \Rightarrow x_\alpha = y_\beta$

does not hold in general. For, if we consider two fuzzy points $b_{0.3}$ and $b_{0.7}$ where b is an element of a BCC-algebra $(X, *, 0)$, then

$$b_{0.3} \odot b_{0.7} = (b * b)_{\min\{0.3,\,0.7\}} = 0_{0.3}$$

and

$$b_{0.7} \odot b_{0.3} = (b * b)_{\min\{0.3,\,0.7\}} = 0_{0.3},$$

but $b_{0.3} \neq b_{0.7}$.

Hence $\mathbb{F}(X)$ may not be a BCC-algebra under the operation "$\odot$". We now call $\mathbb{F}(X)$ the *quasi* BCC-*algebra*.

Definition 3.1. A subset Ω of the quasi BCC-algebra $\mathbb{F}(X)$ is called a *quasi* BCC-*subalgebra* if $x_\alpha \odot y_\beta \in \Omega$ whenever $x_\alpha, y_\beta \in \Omega$.

For any $\alpha \in (0, 1]$, let $\mathbb{F}_\alpha(X)$ denote the set of all fuzzy points in X with the value α. It is easily to check that $(\mathbb{F}_\alpha(X), \odot, 0_\alpha)$ is a BCC-algebra and that $\mathbb{F}_\alpha(X)$ can be identified with X. We call $\mathbb{F}_\alpha(X)$ a *fuzzy point* BCC-*algebra*.

Definition 3.2. Let $\alpha \in (0, 1]$. A subset Ω_α of a fuzzy point BCC-algebra $\mathbb{F}_\alpha(X)$ is called a *fuzzy point* BCC-*subalgebra* if $x_\alpha \odot y_\alpha \in \Omega_\alpha$ whenever $x_\alpha, y_\alpha \in \Omega_\alpha$.

Example 3.3. Let $X = \{0, a, b, c, d\}$ be a BCC-algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	a	0	0
d	d	c	d	c	0

We know that $(\mathbb{F}_{0.5}(X), \odot, 0_{0.5})$ is a fuzzy point BCC-algebra and $\Omega_{0.5} = \{0_{0.5}, a_{0.5}, b_{0.5}\}$ is a fuzzy point BCC-subalgebra of $\mathbb{F}_{0.5}(X)$.

Remark. Let $\alpha \in (0, 1]$. Then $\mathbb{F}_\alpha(X)$ is a quasi BCC-subalgebra of $\mathbb{F}(X)$.

Given a fuzzy set μ in X and $\alpha \in (0, 1]$, we define two sets:

$$\mathbb{F}_\alpha(\mu) := \{x_\alpha \in \mathbb{F}(X) \mid \mu(x) \geq \alpha\}$$

and

$$\mathbb{F}(\mu) := \bigcup_{\alpha \in (0,1]} \mathbb{F}_\alpha(\mu).$$

The following example shows that $\mathbb{F}_\alpha(\mu)$ may not be a fuzzy point BCC-subalgebra of $\mathbb{F}_\alpha(X)$ for some $\alpha \in (0, 1]$.

Example 3.4. Let $X = \{0, a, b, c, d\}$ be a set with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	a
c	c	b	a	0	a
d	d	d	d	d	0

Then $(X, *, 0)$ is a proper BCC-algebra (see [3]). Consider a fuzzy set μ in X defined by $\mu(0) = 0.8$, $\mu(a) = 0.2$, $\mu(b) = 0.3$ and $\mu(c) = \mu(d) = 0.5$. Then $\mathbb{F}_{0.4}(\mu) = \{0_{0.4}, c_{0.4}, d_{0.4}\}$ is not a fuzzy point BCC-subalgebra of $\mathbb{F}_{0.4}(X)$ since $c_{0.4} \odot d_{0.4} = (c * d)_{0.4} = a_{0.4} \notin \mathbb{F}_{0.4}(\mu)$.

Lemma 3.5. *If μ is a fuzzy* BCC-*subalgebra of X, then $\mathbb{F}_\alpha(\mu)$ is a fuzzy point* BCC-*subalgebra of $\mathbb{F}_\alpha(X)$ for every $\alpha \in (0, 1]$.*

Proof. For any $\alpha \in (0, 1]$, let $x_\alpha, y_\alpha \in \mathbb{F}_\alpha(\mu)$. Then $\mu(x) \geq \alpha$ and $\mu(y) \geq \alpha$. Since μ is a fuzzy BCC-subalgebra of X, it follows that

$$\mu(x * y) \geq \min\{\mu(x),\, \mu(y)\} \geq \alpha$$

so that $x_\alpha \odot y_\alpha = (x * y)_\alpha \in \mathbb{F}_\alpha(\mu)$. Hence $\mathbb{F}_\alpha(\mu)$ is a fuzzy point BCC-subalgebra of $\mathbb{F}_\alpha(X)$. □

Theorem 3.6. *Let μ be a fuzzy set in X. If $\mathbb{F}_\alpha(\mu)$ is a fuzzy point* BCC-*subalgebra of $\mathbb{F}_\alpha(X)$ for every $\alpha \in (0, 1]$, then $\mathbb{F}(\mu)$ is a quasi* BCC-*subalgebra of $\mathbb{F}(X)$.*

Proof. Let $x, y \in U(\mu; \beta)$, where $\beta \in (0, 1]$. Then $\mu(x) \geq \beta$ and $\mu(y) \geq \beta$, and so $x_\beta, y_\beta \in \mathbb{F}_\beta(\mu)$. Hence

$$(x * y)_\beta = x_\beta \odot y_\beta \in \mathbb{F}_\beta(\mu)$$

because $\mathbb{F}_\beta(\mu)$ is a fuzzy point BCC-subalgebra. It follows that $\mu(x * y) \geq \beta$ so that $x * y \in U(\mu; \beta)$. Thus $U(\mu; \beta)$ is a BCC-subalgebra of X. Now let $x_\alpha, y_\beta \in \mathbb{F}(\mu)$. Then $\mu(x) \geq \alpha \geq \min\{\alpha, \beta\}$ and $\mu(y) \geq \beta \geq \min\{\alpha, \beta\}$. Therefore $x, y \in U(\mu; \min\{\alpha, \beta\})$, and so $x * y \in U(\mu; \min\{\alpha, \beta\})$ since $U(\mu; \min\{\alpha, \beta\})$ is a BCC-subalgebra. It follows that $\mu(x * y) \geq \min\{\alpha, \beta\}$ so that

$$x_\alpha \odot y_\beta = (x * y)_{\min\{\alpha,\, \beta\}} \in \mathbb{F}(\mu).$$

Consequently, $\mathbb{F}(\mu)$ is a quasi BCC-subalgebra of $\mathbb{F}(X)$. □

Theorem 3.7. *Let μ be a fuzzy set in X such that $\mathbb{F}(\mu)$ is a quasi* BCC-*subalgebra of $\mathbb{F}(X)$. Then*

(i) *μ is a fuzzy* BCC-*subalgebra of X.*

(ii) *$0_\alpha \in \mathbb{F}(\mu)$ for all $\alpha \in \mathrm{Im}(\mu)$.*

Proof. (i) Let $x, y \in X$ be such that $\mu(x) = 0$ or $\mu(y) = 0$. Then

$$\mu(x * y) \geq 0 = \min\{\mu(x),\, \mu(y)\}.$$

Since $x_{\mu(x)}$, $y_{\mu(y)} \in \mathbb{F}(\mu)$ for all $x, y \in X$ with $\mu(x) \neq 0 \neq \mu(y)$, we have

$$(x * y)_{\min\{\mu(x),\, \mu(y)\}} = x_{\mu(x)} \odot y_{\mu(y)} \in \mathbb{F}(\mu).$$

Hence $\mu(x * y) \geq \min\{\mu(x),\, \mu(y)\}$, and so μ is a fuzzy BCC-subalgebra of X.

(ii) Let $\alpha \in \mathrm{Im}(\mu)$. Then there exists $x \in X$ such that $\mu(x) = \alpha$. Thus $x_\alpha \in \mathbb{F}(\mu)$, and so $0_\alpha = (x * x)_\alpha = x_\alpha \odot x_\alpha \in \mathbb{F}(\mu)$. This completes the proof. □

Remark. Let μ be a fuzzy set in X and $\alpha, \beta \in (0, 1]$ with $\alpha \geq \beta$. If $x_\alpha \in \mathbb{F}(\mu)$, then $x_\beta \in \mathbb{F}(\mu)$.

Definition 3.8. Let μ be a fuzzy set in X. Then $\mathbb{F}(\mu)$ is called a *quasi* BCC-*ideal* of $\mathbb{F}(X)$ if it satisfies

- $0_\alpha \in \mathbb{F}(\mu)$ for all $\alpha \in \mathrm{Im}(\mu)$,
- for all $x_\alpha, y_\beta, z_\gamma \in \mathbb{F}(X)$, $(x_\alpha \odot y_\beta) \odot z_\gamma \in \mathbb{F}(\mu)$ and $y_\beta \in \mathbb{F}(\mu)$ imply $x_{\min\{\alpha,\, \beta\}} \odot z_{\min\{\beta,\, \gamma\}} \in \mathbb{F}(\mu)$.

Example 3.9. Let X and μ be as in Example 3.4. Then the set

$$\begin{aligned} \mathbb{F}(\mu) \;:=\; & \{0_\alpha \mid \alpha \in (0, 0.8]\} \cup \{a_\beta \mid \beta \in (0, 0.2]\} \cup \{b_\gamma \mid \gamma \in (0, 0.3]\} \\ & \cup \{c_\delta \mid \delta \in (0, 0.5]\} \cup \{d_\epsilon \mid \epsilon \in (0, 0.5]\} \end{aligned}$$

is not a quasi BCC-ideal of $\mathbb{F}(X)$ because $(b_{0.5} \odot d_{0.4}) \odot d_{0.3} = 0_{0.3} \in \mathbb{F}(\mu)$ and $d_{0.4} \in \mathbb{F}(\mu)$, but

$$b_{\min\{0.5,\, 0.4\}} \odot d_{\min\{0.4,\, 0.3\}} = b_{0.4} \odot d_{0.3} = a_{0.3} \notin \mathbb{F}(\mu).$$

Example 3.10. Let $X = \{0, a, b, c, d, e\}$ be a set with the following Cayley table:

$*$	0	a	b	c	d	e
0	0	0	0	0	0	0
a	a	0	0	0	0	a
b	b	b	0	0	a	a
c	c	b	a	0	a	a
d	d	d	d	d	0	a
e	e	e	e	e	e	0

Then $(X, *, 0)$ is a proper BCC-algebra (see [6]). Let μ be a fuzzy set in X defined by $\mu(e) = 0.3$ and $\mu(x) = 0.5$ for all $x \neq e$. Then μ is a fuzzy BCC-ideal of X, and the set

$$\mathbb{F}(\mu) = \{0_\alpha, a_\beta, b_\gamma, c_\delta, d_\epsilon \mid \alpha, \beta, \delta, \gamma, \epsilon \in (0, 0.5]\} \cup \{e_\rho \mid \rho \in (0, 0.3]\}$$

is a quasi BCC-ideal of $\mathbb{F}(X)$.

Theorem 3.11. *Let μ be a fuzzy set in X. Then μ is a fuzzy* BCC*-ideal of X if and only if $\mathbb{F}(\mu)$ is a quasi* BCC*-ideal of $\mathbb{F}(X)$.*

Proof. Assume that μ is a fuzzy BCC-ideal of X. Since $\mu(0) \geq \mu(x)$ for all $x \in X$, we have $\mu(0) \geq \alpha$ for all $\alpha \in \mathrm{Im}(\mu)$. Hence $0_\alpha \in \mathbb{F}(\mu)$. Let $x_\alpha, y_\beta, z_\gamma \in \mathbb{F}(X)$ be such that $(x_\alpha \odot y_\beta) \odot z_\gamma \in \mathbb{F}(\mu)$ and $y_\beta \in \mathbb{F}(\mu)$. Then $\mu((x * y) * z) \geq \min\{\alpha, \beta, \gamma\}$ and $\mu(y) \geq \beta$. Since μ is a fuzzy BCC-ideal, it follows that

$$\begin{aligned} \mu(x * z) &\geq \min\{\mu((x * y) * z),\, \mu(y)\} \\ &\geq \min\{\min\{\alpha, \beta, \gamma\},\, \beta\} \\ &= \min\{\alpha, \beta, \gamma\} \end{aligned}$$

so that

$$\begin{aligned} x_{\min\{\alpha,\beta\}} \odot z_{\min\{\beta,\gamma\}} &= (x * z)_{\min\{\min\{\alpha,\beta\},\, \min\{\beta,\gamma\}\}} \\ &= (x * z)_{\min\{\alpha,\beta,\gamma\}} \in \mathbb{F}(\mu). \end{aligned}$$

Conversely, suppose that $\mathbb{F}(\mu)$ is a quasi BCC-ideal of $\mathbb{F}(X)$. Obviously $\mu(0) \geq \mu(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $\mu((x * y) * z) = \alpha$ and $\mu(y) = \beta$. Then $y_\beta \in \mathbb{F}(\mu)$ and

$$(x_\alpha \odot y_\beta) \odot z_\alpha = ((x * y) * z)_{\min\{\alpha,\, \beta\}} \in \mathbb{F}(\mu).$$

Since $\mathbb{F}(\mu)$ is a quasi BCC-ideal, it follows that

$$(x * z)_{\min\{\alpha,\beta\}} = x_{\min\{\alpha,\beta\}} \odot y_{\min\{\alpha,\beta\}} \in \mathbb{F}(\mu)$$

so that $\mu(x * z) \geq \min\{\alpha, \beta\} = \min\{\mu((x * y) * z),\, \mu(y)\}$. Hence μ is a fuzzy BCC-ideal of X □

Acknowledgements. The authors are highly grateful to the referee for his/her valuable comments and suggestions for improving the paper.

References

[1] W. A. Dudek, *The number of subalgebras of finite* BCC-*algebras*, Bull. Inst. Math. Academia Sinica, **20**(1992), 129–136.

[2] W. A. Dudek, *On proper BCC-algebras,* Bull. Inst. Math. Academia Sinica, **20**(1992), 137–150.

[3] W. A. Dudek, *Subalgebras in finite* BCC-*algebras*, Bull. Inst. Math. Acad. Sinica, **28**(2000), 201–206.

[4] W. A. Dudek and Y. B. Jun, *Fuzzy* BCC-*ideals in* BCC-*algebras*, Mathematica Montisnigri, **10**(1999), 21–30.

[5] W. A. Dudek, Y. B. Jun and Z. Stojaković, *On fuzzy ideals in* BCC-*algebras*, Fuzzy Sets and Systems, **123**(2001), 251–258.

[6] W. A. Dudek and X. H. Zhang, *On ideals and congruences in* BCC-*algebras*, Czech. Math. J., **48(123)**(1998), 21–29.

[7] J. Hao, *Ideal theory of* BCC-*algebras*, Scientiae Mathematicae, **1(3)**(1998), 373–381.

[8] Y. Imai and K. Iséki, *On axiom system of propositional calculi XIV*, Proc. Japan Academy, **42**(1966), 19–22.

[9] Y. Komori, *The class of* BCC-*algebras is not a variety*, Math. Japon., **29**(1984), 391–394.

[10] A. Wroński, *BCK-algebras do not form a variety*, Math. Japon., **28**(1983), 211–213.