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On Lifting Modules and Weak Lifting Modules

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Abstract. We say that a module M is weak lifting if M is supplemented and every supplement submodule of M is a direct summand. The module M is called lifting, if it is weak lifting and amply supplemented. This paper investigates the structure of weak lifting modules and lifting modules having small radical over commutative noetherian rings.

1. Introduction

In this note all rings are associative with identity elements and all modules are unital left modules. A submodule L of a module M is said *small* in M, written $L \ll M$, provided $M \neq L + X$ for any proper submodule X of M. If every proper submodule of M is small in M, we call M a *hollow* module. The module M will be called a *local* module if $Rad(M)$ is a small maximal submodule of M. Let N be a submodule of a module M. A submodule K of M is called a *supplement* of N in M provided $M = N + K$ and $M \neq N + L$ for any proper submodule L of K. It is easy to check that K is a supplement of N in M if and only if $M = N + K$ and $N \cap K$ is small in K. M is called *supplemented* if every submodule of M has a supplement. On the other hand, the module M is *amply supplemented* if, for any submodules A, B of M with $M = A + B$ there exists a supplement K of A such that $K \leq B$. We say that a module M is $\oplus$*-supplemented* if every submodule has a supplement that is a direct summand of M. The module M is called *completely* $\oplus$*-supplemented* if every direct summand of M is $\oplus$-supplemented. It was shown in [3, Proposition 6] that a direct sum of two hollow modules is always completely $\oplus$-supplemented. We call the module M *lifting*, if M is amply supplemented and every supplement submodule of M is a direct summand.

In this paper we introduce the notion of *weak lifting* modules. An R-module

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M will be called *weak lifting* provided, M is supplemented and every supplement submodule of M is a direct summand. Note that we have the following hierarchy:

Lifting $\Rightarrow$ weak lifting $\Rightarrow$ completely $\oplus$-supplemented (see Corollary 2.4). In Section 2, some relevant counterexamples are indicated to separate these properties. In the third section we will be concerned with the structure of weak lifting and lifting modules. It is shown that we can reduce our investigations about weak lifting and lifting modules over commutative rings to the case of local rings. Then we show that in the class of finitely generated modules over commutative rings, weak lifting and lifting modules are the same. The structure of such modules is given in [16, Folgerung 3.3].

Our main result (Proposition 3.7) describes the structure of lifting and weak lifting modules with small radical over commutative local noetherian rings:

Let R be a commutative noetherian local ring with maximal ideal m. If M is an R-module with $Rad(M) \ll M$, then the following conditions are equivalent:

(i) *M is weak lifting;*

(ii) *M is lifting;*

(iii) *$M \cong \oplus_{k\in K} \frac{R}{I_k}$ where I_k are ideals of R such that*:

 (a) *there exists $e \geq 1$ such that the set $\{k \in K \mid m^e \not\subseteq I_k\}$ is finite,*

 (b) *the ideals $\{I_k \mid k \in K\}$ are linearly ordered by inclusion, and*

 (c) *if $I_i \subseteq I_j$ then $mI_j \subseteq I_i$.*

We conclude this paper by describing the structure of lifting modules over principal ideal rings.

2. Examples

A module M is called *lifting* (or *satisfies* (D_1)) if for every submodule N of M there are submodules K_1 and K_2 of M such that $M = K_1 \oplus K_2$, $K_2 \leq N$ and $N \cap K_1 \ll K_1$. By [6, Proposition 4.8], a module M is lifting if and only if M is amply supplemented and every supplement submodule of M is a direct summand. As a generalization of lifting modules, an R-module M will be called *weak lifting* provided, M is supplemented and every supplement submodule of M is a direct summand. It is clear that hollow modules and semisimple modules are weak lifting.

Remark 2.1. It is easily seen that an R-module M is a weak lifting module if and only if M is $\oplus$-supplemented and every supplement submodule of M is a direct summand. In particular, every weak lifting module is $\oplus$-supplemented.

Example 2.2. Let R be an incomplete rank one discrete valuation ring, with quotient field K. By [6, Lemma A.5], the module $M = K^2$ is supplemented but not amply supplemented. Moreover, from the proof of [6, Lemma A.5] it follows that every supplement submodule of M is a direct summand. Hence the module M is weak lifting but not lifting.

Proposition 2.3. *Any direct summand of a weak lifting module M is also a weak lifting module.*

Proof. It is well known that any direct summand of a supplemented module is also supplemented (see [12, p. 45 Folgerung]). Let N be a direct summand of M with $M = N \oplus K$ for some submodule K of M. The proof is completed by showing that every supplement submodule of N is a direct summand of N. Let L_1 be a submodule of N and N_1 be a supplement of L_1 in N. We thus get $N = L_1 + N_1$ and $L_1 \cap N_1 \ll N_1$. But it is easy to check that $(L_1 + K) \cap N_1 \leq L_1 \cap N_1$. Then $(L_1+K)\cap N_1 \ll N_1$. Since $M = (L_1+K)+N_1$, it follows that N_1 is a supplement of $L_1 + K$ in M. As M is a weak lifting module, N_1 is a direct summand of M. Therefore N_1 is a direct summand of N. □

Corollary 2.4. *Any weak lifting module is completely $\oplus$-supplemented.*

Proof. By Proposition 2.3. □

The following examples show that, in general, a direct sum of two weak lifting modules is not weak lifting. On the other hand, they show also that the converse of Corollary 2.4 is false.

Example 2.5.

(1) Let R be a commutative noetherian local ring with maximal ideal m and let I be an ideal of R such that $I \subset m$ and the ring $\frac{R}{I}$ is a local ring with exactly one additional prime ideal, $\frac{p}{I}$, and such that the integral closure of $\frac{R}{p}$ is also local. Consider the R-module $M = \frac{R}{m} \oplus (\frac{R}{I})_p$ with $(\frac{R}{I})_p$ is the total quotient ring of $\frac{R}{I}$ (e.g. if R is a discrete valuation ring and p=I=0, then $(\frac{R}{I})_p = Q(R)$ the quotient field of R). It is clear by [6, Proposition 5.10] that $\frac{R}{m}$ and $(\frac{R}{I})_p$ are hollow modules. So they are weak lifting modules. By [6, Proposition 5.10 and Lemma 5.11], $\frac{R}{m}$ and $(\frac{R}{I})_p$ are not relatively projective. Since $(\frac{R}{I})_p$ is amply supplemented, it follows that M is amply supplemented by [7, Proposition 4.6(b)]. Suppose that the module M is weak lifting. Then M is lifting. So $\frac{R}{m}$ and $(\frac{R}{I})_p$ are relatively projective by [5, Corollary 7], a contradiction. Consequently, M is not weak lifting.

(2) Let p be any prime integer. Consider the $\mathbb{Z}$-module, $M = \frac{\mathbb{Z}}{p\mathbb{Z}} \oplus \frac{\mathbb{Z}}{p^3\mathbb{Z}}$. It is well known that $\frac{\mathbb{Z}}{p\mathbb{Z}}$ and $\frac{\mathbb{Z}}{p^3\mathbb{Z}}$ are hollow local modules. Then they are weak lifting. On the other hand, let $L = 0 \oplus \frac{\mathbb{Z}}{p^3\mathbb{Z}}$ and $N = \mathbb{Z}(1+p\mathbb{Z}, p+p^3\mathbb{Z})$. Then $M = L+N$, $N \cap L = 0 \oplus \frac{p^2\mathbb{Z}}{p^3\mathbb{Z}}$ and $N \cong \frac{\mathbb{Z}}{p^2\mathbb{Z}}$. Hence N is hollow and $N \cap L$ is small in N. Therefore N is a supplement submodule of M. But it is easy to see that N is not a direct summand of M. Consequently, M is not weak lifting.

(3) Let R be a local commutative ring which is not a valuation ring and let m

be the maximal ideal of R. Let a and b be ideals of R, neither of them contains the other. We consider the R-module $M = \frac{R}{a} \times \frac{R}{b} = Rx_1 \oplus Rx_2$ with $Ann_R(x_1) = a$ and $Ann_R(x_2) = b$. It is clear that $\frac{R}{a}$ and $\frac{R}{b}$ are local modules. Hence they are weak lifting. Now let $L = R(x_1 - x_2)$. Since $L + Rx_2 = M$, it follows that L is not small in M. As R is local, L is a local module. Therefore L is a supplement submodule of M. Suppose that M is a weak lifting module. Then L will be a direct summand of M. By the Krull Schmidt Azumaya theorem, $M = L \oplus Rx_1$ or $M = L \oplus Rx_2$. Therefore $L \cong Rx_1$ or $L \cong Rx_2$. Since $L = R(x_1 - x_2)$, we have $ax_2 = 0$ or $bx_1 = 0$. Hence $a \subseteq b$ or $b \subseteq a$. This contradicts our assumption. Consequently, M is not a weak lifting module.

Note that in each of these examples, M is a direct sum of two hollow modules. Thus M is completely $\oplus$-supplemented by [3, Proposition 6].

3. Lifting modules with small radical

Throughout this section R will denote a commutative ring. Let Ω be the set of all maximal ideal of R. If $m \in \Omega$, M an R-module, we denote as in [15, p. 53] by $K_m(M) = \{x \in M \mid x = 0 \text{ or the only maximal ideal over } Ann_R(x) \text{ is } m\}$ as the *m-local component* of M. We call M *m-local* if $K_m(M) = M$. In this case M is an R_m-module by the following operation: $(\frac{r}{s})x = rx'$ with $x = sx'$ $(r \in R, s \in R - m)$. The submodules of M over R and over R_m are identical.

For $K(M) = \{x \in M \mid Rx \text{ is supplemented}\}$ it is easily seen that $K(M) = \{x \in M \mid \frac{R}{Ann_R(x)} \text{ is semiperfect}\}$, and we always have the decomposition $K(M) = \oplus_{m\in\Omega} K_m(M)$ (see [15, Satz 2.3]).

The next result shows that in studying of weak lifting or lifting modules with $M = K(M)$, one may restrict to the case of modules over local rings.

Proposition 3.1. *Let M be an R-module. Then:*

(i) *$K(M)$ is weak lifting if and only if $K_m(M)$ is weak lifting for all $m \in \Omega$.*

(ii) *$K(M)$ is lifting if and only if $K_m(M)$ is lifting for all $m \in \Omega$.*

Proof. It is an immediate consequence of the fact that for every submodule N of $K(M)$ we have $N = \oplus_{m\in\Omega} N \cap K_m(M)$. □

Lemma 3.2. *Let M be a finitely generated R-module. The following statements are equivalent*:

(i) *$K(M) = M$;*

(ii) *M is supplemented;*

(iii) *M is amply supplemented;*

(iv) *the ring $\frac{R}{Ann_R(M)}$ is semiperfect.*

Proof. (i) ⇔ (ii) ⇔ (iv). See [15, Satz 1.6].
(ii) ⇔ (iii). By [15, p. 52 Folgerung]. □

The last result shows that in the class of finitely generated modules over commutative rings, weak lifting and lifting modules are the same. The structure of such modules is given in [16, Folgerung 3.3] (or see [6, Lemma A.4]). We thus get the following proposition.

Proposition 3.3. *Let R be a commutative local ring with maximal ideal m. The following are equivalent for a finitely generated R-module M :*

(i) *M is a weak lifting module;*

(ii) *M is lifting.*

(iii) $M \cong \frac{R}{a_1} \times \cdots \times \frac{R}{a_n}$ *with* $a_1 \subseteq a_2 \subseteq \cdots \subseteq a_n \subset R$ *and* $ma_n \subseteq a_1$.

Our next goal is to describe lifting and weak lifting modules with small radical over commutative noetherian rings.

Lemma 3.4. *Let R be a commutative noetherian ring and M an R-module with $Rad(M) \ll M$. The following are equivalent*:

(i) *M is supplemented;*

(ii) *M is amply supplemented;*

(iii) *$M = K(M)$ and M is coatomic.*

Proof. (i) ⇒ (iii). See [15, Satz 2.5] and [12, Lemma 1.5(c)].
(iii) ⇒ (ii). By [7, Proposition 2.2(c) and (d)], every submodule of M is supplemented. Therefore M is amply supplemented [9, Lemma 2.19].
(ii) ⇒ (i). Clear. □

We conclude from the last Lemma that in the class of modules with small radical over commutative noetherian rings, there is no distinction between lifting and weak lifting modules.

Let A be a submodule of a module M. Then A is called a *coclosed* submodule of M if $\frac{A}{B}$ is not small in $\frac{M}{B}$ for any proper submodule B of A.

Proposition 3.5. *Let R be a commutative noetherian local ring with maximal ideal m. If M is a coatomic R-module, then every local summand of M is a supplement submodule of M.*

Proof. Since M is coatomic, M is supplemented by [7, Proposition 2.2(c)]. Hence for every submodule N of M, N is a supplement of some submodule in M if and only if N is coclosed (see [4, Proposition 3]). Taking into account [15, Lemma 3.1(a)] and [14, Lemma 1.1], this is equivalent to $mN = N \cap mM$. Let $X = \Sigma_{\lambda\in\Lambda} X_\lambda$ be a local summand of M. It is clear that $mX \leq X \cap mM$. Let $x \in X \cap mM$. Then there are $x_1 \in X_{\lambda_1}, \cdots, x_k \in X_{\lambda_k}$ such that $x = x_1 + \cdots + x_k$. Hence

$x - x_1 = x_2 + \cdots + x_k \in X_{\lambda_1} + mM$. Now, Since X is a local summand, there is a submodule K of M such that $X_{\lambda_1} \oplus X_{\lambda_2} \oplus \cdots \oplus X_{\lambda_k} \oplus K = M$. Thus $mX_{\lambda_1} \oplus mX_{\lambda_2} \oplus \cdots \oplus mX_{\lambda_k} \oplus mK = mM$. So $mM + X_{\lambda_1} = X_{\lambda_1} \oplus mX_{\lambda_2} \oplus \cdots \oplus mX_{\lambda_k} \oplus mK$. It follows that $(mM + X_{\lambda_1}) \cap (X_{\lambda_2} \oplus \cdots \oplus X_{\lambda_k}) = mX_{\lambda_2} \oplus \cdots \oplus mX_{\lambda_k}$. This clearly forces $(mM + X_{\lambda_1}) \cap (X_{\lambda_2} \oplus \cdots \oplus X_{\lambda_k}) \leq mM$, and so $x - x_1 \in mM$. But $x \in mM$, then $x_1 \in mM$. Therefore $x_1 \in mM \cap X_{\lambda_1}$. By [4, Proposition 4], we have $mM \cap X_{\lambda_1} = mX_{\lambda_1}$. We thus get $x_1 \in mX$. In the same manner we can see that $x_i \in mX$ for all $i = 2, \cdots, k$. This gives $x \in mX$. Consequently, $X \cap mM = mX$, and finally X is a supplement submodule of M. □

Definition 3.6. A family of modules $\{M_\alpha : \alpha \in \Lambda\}$ is called *locally-semi-transfinitely-nilpotent* (lsTn) if for any subfamily $M_{\alpha_i} (i \in \mathbb{N})$ with distinct α_i and any family of non-isomorphisms $f_i : M_{\alpha_i} \longrightarrow M_{\alpha_{i+1}}$, and for every $x \in M_{\alpha_1}$, there exists $n \in \mathbb{N}$ (depending on x) such that $f_n \cdots f_2 f_1(x) = 0$.

Proposition 3.7. *Let R be a commutative noetherian local ring with maximal ideal m. The following statements are equivalent for an R-module M with $Rad(M) \ll M$:*

(i) *M is weak lifting;*

(ii) *M is lifting;*

(iii) *$M \cong \oplus_{k \in K} \frac{R}{I_k}$ where I_k are ideals of R such that:*

  (a) *there exists $e \geq 1$ such that the set $\{k \in K \mid m^e \not\subseteq I_k\}$ is finite,*

  (b) *the ideals $\{I_k \mid k \in K\}$ are linearly ordered by inclusion, and*

  (c) *if $I_i \subseteq I_j$ then $mI_j \subseteq I_i$.*

Proof. (i) $\Leftrightarrow$(ii) This is clear by Lemma 3.4.

(ii) $\Rightarrow$ (iii) Suppose that M is lifting. Then M is coatomic by Lemma 3.4. Since every supplement submodule of M is a direct summand, Proposition 3.5 shows that every local summand of M is a direct summand. By [6, Theorem 2.17], M is a direct sum of indecomposable modules. By [6, Lemma 4.7 and Corollary 4.9], M is a direct sum of hollow local modules. So $M \cong \oplus_{k \in K} \frac{R}{I_k}$ for some ideals I_k $(k \in K)$ of R. Further, by [14, Satz 2.4], there exists $e \geq 1$ such that $m^e M$ is finitely generated. Hence the set $\{j \in K \mid m^e \not\subseteq I_j\}$ is finite. Now, let k_1, k_2 be two elements in K. Since $\frac{R}{I_{k_1}} \oplus \frac{R}{I_{k_2}}$ is lifting, taking into account Proposition 3.3, [2, Theorem 4.1] and [1, Corollary 12.7], we have $I_{k_1} \subseteq I_{k_2}$ and $mI_{k_2} \subseteq I_{k_1}$ or $I_{k_2} \subseteq I_{k_1}$ and $mI_{k_1} \subseteq I_{k_2}$.

(iii) $\Rightarrow$ (ii) Suppose that M satisfies the stated conditions. Then M can be written as $M = \oplus_{k \in K} Rx_k$ with $Ann_R(x_k) = I_k$. By [13, Satz 3.1], we need to show the following two conditions:

(α) every non-small submodule of M contains a nonzero direct summand of M, and

(β) every submodule of M contains a maximal direct summand of M.

(α) Let L be any non-small submodule of M. Since $Rad(M) \ll M$, there is $x \in L$ such that $x \notin Rad(M)$. Thus $Rx \not\ll M$. On the other hand, there are k_1, k_2, $\cdots$, k_n in K such that $x \in \oplus_{i=1}^n Rx_{k_i}$. But by proof of [16, Folgerung 3.3], Rx is a direct summand of $\oplus_{i=1}^n Rx_{k_i}$. Hence Rx is a direct summand of M, and (α) is proved.

(β) It is clear that every direct summand of M has the structure described in (iii) (see [10, Theorem 1]). By [6, Lemma 2.16], we shall have established the proposition if we prove that every local summand of M is a direct summand of M. By [6, Theorem 2.25] and [2, Theorem 4.1], the proof is completed by showing that the family $\{Rx_k : k \in K\}$ is lsTn. Let $f : Rx_i \longrightarrow Rx_j$ be a non-isomorphism and $a \in R$ such that $f(x_i) = ax_j$. There are three cases: $I_i \subset I_j$ or $I_j \subset I_i$ or $I_i = I_j$.

(1) If $I_j \subset I_i$, then $aI_i \subseteq I_j$. Hence $a \in m$.

(2) If $I_i \subset I_j$, then f is not a monomorphism, because if $\alpha \in I_j - I_i$ then $\alpha x_i \neq 0$ but $f(\alpha x_i) = a\alpha x_j = 0$.

(3) If $I_i = I_j$, then we must have $a \in m$, for otherwise f will be an isomorphism.

Let $Rx_{\alpha_i}(i \in \mathbb{N})$ be a subfamily of $Rx_k(k \in K)$ with distinct α_i and let $f_i : Rx_{\alpha_i} \longrightarrow Rx_{\alpha_{i+1}}$ be a family of non-isomorphisms. Let $b_i \in R$ such that $f_i(x_{\alpha_i}) = b_i x_{\alpha_{i+1}}$. Thus for every $n \in \mathbb{N}$, we have $f_n \cdots f_2 f_1(x_{\alpha_1}) = b_n \cdots b_2 b_1 x_{\alpha_{n+1}}$. Since R is noetherian and $\{k \in K \mid m^e \not\subseteq I_k\}$ is finite, there exists $l \in \mathbb{N}$ such that $f_l \cdots f_2 f_1(x_{\alpha_1}) = 0$. Therefore $\{Rx_k : k \in K\}$ is lsTn. □

Corollary 3.8. *Let R be a commutative noetherian local ring with maximal ideal m. The following statements are equivalent for an R-module M with $Rad(M) \ll M$:*

(i) *M is lifting;*

(ii) *M is weak lifting;*

(iii) *$M = \oplus_{i \in I} Rx_i$ and for every pair $(j,k) \in I \times I$, $Rx_j \oplus Rx_k$ is weak lifting;*

(iv) *$M = \oplus_{i \in I} Rx_i$ and for every pair $(j,k) \in I \times I$, $Rx_j \oplus Rx_k$ is lifting.*

Proof. (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii)$\Rightarrow$(iv). By Proposition 3.7.

(iv) $\Rightarrow$ (i) Let $I_i = Ann_R(x_i)$ $(i \in I)$. By Proposition 3.7, the proof is completed by showing that there exists $e \geq 1$ such that the set $\{i \in I \mid m^e \not\subseteq I_i\}$ is finite. Since M is a direct sum of local modules and $Rad(M) \ll M$, [12, Satz 1.4] shows that M is supplemented. Hence M is coatomic by Lemma 3.4. The assertion follows from [14, Satz 2.4]. □

4. Lifting modules over principal ideal rings

Throughout this section R will denote a commutative principal ideal ring (PIR) (not necessarily a domain). As in [11, p. 245] a PIR is called *special* if it has only one prime ideal $p \neq R$ and if p is nilpotent. From [11, Ch. IV, §15, Theorem 33] we conclude that a local PIR is either a principal ideal domain (PID), or else a special

PIR.

Notation. Let m be a maximal ideal of R and n a non-negative integer. We will denote by $B_m(n, n+1)$ the direct sum of arbitrarily many copies of $\frac{R}{m^n}$ and $\frac{R}{m^{(n+1)}}$.

Proposition 4.1. *Let R be a local PIR with maximal ideal m. If M is an R-module with $Rad(M) \ll M$, then the following are equivalent*:

(i) *M is weak lifting;*

(ii) *M is lifting;*

(iii) *$M \cong B_m(n, n+1)$ or $M \cong R^{(a)}$ for some non-negative integers a and n.*

Proof. By Proposition 3.7 and [8, Lemma 6.3]. □

Corollary 4.2. *Let R be a special PIR with maximal ideal m and let M be an R-module. The following are equivalent*:

(i) *M is weak lifting;*

(ii) *M is lifting;*

(iii) *$M \cong B_m(n, n+1)$ for some non-negative integer n.*

We finally give the structure of lifting modules with small radical over PIR's.

Proposition 4.3. *Let R be a PIR and let M be an R-module with $Rad(M) \ll M$. The following are equivalent*:

(i) *M is weak lifting;*

(ii) *M is lifting;*

(iii) *$M \cong [\oplus_{i\in I} B_{m_i}(n_i, n_i+1)] \oplus [\oplus_{j\in J}(\frac{R}{p_j})^{(a_j)}]$ with :*

  (a) *the $m_i (i \in I)$ are maximal ideals of R, the $p_j (j \in J)$ are non-maximal prime ideals of R and $\{n_i, a_j\}_{(i,j)\in I\times J}$ is a family of positive integers,*

  (b) *the ring $\frac{R}{p_j}$ is local for all $j \in J$, and*

  (c) *given any two elements of the family $\{m_i, p_j\}_{(i,j)\in I\times J}$, then neither of them contains the other.*

Proof. By [11, p. 245 Lemma and Theorem 33] and Proposition 4.1. □

Example 4.4. Let M be a $\mathbb{Z}$-module with $Rad(M) \ll M$. By Proposition 4.3, M is lifting if and only if $M \cong \oplus_{i\in I} B_{p_i\mathbb{Z}}(n_i, n_i+1)$, where the $n_i (i \in I)$ are positive integers and the $p_i (i \in I)$ are prime integers.

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References

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Berlin, Heidelberg, New-York, Springer-Verlag, 1974.

[2] Patrick Fleury, *Hollow modules and local endomorphism rings*, Pacific J. Math., **53(2)**(1974), 379-385.

[3] A. Idelhadj and R. Tribak, *A dual notion of CS-modules generalization*, Lecture Notes in Pure and Appl. Math., **208**(2000), 149-155.

[4] Tomio Inoue, *Sum of hollow modules*, Osaka J. Math., **20**(1983), 331-336.

[5] Derya Keskin, *Finite Direct Sums of* (*D1*)*-Modules*, Tr. J. of Mathematics, **22**(1998), 85-91.

[6] S. H. Mohamed and B. J. Müller, Continuous and Discrete Modules, London Math. Soc. Lecture Note Ser. 147, Cambridge: Cambridge Univ. Press, 1990.

[7] P. Rudlof, *On the Structure of Couniform and Complemented Modules*, J. Pure Appl. Algebra, **74**(1991), 281-305.

[8] D. W. Sharpe and P. Vamos, Injective Modules, Cambridge: Cambridge Univ. Press, 1972.

[9] K. Varadarajan, *Dual Goldie dimension*, Comm. Algebra, **7(6)**(1979), 565-610.

[10] R. B. Warfield, *A Krull-Schmidt theorem for infinite sums of modules*, Proc. Amer. Math. Soc., **22**(1969), 460-465.

[11] O. Zarisky and P. Samuel, Commutative Algebra, Vol. 1, New-York, Heidelberg, Berlin, Springer-Verlag, 1979.

[12] H. Zöschinger, *Komplementierte Moduln über Dedekindringen*, J. Algebra, **29**(1974), 42-56.

[13] H. Zöschinger, *Komplemente als direkte Summanden*, Arch. Math. (Basel), **25**(1974), 241-253.

[14] H. Zöschinger, *Koatomare Moduln*, Math. Z., **170**(1980), 221-232.

[15] H. Zöschinger, *Gelfandringe und koabgeschlossene Untermoduln*, Bayer. Akad. Wiss. Math.-Natur. Kl. Sitzungsber., **3**(1982), 43-70.

[16] H. Zöschinger, *Komplemente als direkte Summanden II*, Arch. Math. (Basel), **38**(1982), 324-334.