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On Semirings which are Distributive Lattices of Rings

S. K. Maity
Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kolkata-700019, India
e-mail : sskmaity@yahoo.com

Abstract. We introduce the notions of nilpotent element, quasi-regular element in a semiring which is a distributive lattice of rings. The concept of Jacobson radical is introduced for this kind of semirings.

1. Introduction

Recall that a semiring $(S, +, \cdot)$ is a type $(2, 2)$ algebra whose semigroup reducts $(S, +)$ and $(S, \cdot)$ are connected by distributivity, that is, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in S$. We call a semiring $(S, +, \cdot)$ additive regular if for every element $a \in S$ there exists an element $x \in S$ such that $a + x + a = a$. Additive regular semirings were first studied by J. Zeleznekow [6] in 1981. We call a semiring $(S, +, \cdot)$ an additive inverse semiring if $(S, +)$ is an inverse semigroup, i.e., for each $a \in S$ there exists a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$. Additive inverse semirings were first studied by Karvellas [4] in 1974.

Throughout this paper, $E^+(S)$ will always denote the set of all additive idempotents of the semiring S. As usual, we denote the Green's relations on the semiring $(S, +, \cdot)$ by $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{J}$ and $\mathcal{H}$ and correspondingly, the $\mathcal{L}$-relation, $\mathcal{R}$-relation, $\mathcal{D}$-relation, $\mathcal{J}$-relation and $\mathcal{H}$-relation on $(S, +)$ are denoted by $\mathcal{L}^+$, $\mathcal{R}^+$, $\mathcal{D}^+$, $\mathcal{J}^+$ and $\mathcal{H}^+$, respectively. In fact, the relations $\mathcal{L}^+$, $\mathcal{R}^+$, $\mathcal{D}^+$, $\mathcal{J}^+$ and $\mathcal{H}^+$ are all congruence relations on the multiplicative reduct $(S, \cdot)$. Thus if any one of them is a congruence on $(S, +)$, then it will be a congruence on the semiring $(S, +, \cdot)$. For any $a \in S$, let J_a^+ be the $\mathcal{J}^+$-class in $(S, +)$ containing a. A subsemiring I of a semiring $(S, +, \cdot)$ is called an ideal of S if $SI,\ IS \subseteq I$. An ideal I is called a full ideal if $E^+(S) \subseteq I$. Furthermore, an ideal I of a semiring S is called a k-ideal of S if $a \in I$ and either $a + x \in I$ or $x + a \in I$ for some $x \in S$ imply $x \in I$.

We now give the following definitions in an additive inverse semiring.

Definition 1.1. Let S be an additive inverse semiring. An element $a \in S$ is called

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nilpotent if there exists a positive integer n such that $a^n = a + a'$.

Naturally, we call an additive inverse semiring S a nilpotent semiring if every element of the additive inverse semiring S is nilpotent.

We can easily verify that the direct product of a nilpotent ring and a distributive lattice is a nilpotent semiring.

Definition 1.2. An ideal I of an additive inverse semiring S is called *nil* if every element of I is nilpotent.

Definition 1.3. An ideal I of an additive inverse semiring is called *nilpotent* if there exists a positive integer n such that $I^n = E^+(S)$.

Example 1.4. We consider the semiring $S = \mathbb{Z}_4 \times D$, where D is a distributive lattice. Then $I = \{[0],\ [2]\} \times D$ is a non-trivial nilpotent ideal of S.

Remark 1.5. It is remarkable to note that similar to ring theory one can easily verify that every nilpotent ideal of an additive inverse semiring is nil. But the converse is not true in general. For this let D be a distributive lattice, R be a ring and I be an ideal of R such that I is nil but not nilpotent. Then $I \times D$ is a nil ideal of the additive inverse semiring $S = R \times D$ but $I \times D$ is not a nilpotent ideal of S.

In section 2, we study those additive inverse semirings which are distributive lattices of skew-rings.

In section 3, we define quasi-regular elements, quasi-regular ideals and regular ideals in a semiring. Finally, we show that for an additive inverse semiring S which is a distributive lattice of rings, the intersection of all regular ideals is a quasi-regular ideal.

In section 4, we define the Jacobson radical of a semiring. We introduce the concepts of semisimple semiring and Artinian semiring in this section. We show that in an Artinian semiring S, the Jacobson radical is a nilpotent ideal of S.

2. Distributive lattices of skew-rings

According to M. P. Grillet [3], a semiring $(S, +, \cdot)$ is called a skew-ring if its additive reduct $(S, +)$ is a group.

Next we give the following definition.

Definition 2.1. A congruence ρ on a semiring S is called a *distributive lattice congruence* if S/ρ is a distributive lattice. A semiring S is called a distributive lattice D of skew-rings $R_\alpha (\alpha \in D)$ if S admits a distributive lattice congruence ρ on S such that $D = S/\rho$ and each R_α is a ρ-class.

We now give the necessary and sufficient condition such that a semiring becomes a distributive lattice of skew-rings.

Theorem 2.2. *A semiring S is a distributive lattice of skew-rings if and only if S is an additive inverse semiring satisfying the following conditions*:

(i) $a + a' = a' + a$,

(ii) $a(a + a') = a + a'$,

(iii) $a(b + b') = (b + b')a$, *and*

(iv) $a + a(b + b') = a$ *for all* $a, b \in S$.

Proof. First, assume that $(S, +, \cdot)$ is an additive inverse semiring satisfying the given conditions. Then $(S, +)$ is completely regular semigroup and hence we find from Theorem II.1.4. in [5] that $(S, +)$ is semilattice $(Y, +)$ of completely simple semigroups $(S_\alpha, +)$, where $Y = S/\mathcal{J}^+$ and $\mathcal{J}^+$ is a congruence on the semiring S. We first show that each S_α is a skew-ring. Let $a \in S$. Then $a = a + a' + a = a(a + a') + a = aa' + a^2 + a$ and $a^2 = a(a + a' + a) = a + a' + a^2 = a' + a + a^2$. Thus $a^2\mathcal{J}^+a$. Let $b, c \in J_a^+$. Then $b\mathcal{J}^+a$ and $c\mathcal{J}^+a$. Hence $bc\mathcal{J}^+a^2\mathcal{J}^+a$. Thus $(S_\alpha, +, \cdot)$ is a semiring. Let $e, f \in S_\alpha$ be two additive idempotents. Then $e\mathcal{J}^+f$. Since $(S_\alpha, +)$ is a completely simple semigroup, we have that $\mathcal{J}^+ = \mathcal{D}^+$ on S_α. So $e\mathcal{D}^+f$ and hence there exists an element $x \in S_\alpha$ such that $e\mathcal{L}^+x$ and $x\mathcal{R}^+f$. Since $(S, +)$ is an inverse semigroup, we have $e = x + x' = f$ and hence $(S_\alpha, +, \cdot)$ is a skew-ring.

Now $ab = a(b+b'+b) = a(b+b')+ab = (b+b')a+ab = b'a+ba+ab$. Similarly, $ba = a'b+ab+ba$. Thus, $ab\mathcal{J}^+ba$. Again $a = a+a(b+b') = (a+a')+a+ab+(ab')$ and $a + ab = (a + a') + a + (ab)$. Hence $a + ab\mathcal{J}^+a$. Thus, $\mathcal{J}^+$ is a distributive lattice congruence on S. Consequently, S is a distributive lattice of skew-rings.

Conversely, if a semiring S is a distributive lattice of skew-rings then one can easily show that it is an additive inverse semiring such that the given conditions (i)-(iv) are satisfied. Thus the proof is completed. □

Corollary 2.3. *Let S be an additive commutative semiring. S is a distributive lattice of rings if and only if it is an additive inverse semiring satisfying the following conditions*:

(i) $a(a + a') = a + a'$,

(ii) $a(b + b') = (b + b')a$, *and*

(iii) $a + a(b + b') = a$ *for all* $a, b \in S$.

Example 2.4. Let $S = \{0, a, b\}$ be a semiring with the following Cayley tables:

$+$	0	a	b
0	0	a	b
a	a	0	b
b	b	b	b

$\cdot$	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

Then we can easily see that the additive reduct $(S, +)$ is a commutative inverse semigroup. Also we can show that for all $a, b \in S$, we have

(i) $a(a+a') = a+a'$,

(ii) $a(b+b') = (b+b')a$, and

(iii) $a + a(b+b') = a$.

Hence S is a distributive lattice of rings.

3. Quasi-regular ideals

Throughout the rest of the paper, S is going to denote an additive commutative semiring which is a distributive lattice of rings.

Definition 3.1. Let S be a semiring and $a \in S$. If there exists an element $b \in S$ such that $a = b' + ab$, a is said to be *right quasi-regular* and that b is a right quasi-inverse of a. Left quasi-inverse elements are defined similarly. The element a is said to be *quasi-regular* if it is both left quasi-regular and right quasi-regular.

Definition 3.2. A right ideal (left ideal or ideal resp.) in S is said to be *right quasi-regular* (*left quasi-regular* or *quasi-regular resp.*) if each of its elements is right quasi-regular (left quasi-regular or quasi-regular resp.)

We now proceed to establish several lemmas which will be useful in the sequel.

We now observe that if an element of S has a right quasi-inverse and left quasi-inverse then these inverses are equal.

Lemma 3.3. *If an element a of S has a right quasi-inverse b and also a left quasi-inverse c, then $b = c$.*

Proof. Since b is right quasi inverse of a, we have $a = b' + ab$. Similarly, $a = c' + ca$. Hence $b' = a + a'b$ and $c' = a + c'a$.

Now

$$\begin{aligned} b' &= a + a'b \\ &= a + ab' \\ &= a + (c' + ca)b' \\ &= a + c'b' + cab' \\ &= a + c'(a + ab') + cab' \\ &= a + c'a + c'ab' + cab' \\ &= a + c'a + cab + cab' \\ &= c' + cab + cab' \\ &= c'. \end{aligned}$$

Hence $b = c$. □

Lemma 3.4. *Every nilpotent element of a semiring S is quasi-regular.*

Proof. Let a be a nilpotent element in S. Then there exists a positive integer n such that $a^n = a + a'$. Let $b = a' + (a^2)' + \cdots + (a^{n-1})'$.

Now

$$\begin{aligned} b' + ab &= a + a^2 + \cdots + a^{n-1} + aa' + a(a^2)' + \cdots + a(a^{n-1})' \\ &= a + a^2 + \cdots + a^{n-1} + aa' + a^2a' + \cdots + a^{n-2}a' + (a^n)' \\ &= a + a(a + a') + a^2(a + a') + \cdots + a^{n-2}(a + a') + (a + a')' \\ &= a. \end{aligned}$$

Similarly, we have that $b' + ba = a$. Hence a is quasi-regular. □

Lemma 3.5. *An element a of the semiring S is right quasi-regular if and only if the right ideal $A = \{x + (ax)' : x \in S\}$ coincides with S.*

Proof. First, let $A = S$. Then there exists a y such that $a = y+(ay)' = (y')'+(ay') = z' + az$ (where $y' = z$) and hence a is right quasi-regular.

Conversely, let a be a right quasi-regular element in a semiring S. Then there exists a $y \in S$ such that $a = y' + ay$. Then for any $x \in S$ we have $ax = y'x + ayx = (y'x) + (ay'x)'$ and hence $ax \in A$. But $x + (ax)' \in A$. Clearly A is a right ideal of S. Thus $x + (ax)' + ax \in A$, i.e., $x \in A$. Hence $S \subseteq A$ and therefore $A = S$. □

Lemma 3.6. *If $a, b \in S$ and ab is right quasi-regular, then ba is also right quasi-regular.*

Proof. Since ab is right quasi-regular, there exists an element $c \in S$ such that $ab = c' + abc$. Let $d = b'a + bca$.

Now

$$\begin{aligned} d' + bad &= ba + bc'a + ba(b'a + bca) \\ &= ba + bab'a + bc'a + babca \\ &= ba + bab'a + b(c' + abc)a \\ &= ba + bab'a + baba \\ &= ba \end{aligned}$$

Hence ba is right quasi-regular. □

Lemma 3.7. *Let A be a right ideal of S such that $a' \in A$ for every $a \in A$. If A is right quasi-regular, then it is quasi-regular.*

Proof. Let $a \in A$. Then there exists an element $b \in A$ such that $a = b' + ab, i.e., b' = a + ab'$. Thus $b' \in A$ and hence $b \in A$. Now from $b' = a + ab'$ we have $b = a' + ab$. So b has a left quasi-inverse a. Applying the same argument to b, we deduce that b has a right quasi-inverse c. Hence by Lemma 3.3, we have $a = c$.

Now, $b = c' + bc = a' + ba$ gives $a' = b + b'a$, i.e., $a = b' + ba$. Hence a has b as left quasi-inverse. Consequently, a is quasi-regular. □

Definition 3.8. A right ideal I of a semiring S is said to be *regular* if there exists an element $e \in S$ such that $r + (er)' \in I$ for all $r \in S$. Similarly, a left ideal I of semiring S is said to be regular if there exists an element $e \in S$ such that $r + (re)' \in I$ for all $r \in S$.

Lemma 3.9. *If I is a regular right ideal of a semiring S, then I is contained in a* *maximal right ideal which is regular.*

Proof. Since I is a regular right ideal, there exists an element $e \in S$ such that $r + (er)' \in I$ for all $r \in S$. Thus any right ideal J containing I is also regular (with the same element e). If $e \in J$, then $er \in J$ and $r + (er)' \in J$ for all $r \in S$ gives $r = r + (er)' + er \in J$, whence $J = S$.

Let $\mathcal{C} = \{J : J$ is a regular right ideal of S such that $I \subseteq J$ and $e \notin J\}$. Then $\mathcal{C}$ is a partially ordered set under set inclusion relation. Applying Zorn's lemma, we have a maximal element K in $\mathcal{C}$. Then K is a maximal right ideal of S containing I which is regular. □

Example 3.10. We consider the ring of integers $\mathbb{Z}$ and let D be a distributive lattice. Then $3\mathbb{Z} \times D$ is a regular maximal ideal of $\mathbb{Z} \times D$. But $3\mathbb{Z} \times D$ is not a quasi-regular ideal. □

Theorem 3.11. *Let S be a semiring and K be the intersection of all regular maximal right ideals of S. Then K is a quasi-regular right ideal of S.*

Proof. Clearly, K is a right ideal of S. Given an $a \in K$, let $T = \{r + (ar)' : r \in S\}$. We must show $T = S$.

Clearly, T is a regular right ideal of S. If $T \neq S$, then by Lemma 3.9, T is contained in a regular maximal right ideal I_0. Since $a \in T \subseteq I_0$, $ar \in I_0$ for all $r \in S$. Again from $r + (ar)' \in T \subseteq I_0$, we must have $r = r + (ar)' + ar \in I_0$ for all $r \in S$. Consequently, $S = I_0$, which contradicts the maximality of I_0. Therefore $T = S$ and a is right quasi-regular by Lemma 3.5. Hence, K is a quasi-regular right ideal of S. □

4. The Jacobson radical

Definition 4.1. The Jacobson radical $J(S)$ of a semiring S is defined as follows:

$$J(S) = \{a \in S : aS \text{ is right quasi-regular}\}.$$

Furthermore, if I is an ideal of a semiring S, then the Jacobson radical of I is defined by

$$J(I) = \{a \in I : aI \text{ is right quasi-regular}\}.$$

It is important to keep in mind that, in view of Lemma 3.7, this is equivalent to require that aS be quasi-regular in the definition of $J(S)$.

Theorem 4.2. *$J(S)$ is a quasi-regular ideal in S which contains every quasi-regular right ideal and every quasi-regular left ideal in S.*

Proof. First we show that $J(S)$ is an ideal of S. Let $a \in J(S)$ such that aS is right quasi-regular. Then for each $x \in S$, we have $(ax)S \subseteq aS$, so $(ax)S$ is right quasi-regular and $ax \in J(S)$. Now for each $x, y \in S$, it follows from what we have just proved that ayx is right quasi-regular and Lemma 3.6 shows that xay is right quasi-regular. That is, $(xa)S$ is right quasi-regular and $xa \in J(S)$. Let $a, b \in J(S)$.

We show that $(a+b) \in J(S)$. If $x \in S$ then ax is right quasi-regular. Suppose that ax has r_1 as right quasi-inverse so that $ax = r_1' + axr_1$. Since $b \in J(S)$, $b(x+x'r_1)$ is right quasi-regular and hence there exists an element $r_2 \in S$ such that $b(x+x'r_1) = r_2' + b(x+x'r_1)r_2$. Let $z = r_1 + r_2 + r_1'r_2$. Then

$$\begin{aligned}
z' + (ax+bx)z &= (r_1 + r_2 + r_1'r_2)' + (ax+bx)(r_1 + r_2 + r_1'r_2) \\
&= r_1' + r_2' + r_1'r_2' + axr_1 + axr_2 + axr_1'r_2 + bxr_1 + bxr_2 + bxr_1'r_2 \\
&= ax + b(x + x'r_1) + (r_1' + axr_1)r_2' + axr_2 + bxr_1 \\
&= ax + bx + bx'r_1 + axr_2' + axr_2 + bxr_1 \\
&= ax + bx \\
&= (a+b)x.
\end{aligned}$$

This proves that $(a+b)S$ is right quasi-regular and hence $a+b \in J(S)$. Therefore $J(S)$ is an ideal of S.

We now show that $J(S)$ is right quasi-regular. Let $a \in J(S)$. Then aS is right quasi-regular. So in particular a^2 is right quasi-regular. Hence there exists an element $c \in S$ such that $a^2 = c' + a^2c$. Let $d = a' + c + ac$. Then

$$\begin{aligned}
d' + ad &= a + c' + a'c + a(a' + c + ac) \\
&= a + c' + a'c + aa' + ac + a^2c \\
&= a + c' + aa' + a^2c \\
&= a + aa' + a^2 \\
&= a + a + a' \\
&= a.
\end{aligned}$$

Hence a is right quasi-regular and thus $J(S)$ is right quasi-regular ideal of S. Again for all $a \in J(S)$, we have $a' \in J(S)$. Hence by Lemma 3.7, we have $J(S)$ is quasi-regular.

If A is a quasi-regular right ideal of S and $a \in A$, then $aS \subseteq A$ and by definition of $J(S)$ we have $a \in J(S)$. Hence $A \subseteq J(S)$.

If A is a quasi-regular left ideal of S and $a \in A$, then $Sa \subseteq A$ and Sa is right quasi-regular. By Lemma 3.6, we have aS is right quasi-regular and hence $a \in J(S)$. Thus $A \subseteq J(S)$. This completes the proof of the theorem. □

The following result is an immediate consequence of Lemma 3.4.

Corollary 4.3. *$J(S)$ contains every nil right (left) ideal in S.*

We now prove an interesting theorem.

Theorem 4.4. *$J(S)$ is a k-ideal of S.*

Proof. We have already proved that $J(S)$ is an ideal of S. To complete the proof, let $a+b, b \in J(S)$. Hence $a+b, b' \in J(S)$ and $a+b+b' \in J(S)$. So $(a+b+b')S$ is right quasi-regular. Let $x \in S$. Then $(a+b+b')xax$ is right quasi-regular, i.e., $(ax)^2 + (bx + bx')ax$ is right quasi-regular. Let $c = ax$ and $d = bx$. Then $c^2 + (d + d')c$ is right quasi-regular. Hence there exists an element $z \in S$ such that $c^2 + (d + d')c = z' + (c^2 + dc + d'c)z$. Let $y = c' + z + cz$.

Then

$$y' + cy = c + z' + c'z + cc' + cz + c^2 z$$

$$\begin{aligned} &= c + cc' + z' + c^2 z \\ &= c + cc' + z' + \big(c^2 + (d + d')c^2\big)z \\ &= c + cc' + z' + \big(c^2 + (d + d')c\big)z \\ &= c + cc' + c^2 + (d + d')c \\ &= c. \end{aligned}$$

Hence $c = ax$ is right quasi-regular for all $x \in S$. Thus aS is right quasi-regular and hence $a \in J(S)$. Consequently, $J(S)$ is a k-ideal of S. □

It is worth to note that since every additive idempotent is quasi-regular we have $E^+(S) \subseteq J(S)$ and hence $S/J(S)$ is a ring.

Definition 4.5. A semiring S is called *semisimple semiring* if $J(S) = E^+(S)$.

Theorem 4.6. *If $J(S)$ is the Jacobson radical of the semiring S, then $J\big(S/J(S)\big) = \{0\}$.*

Proof. Let $a + J(S) \in J\big(S/J(S)\big)$. Then for any $x \in S$ we have $(a + J(S))(x + J(S))$ is right quasi-regular i.e., $ax + J(S)$ is right quasi-regular. Hence there exists an element $s + J(S)$ such that $(ax + J(S)) + (s + J(S)) = (ax + J(S))(s + J(S))$, i.e., $ax + s + a'xs \in J(S)$, i.e., $(ax + s + a'xs)ax$ is right quasi-regular. Then by Lemma 3.6, $ax(ax + s + a'xs)$ is right quasi-regular, i.e., $(ax)^2 + axs + axa'xs$ is right quasi-regular. Then there exists an element $z \in S$ such that $(ax)^2 + axs + axa'xs = z' + (ax)^2 z + axsz + axa'xsz$. Let $y = z + a'x + axz + axs + a'xsz$.

Then

$$\begin{aligned} y' + (ax)y &= z' + ax + a'xz + axs' \\ &\quad + axsz + axz + axa'x + axaxz + axaxs + axa'xsz \\ &= z' + ax(z + z') + ax + axs' \\ &\quad + axa'x + axsz + axaxz + axaxs + axa'xsz \\ &= z' + ax + axs' + axa'x + (ax)^2 z + axsz + axaxs + axa'xsz \\ &= ax + axs' + axa'x + axaxs + (ax)^2 + axs + axa'xs \\ &= ax. \end{aligned}$$

Hence ax is right quasi-regular for all $x \in S$. Hence aS is right quasi-regular. Consequently, $a \in J(S)$ and $J\big(S/J(S)\big) = \{0\}$. □

Corollary 4.7. *If $a \in S$ such that $SaS \subseteq J(S)$, then $a \in J(S)$.*

Proof. Let $x \in S$. Then $(ax)^2 \subseteq (aS)^2 \subseteq J(S)$. Hence $(ax)^2$ is right quasi-regular and thus ax is right quasi-regular. Consequently, $a \in J(S)$. □

Theorem 4.8. *If I is an ideal in the semiring S such that $a' \in I$ for all $a \in I$,* *then* $J(I) = I \cap J(S)$.

Proof. Suppose, first, $a \in I \cap J(S)$. Since $a \in J(S)$, ax is right quasi-regular for each $x \in S$ and there exists $y \in S$ such that $ax = y' + axy, i.e., y' = ax + axy'$ and hence $y = ax' + axy \in I$. In particular aI is right quasi-regular in the semiring I and therefore $a \in J(I)$. This shows that $I \cap J(S) \subseteq J(I)$.

Conversely, suppose that $a \in J(I)$. Since $(aS)^2 \subseteq aI$, $(aS)^2$ is a quasi-regular right ideal in S and hence $(aS)^2 \subseteq J(S)$. Let $x \in S$. Then $(ax)^2 \in (aS)^2 \subseteq J(S)$ shows that $(ax)^2$ is quasi-regular and hence ax is quasi-regular for every $x \in S$. Thus, $a \in J(S)$ and $J(I) \subseteq I \cap J(S)$ and the proof is completed. □

Definition 4.9. An additive inverse semiring S is called an *Artinian semiring* if any descending chain of full ideals of S terminates i.e., for any descending chain of full ideals $I_1 \supseteq I_2 \supseteq \cdots$. there exists a positive integer n such that $I_n = I_{n+1} = I_{n+2} = \cdots$.

We can easily prove that a semiring S is Artinian if and only if any non empty collection of full ideals contains a minimal element.

Theorem 4.10. *If S is an Artinian semiring then the Jacobson radical $J(S)$ is a nilpotent ideal of S. Consequently, every nil left (right) ideal of S is nilpotent and $J(S)$ is the unique maximal nilpotent ideal of S.*

Proof. We consider the descending chain of full ideals

$$J(S) \supseteq (J(S))^2 \supseteq (J(S))^3 \supseteq \ldots \ .$$

Since S is Artinian, there exists a positive integer n such that $(J(S))^n = (J(S))^{n+1} = \ldots$. Let $I = (J(S))^n$. Then $I \subseteq J(S)$ and $I^2 = I$. Clearly, $E^+(S) \subseteq I$. To complete the proof it suffices to show that $I \subseteq E^+(S)$. Assume that $I \nsubseteq E^+(S)$ and consider the collection

$$\mathcal{C} = \{J : J \text{ is a full ideal of } S \text{ such that } J \subseteq I \text{ and } JI \nsubseteq E^+(S)\}$$

Then $\mathcal{C} \neq \emptyset$, since $I \in \mathcal{C}$. Hence $\mathcal{C}$ has a minimal element K. So $KI \nsubseteq E^+(S)$. Hence $aI \nsubseteq E^+(S)$ for some non additive idempotent $a \in K$. Thus $(aI + E^+(S))I \subseteq aI + E^+(S)I \subseteq aI + E^+(S) \nsubseteq E^+(S)$ with $aI + E^+(S) \subseteq K \subseteq I$. Hence $aI + E^+(S) = K$ by the minimality of K. So there exist elements $b \in I$ and $e \in E^+(S)$ such that $a = ab + e$. But $b \in I \subseteq J(S)$. So there exists $c \in S$ such that $b = c' + bc$.

Now $a = ab + e = ac' + abc + e = (ab + e)c' + abc + e = abc' + ec' + abc + e = ab(c + c') + e \in E^+(S)$. Therefore $aI \subseteq E^+(S)$, contradicting the fact that $aI \nsubseteq E^+(S)$. Hence $I = (J(S))^n \subseteq E^+(S)$, as required. □

Theorem 4.11. *Every regular maximal right ideal of S is a full right k-ideal of S.*

Proof. Let A be a regular maximal right ideal of S. Since A is regular there exists an element $e \in S$ such that $r + (er)' \in A$ for all $r \in S$. Clearly, A is a full ideal and $e \notin A$. Let $a, a + b \in A$. We show that $b \in A$. Assume that

$b \notin A$ and consider the right ideal $A_1 =< A, b >_r$, the smallest ideal generated by A and b. Then $A \subsetneqq A_1$. Since A is maximal we have $A_1 = S$. Then $e = c + bt$ for some $c \in A$ and $t \in S$. Now, from $a, a + b \in A$ we have $btb \in A$. Now $b = b + (eb)' + eb = b + (eb)' + (c + bt)b = b + (eb)' + cb + btb \in A$, a contradiction to our assumption that $b \notin A$. Hence $b \in A$ and hence A is a full k-ideal of S. □

Definition 4.12. Let S be a semiring and A, a non-empty subset of S. We define $(A : S)$ as $(A : S) = \{r \in S : Sr \subseteq A\}$.

It should be noted that if A is a right k-ideal of S then $(A : S)$ is a k-ideal of S. Thus if A is a maximal regular right ideal of S then $(A : S)$ is a full k-ideal of S and hence $S/(A : S)$ is a ring.

Lemma 4.13. *If A is a regular maximal right ideal of S, then $J(S) \subseteq (A : S)$.*

Proof. Let $a \in J(S)$. We show that $a \in (A : S)$. If possible let $a \notin (A : S)$. Then $xa \notin A$ for some $x \in S$. We consider $A_1 = A + xJ(S)$. Then A_1 is a right ideal of S such that $A \subseteq A_1$ and $xa \in A_1$. Hence $A \subsetneqq A_1$. Therefore, $A_1 = S$. This implies $x = a_1 + xb$ for some $a_1 \in A$ and $b \in J(S)$. This leads to $a_1 + xb + x' = x + x' \in A$. Also, $a_1 \in A$. Since A is a right k-ideal so $xb + x' \in A$. Let $xb + x' = a_2$. Now $b \in J(S)$ implies there exists an element $c \in S$ such that $b = c' + bc$. Hence $xb = x(c' + bc) = xc' + xbc = (x' + xb)c = a_2c$. Then $xa = (a_1 + xb)a = a_1a + xba = a_1a + a_2ca \in A$, a contradiction. This contradiction leads to $a \in (A : S)$. Consequently, $J(S) \subseteq (A : S)$. □

Lemma 4.14. *If $a \in S$ and a is not right quasi-regular, there exists a regular maximal right ideal A of S such that $a \notin A$.*

Proof. Since a is not right quasi-regular, the right ideal $B = \{r + (ar)' : r \in S\}$ does not contain a. Clearly, B is a regular right ideal.

Let $\mathcal{C} = \{I : I$ is a regular right ideal of S such that $a \notin I\}$. Clearly, $\mathcal{C} \neq \emptyset$, since $B \in \mathcal{C}$. By Zorn's Lemma, $\mathcal{C}$ has a maximal element A and hence A is a regular maximal right ideal of S such that $a \notin A$. □

Lemma 4.15. *If $b \in S$ such that $b \notin J(S)$, then there exists a regular maximal right ideal of S which does not contain b.*

Proof. Since $b \notin J(S)$, there exists $t \in S$ such that bt is not right quasi-regular. Hence by Lemma 4.14, there is a regular maximal right ideal A such that $bt \notin A$. Hence $b \notin A$ as required. □

Theorem 4.16. *Let S be a semiring such that $S \neq J(S)$ and let $\{A_i\}_{i\in\Lambda}$ be the family of all regular maximal right ideal of S. Then*

(a) $J(S) = \bigcap_{i\in\Lambda} A_i$,

(b) $J(S) = \bigcap_{i\in\Lambda} (A_i : S)$.

Proof. (a) By Theorem 3.11 and Theorem 4.2, it follows that $\bigcap_{i\in\Lambda} A_i \subseteq J(S)$. Now by Lemma 4.13, $J(S) \subseteq (A_i : S)$ for each $i \in \Lambda$. Thus, $J(S) \subseteq \bigcap_{i\in\Lambda}(A_i : S)$ and hence $SJ(S) \subseteq A_i$ for each $i \in \Lambda$. Since each A_i is regular, it follows that $J(S) \subseteq A_i$ for each $i \in \Lambda$ and therefore that $J(S) \subseteq \bigcap_{i\in\Lambda} A_i$.

(b) Since A_i is regular, $\bigcap_{i\in\Lambda}(A_i : S) \subseteq \bigcap_{i\in\Lambda} A_i = J(S)$. Also, $J(S) \subseteq (A_i : S)$ for every $i \in \Lambda$. Hence $J(S) \subseteq \bigcap_{i\in\Lambda}(A_i : S)$. Thus $J(S) = \bigcap_{i\in\Lambda}(A_i : S)$. □

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