KYUNGPOOK Math. J. 45(2005), 211-220

Modular Tranformations for Ramanujan's Tenth Order Mock Theta Functions

Bhaskar Srivastava
Department of Mathematics, Lucknow University, Lucknow, India
e-mail : bhaskarsrivastav@yahoo.com

Abstract. In this paper we obtain the transformations of the Ramanujan's tenth order mock theta functions under the modular group generators $\tau \to \tau + 1$ and $\tau \to -1/\tau$ where $q = e^{\pi i t}$.

1. Introduction

The mock theta functions were named and first studied by Ramanujan. In early 1920, four months before his death Ramanujan wrote his last letter to Hardy [7, p. 354-355]. In the course of it he said "I discovered very interesting functions recently which I call "Mock theta functions" unlike the 'false' theta functions (studied by L. J. Rogers) they enter into mathematics as beautifully as the ordinary theta functions". The last gift of Ramanujan to mathematical world was The Mock Theta Functions. He also commented in the letter that the asymptotic representation of certain q-series with exponential singularities can be written in a closed exponential form only in a limited number of cases in the neighborhood of rational point of the unit circle. On the other hand in a majority of cases they cannot be written in a closed form at the essential singularities on its natural boundary.

Ramanujan listed seventeen mock theta functions and called them of order three, five and seven. Watson [9] found three more mock theta functions of order three and two more appear in the Lost Notebook [9, p. 9]. In the Lost Notebook there are seven more functions [9] and Andrews and Hickerson [2] called them of order six.

There remains a profound mystery about these functions, no one, including Ramanujan, has ever proved that these functions are indeed mock theta functions and not just some clever combination of theta functions.

Recently Choi [4] considered four functions found in the Lost Notebook of Ramanujan [9, p. 9] and called them mock theta function of order ten. In [4] Choi gave generalized Lambert Series for the tenth order mock theta functions $\Phi_R(q)$ and $\Psi_R(q)$. In this paper we obtain the transformation of Ramanujan's tenth order mock theta functions $\Phi_R(q)$ and $\Psi_R(q)$ under the modular group generators

Received February 23, 2004, and, in revised form, May 6, 2004.
2000 Mathematics Subject Classification: 33D.
Key words and phrases: mock theta functions, basic hypergeometric series.

$\tau \to \tau + 1$ and $\tau \to -1/\tau$, where $q = e^{\pi i \tau}$.

By studying the modular transformation of these mock theta functions, we feel we will be able to know about the behavior of these functions.

2. Notations

We shall use the following usual basic hypergeometric notations:
For $|q^k| < 1$,

$$\begin{aligned} (a;q^k)_n &= (1-a)(1-aq^k)\cdots(1-aq^{k(n-1)}), \quad n \geq 1, \\ (a;q^k)_0 &= 1, \\ (a;q^k)_\infty &= \Pi_{j=0}^{\infty}(1-aq^{kj}), \\ (a_1, a_2, \cdots, a_m; q^k)_n &= (a_1;q^k)_n (a_2;q^k)_n \cdots (a_m;q^k)_n, \\ (a;q)_n &= (a)_n. \\ f(b,c) &= \sum_{j=-\infty}^{\infty} b^{\frac{j(j+1)}{2}} c^{\frac{j(j-1)}{2}}. \end{aligned}$$

3. Tenth order mock theta functions

The four tenth order mock theta functions, as defined by Ramanujan, are

$$\Phi_R(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}}, \tag{1}$$

$$\Psi_R(q) = \sum_{n=0}^{\infty} \frac{q^{(n+1)(n+2)/2}}{(q;q^2)_{n+1}}, \tag{2}$$

$$X_R(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}}, \tag{3}$$

and

$$\chi_R(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)^2}}{(-q;q)_{2n+1}}. \tag{4}$$

4. Modular transformation formula

To obtain transformation formulas for tenth order mock theta functions $\Phi_R(q)$ and $\Psi_R(q)$, we use the following generalized Lambert series given by Choi [4],

$$\Psi_R(q) = a_1(q) + 2qh(q, q^5) \tag{5}$$

and

$$\Phi_R(q) = a_2(q) + 2qh(q^2, q^5), \tag{6}$$

$$a_1(q) = -\frac{q(q^5;q^5)_\infty (q^{10};q^{10})_\infty f(-q^2;-q^8)}{f(-q,-q^4)f(-q^4,-q^6)}, \tag{7}$$

$$a_2(q) = \frac{(q^5;q^5)_\infty (q^{10};q^{10})_\infty f(-q^4;-q^6)}{f(-q^2,-q^3)f(-q^2,-q^8)}, \tag{8}$$

and

$$h(x,q) = \frac{1}{f(-q,-q)} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)}}{1-q^n x}. \tag{9}$$

5. Transformation formula for $M(r,q)$

Let

$$M(r,q) = a_r(q) + 2qh(q^r, q^5). \tag{10}$$

For obtaining the transformation formula for $\Phi_R(q)$ and $\Psi_R(q)$, we define a function $M(r,q)$, which for $r=1$ and $r=2$ becomes $\Psi_R(q)$ and $\Phi_R(q)$, respectively. We shall find the transformation formula for $h(q^r, q^5)$.

Now

$$f(-q^5,-q^5)h(q^r,q^5) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)}}{1-q^{5n+r}}. \tag{11}$$

In (11) put $q = e^{-\alpha}$, $a > 0$ and consider the contour integral

$$\begin{aligned} I &= \frac{1}{2\pi i}\int_{-\infty-\varepsilon i}^{-\infty-\varepsilon i} \frac{\pi}{\sin \pi z} \frac{e^{-5z\alpha(z+1)}}{1-e^{-\alpha(5z+r)}} dz + \frac{1}{2\pi i}\int_{-\infty+\varepsilon i}^{-\infty+\varepsilon i} \frac{\pi}{\sin \pi z} \frac{e^{-5z\alpha(z+1)}}{1-e^{-\alpha(5z+r)}} dz \\ &= I_1 + I_2 \ \text{(say)}, \end{aligned}$$

where $\varepsilon > 0$ is so small that the only poles inside the contour are zero of $\sin \pi z$ and the zero of $1-e^{-\alpha(5z+r)}$. Now $(-1)^n$ is the residue of a simple pole of $\frac{\pi}{\sin \pi z}$ for each integer n and $\frac{1}{5\alpha}$ is the residue of a simple pole of $\frac{1}{1-e^{-\alpha(5z+r)}}$ at $z = -\frac{r}{5}$, hence

$$I = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{5n(n+1)}}{1-q^{5n+r}} + \frac{\pi}{\sin(-\frac{\pi r}{5})} \frac{q^{-r(1-\frac{r}{5})}}{5\alpha}. \tag{12}$$

We shall evaluate I_2 first :

In the upper half plane

$$\frac{1}{\sin \pi z} = -2i \sum_{n=0}^{\infty} e^{(2n+1)\pi i z}.$$

So

$$\begin{aligned} (13) \qquad I_2 &= \sum_{n=0}^{\infty} \int_{-\infty+\varepsilon i}^{\infty+\varepsilon i} \frac{e^{(2n+1)\pi i z - 5\alpha z - 5\alpha z^2}}{1-e^{-\alpha(5z+r)}} dz \\ &= \sum_{n=0}^{\infty} J_n(z) \quad \text{(say)}. \end{aligned}$$

The integrand of $J_n(z)$, in the upper half plane, has poles at the points

$$z_m = -\frac{r}{5} + \frac{2m\pi i}{5\alpha}, \quad m = 1, 2, \cdots .$$

The residue at z_m is

$$(14) \qquad = \frac{2\pi i}{5\alpha} q^{-r+\frac{r^2}{5}} e^{-(2n+1)\frac{\pi i r}{5}} q_1^{\frac{2m(2n+1)}{5}} q_1^{-\frac{4}{5}m^2} e^{\frac{4m\pi i r}{5}} = \lambda_{n,m} \quad \text{(say)}$$

where $q = e^{-\alpha}$, $q_1 = e^{-\beta}$, $\alpha\beta = \pi^2$.

Symmetrizing the denominator of the integrand of $J_n(z)$, we have the integrand

$$\begin{aligned} (15) \qquad &= \frac{e^{(2n+1)\pi i z - 5\alpha z - 5\alpha z^2} e^{\frac{a}{2}(5z+r)}}{e^{\frac{a}{2}(5z+r)} - e^{-\frac{a}{2}(5z+r)}} \\ (16) \qquad &= \frac{e^{\alpha r} + e^{-5\alpha z}}{e^{\alpha(5z+r)} - e^{-\alpha(5z+r)}} e^{(2n+1)\pi i z - 5\alpha z^2}. \end{aligned}$$

To get the saddle point, we differentiate the last term, i.e.,

$$z = \frac{(2n+1)\pi i}{10\alpha} = w_n \quad \text{(say)}.$$

Now we move the upper contour of J_n up to the horizontal line through w_n, getting J_n'. By Cauchy's residue theorem,

$$J_n = J_n' + \text{ sum of the residues of the poles between the two contours.}$$

The poles will be at the points

$$z_m = -\frac{r}{5} + \frac{2m\pi i}{5\alpha} \text{ for which } \quad 0 < 2m < \frac{(2n+1)}{2},$$

which is equivalent to $0 < m \le n/2$. Hence

$$J_n = J_n' + \sum_{0<m\le\frac{n}{2}}^{\infty} \lambda_{n,m}.$$

Summing both sides over n from 0 to ∞, we have

$$I_2 = \sum_{n=0}^{\infty} J_n = \sum_{n=0}^{\infty} J'_n + \sum_{m=1}^{\infty} \sum_{n=2m}^{\infty} \lambda_{n,m}.$$

Now we evaluate the double summation on the right-hand side, for this we have from (14)

$$\lambda_{n+1,m} = e^{-\frac{2\pi nir}{5}} q_1^{\frac{4m}{5}} \lambda_{n,m}.$$

So

$$\begin{aligned} \sum_{m=1}^{\infty} \sum_{n=2m}^{\infty} \lambda_{n,m} &= \frac{\lambda_{2m,m}}{1 - e^{-\frac{2\pi i}{5}} q_1^{\frac{4m}{5}}} \\ &= \frac{2\pi i}{5\alpha} \frac{e^{-(4m+1)\frac{\pi ir}{5}} q_1^{\frac{2m(4m+1)}{5}} q^{-r+\frac{r^2}{5}} q_1^{-\frac{4m^2}{5}} e^{\frac{4m\pi ri}{5}}}{1 - e^{-\frac{2\pi ir}{5}} q_1^{\frac{4m}{5}}} \end{aligned} \tag{17}$$

Hence

$$I_2 = \frac{2\pi i}{5\alpha} q^{-r+\frac{r^2}{5}} e^{-\frac{\pi ir}{5}} \sum_{m=1}^{\infty} \frac{q_1^{\frac{2m(2m+1)}{5}}}{1 - e^{-\frac{2\pi ir}{5}} q_1^{\frac{4m}{5}}} + \sum_{n=0}^{\infty} J'_n(z). \tag{18}$$

Evaluating the integral I_1 which is over the lower contour and observing that on the lower contour

$$\frac{1}{\sin \pi z} = 2i \sum_{n=0}^{\infty} e^{-(2n+1)\pi iz}, \tag{19}$$

which is the complex conjugate of the expansion used in the upper contour, so

$$I_1 = \sum_{n=0}^{\infty} K_n \quad \text{(say)},$$

where

$$K_n = \int_{-\infty-\varepsilon i}^{\infty-\varepsilon i} \frac{e^{-(2n+1)\pi iz - 5\alpha z - 5\alpha z^2}}{1 - e^{-\alpha(5z+r)}} dz.$$

Moving the lower contour down to the horizontal line through $\overline{w}_m$, we have

$$K_n = \bar{J}'_n + \sum_{0<m\leq\frac{n}{2}}^{\infty} \bar{\lambda}_{n,m}, \tag{20}$$

in view of (19), the sum on the right-hand side is just the complex conjugate of the summation evaluated earlier. So from (18)

$$I_1 = -\frac{2\pi i}{5\alpha} q^{-r+\frac{r^2}{5}} e^{\frac{\pi i r}{5}} \sum_{m=1}^{\infty} \frac{q_1^{\frac{2m(2m+1)}{5}}}{1 - e^{\frac{2\pi i r}{5}} q_1^{\frac{4m}{5}}} + \sum_{n=0}^{\infty} \bar{J}'_n. \tag{21}$$

Adding (18) and (21), we have

$$\begin{aligned} (22)\quad & I \\ = \quad & I_1 + I_2 \\ = \quad & \frac{2\pi i}{5\alpha} q^{-r+\frac{r^2}{5}} \sum_{m=1}^{\infty} q_1^{\frac{2m(m+1)}{5}} \left[\frac{e^{-\frac{\pi i r}{5}}}{1 - e^{-\frac{2\pi i r}{5}} q_1^{\frac{4m}{5}}} - \frac{e^{\frac{\pi i r}{5}}}{1 - e^{\frac{2\pi i r}{5}} q_1^{\frac{4m}{5}}} \right] + \sum_{n=0}^{\infty} (J'_n + \bar{J}'_n) \\ = \quad & \frac{4\pi}{5\alpha} q^{-r+\frac{r^2}{5}} \sin \frac{\pi i r}{5} \sum_{m=1}^{\infty} \frac{q_1^{\frac{2m(2m+1)}{5}} (1 + q_1^{\frac{4m}{5}})}{1 - 2 q_1^{\frac{4m}{5}} \cos \frac{2\pi r}{5} + q_1^{\frac{8m}{5}}} + \sum_{n=0}^{\infty} (J'_n + \bar{J}'_n). \end{aligned}$$

From (11) and (12),

$$\begin{aligned} (23)\qquad & f(-q^5, -q^5) h(q^r, q^5) \\ = \quad & I + \frac{\pi}{\sin \frac{\pi r}{5}} \frac{q^{-r(1-\frac{r}{5})}}{5\alpha} \\ = \quad & \frac{\pi}{\sin \frac{\pi r}{5}} \frac{q^{-\frac{r}{5}(5-r)}}{5\alpha} + \frac{4\pi}{5\alpha} q^{-r+\frac{r^2}{5}} \sin \frac{\pi r}{5} \sum_{m=1}^{\infty} \frac{q_1^{\frac{2m(2m+1)}{5}} (1 + q_1^{\frac{4m}{5}})}{1 - 2 q_1^{\frac{4m}{5}} \cos \frac{2\pi r}{5} + q_1^{\frac{8m}{5}}} \\ & + \sum_{n=0}^{\infty} (J'_n + \bar{J}'_n) \\ = \quad & \frac{\pi}{\sin \frac{\pi r}{5}} \frac{q^{-\frac{r}{5}(5-r)}}{5\alpha} \left[1 + \sum_{m=1}^{\infty} \frac{(2 - 2\cos \frac{2\pi r}{5}) q_1^{\frac{2m(2m+1)}{5}} (1 + q_1^{\frac{4m}{5}})}{1 - 2 q_1^{\frac{4m}{5}} \cos \frac{2\pi r}{5} + q_1^{\frac{8m}{5}}} \right] \\ & + \sum_{n=0}^{\infty} (J'_n + \bar{J}'_n). \end{aligned}$$

We now evaluate $\sum_{n=0}^{\infty} (J'_n + \bar{J}'_n)$.

First we evaluate the integral J'_n. Putting $z = -\frac{r}{5} + p + x$, where $p = \frac{(2n+1)\pi i}{10\alpha}$ in the integrand and simplifying, we have

$$\begin{aligned} (24)\ J'_n \quad = \quad & e^{\alpha r(1-\frac{r}{5})} q_1^{\frac{(2n+1)^2}{20}} \int_{-\infty}^{\infty} \frac{e^{-5\alpha x^2} e^{2\alpha r x} + e^{-\frac{(2n+1)\pi i}{2}} e^{-5\alpha x^2} e^{-\alpha x(-2r+5)}}{2i(-1)^n \cosh 5\alpha x} dx \\ = \quad & P_n + Q_n. \end{aligned}$$

Observe P_n is purely imaginary. Therefore,

$$J'_n + \bar{J}'_n = Q_n + \bar{Q}_n.$$

So

$$J'_n + \bar{J}'_n = -q^{r(1-\frac{r}{5})} q_1^{\frac{(2n+1)^2}{20}} \int_{-\infty}^{\infty} \frac{e^{-5\alpha x^2} \cosh(5-2r)\alpha x}{\cosh 5\alpha x} dx.$$

Hence

$$\sum_{n=0}^{\infty} (J'_n + \bar{J}'_n) = -\frac{1}{2} q^{-r(1-\frac{r}{5})} \partial_1(\frac{\pi}{2}, q_1^{\frac{1}{5}}) \int_{-\infty}^{\infty} \frac{e^{-5\alpha x^2} \cosh(5-2r)\alpha x}{\cosh 5\alpha x} dx.$$

By (23),

$$\begin{aligned}
(25) \qquad & f(-q^5, -q^5) h(q^r, q^5) \\
= \quad & \frac{\pi}{\sin \frac{\pi r}{5}} \frac{q^{-\frac{r}{5}(5-r)}}{5\alpha} \left[1 + \sum_{m=1}^{\infty} \frac{(2 - 2\cos \frac{2\pi r}{5}) q_1^{\frac{2m(2m+1)}{5}} (1 + q_1^{\frac{4m}{5}})}{1 - 2q_1^{\frac{4m}{5}} \cos \frac{2\pi r}{5} + q_1^{\frac{8m}{5}}} \right] \\
& -\frac{1}{2} q^{-\frac{r}{5}(5-r)} \partial_1(\frac{\pi}{2}, q_1^{\frac{1}{5}}) \int_{-\infty}^{\infty} \frac{e^{-5\alpha x^2} \cosh(5-2r)\alpha x}{\cosh 5\alpha x} dx.
\end{aligned}$$

Putting $r = 1$ we have from (10), (11) and (25)

$$\begin{aligned}
(26) \quad & \Psi_R(q) \\
= \quad & M(1, q) \; = \; a_1(q) + 2a h(q, q^5) \\
= \quad & a_1(q) + \frac{1}{f(-q^5, -q^5)} [\frac{2\pi}{\sin \frac{\pi}{5}} \frac{q^{\frac{1}{5}}}{5\alpha} \left\{ 1 + \sum_{m=1}^{\infty} \frac{(2 - 2\cos \frac{2\pi}{5}) q_1^{\frac{2m(2m+1)}{5}} (1 + q_1^{\frac{4m}{5}})}{1 - 2q_1^{\frac{4m}{5}} \cos \frac{2\pi}{5} + q_1^{\frac{8m}{5}}} \right\} \\
& -q^{\frac{1}{5}} \vartheta_1(\frac{\pi}{2}, q_1^{\frac{1}{5}}) \int_{-\infty}^{\infty} e^{-5\alpha x^2} \frac{\cosh 3\alpha x}{\cosh 5\alpha x} dx].
\end{aligned}$$

We now find the transformation formula for $f(-q, -q)$.

By definition,

$$f(-q, -q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2} = \partial_4(0, q).$$

Using the general transformation formula

$$q^{\frac{B^2}{4A}} \sum_{n=-\infty}^{\infty} (-1)^n q^{An^2 + Bn} = \sqrt{\frac{4\pi}{A\alpha}} \sum_{n=1}^{\infty} q_1^{\frac{(2n-1)^2}{4A}} \cos \frac{(2n-1)B\pi}{2A},$$

we have the transformation formula for $\partial_4(0, q)$, which is

$$(27) \qquad \partial_4(0, q) = \sqrt{\frac{4\pi}{\alpha}} q_1^{\frac{1}{4}} \psi(q_1^2)$$

where

$$\psi(q) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}}.$$

Eliminating $f(-q^5, -q^5) = \partial_4(0, q^5)$ from (26) with the help of (27), we have

$$(28)\; q^{-\frac{1}{5}}\Psi_R(q) = q^{-\frac{1}{5}}a_1(q) + \frac{q_1^{-\frac{1}{20}}}{\psi(q_1^{10})} \left[\frac{1}{5}\sqrt{\frac{\pi}{\alpha}}\csc\frac{\pi}{5}\left\{1 + \sum_{m=1}^{\infty}\frac{(2-2\cos\frac{2\pi}{5})q_1^{\frac{2m(2m+1)}{5}}(1+q_1^{\frac{4m}{5}})}{1-2q_1^{\frac{4m}{5}}\cos\frac{2\pi}{5}+q_1^{\frac{8m}{5}}}\right\} - \frac{1}{2}\sqrt{\frac{\alpha}{\pi}}\partial_1(\frac{\pi}{2}, q_1^{\frac{1}{5}})J_2(\alpha)\right]$$

where $J_2(\alpha)$ is the Mordell integral,

$$J_2(\alpha) = \int_{-\infty}^{\infty} e^{-5\alpha x^2}\frac{\cosh 3\alpha x}{\cosh 5\alpha x}dx.$$

Now we simplify the summation on the right-hand side of (23).

$$\frac{1}{1-2q_1^{\frac{4m}{5}}\cos\frac{2\pi}{5}+q_1^{\frac{8m}{5}}} = \frac{1}{(1-q_1^{\frac{4m}{5}}e^{\frac{2\pi i}{5}})(1-q_1^{\frac{4m}{5}}e^{-\frac{2\pi i}{5}})}$$

$$= (1+q_1^{\frac{4m}{5}}e^{\frac{2\pi i}{5}}+q_1^{\frac{8m}{5}}e^{\frac{4\pi i}{5}}+\cdots)(1+q_1^{\frac{4m}{5}}e^{-\frac{2\pi i}{5}}+q_1^{\frac{8m}{5}}e^{-\frac{4\pi i}{5}}+\cdots)$$

$$= 1 + q_1^{\frac{4m}{5}}2\cos\frac{2\pi}{5} + (1+2\cos\frac{4\pi}{5})q_1^{\frac{8m}{5}} + (2\cos\frac{2\pi}{5}+2\cos\frac{6\pi}{5})q_1^{\frac{12m}{5}}$$

$$+(1+2\cos\frac{4\pi}{5}+2\cos\frac{8\pi}{5})q_1^{\frac{16m}{5}}+\cdots$$

$$= 1+\tau q_1^{\frac{4m}{5}} - \tau q_1^{\frac{8m}{5}} - q_1^{\frac{12m}{5}} + q_1^{\frac{20m}{5}} + \tau q_1^{\frac{24m}{5}} - \tau q_1^{\frac{28m}{5}} - q_1^{\frac{32m}{5}} + \cdots$$

$$= (1+\tau q_1^{\frac{4m}{5}} - \tau q_1^{\frac{8m}{5}} - q_1^{\frac{12m}{5}})(1+q_1^{\frac{20m}{5}}+q_1^{\frac{40m}{5}}+\cdots)$$

$$= \frac{1+\tau q_1^{\frac{4m}{5}} - \tau q_1^{\frac{8m}{5}} - q_1^{\frac{12m}{5}}}{1-q_1^{4m}}$$

where

$$\tau = 2\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{2}.$$

So

$$
\begin{aligned}
(29)\quad & 1+\sum_{m=1}^{\infty}\frac{q_1^{\frac{2m(2m+1)}{5}}(1+q_1^{\frac{4m}{5}})(2-2\cos\frac{2\pi}{5})}{1-2q_1^{\frac{4m}{5}}\cos\frac{2\pi}{5}+q_1^{\frac{8m}{5}}}\\
= \quad & 1+\sum_{m=1}^{\infty}\frac{(1+q_1^{\frac{4m}{5}})\sqrt{5}\tau(1+\tau q_1^{\frac{4m}{5}}-\tau q_1^{\frac{8m}{5}}-q_1^{\frac{12m}{5}})}{1-q_1^{4m}}q_1^{\frac{2m(2m+1)}{5}}\\
= \quad & 1+\sum_{m=1}^{\infty}\frac{\sqrt{5}(\tau+q_1^{\frac{4m}{5}}-q_1^{\frac{12m}{5}}-\tau q_1^{\frac{16m}{5}})}{1-q_1^{4m}}q_1^{\frac{2m(2m+1)}{5}}\\
= \quad & \sqrt{5}\tau\left[\frac{2}{5}+\sum_{m=1}^{\infty}\frac{(1-q_1^{\frac{16m}{5}})}{1-q_1^{4m}}q_1^{\frac{2m(2m+1)}{5}}\right]+\sqrt{5}\left[\frac{1}{5}+\sum_{m=1}^{\infty}\frac{(1-q_1^{\frac{8m}{5}})}{1-q_1^{4m}}q_1^{\frac{2m(2m+3)}{5}}\right].
\end{aligned}
$$

Putting this value in (28), we have

$$
\begin{aligned}
(30)\qquad & q^{-\frac{1}{5}}\Psi_R(q)\\
= \quad & q^{-\frac{1}{5}}a_1(q)+\frac{q_1^{-\frac{1}{20}}}{\psi(q_1^{10})}\left[\frac{1}{5}\sqrt{\frac{\pi}{\alpha}}\csc\frac{\pi}{5}\{\sqrt{5}\tau\left(\frac{2}{5}+\sum_{m=1}^{\infty}\frac{(1-q_1^{\frac{16m}{5}})}{1-q_1^{4m}}q_1^{\frac{2m(2m+1)}{5}}\right)\right.\\
& \left.+\sqrt{5}\left(\frac{1}{5}+\sum_{m=1}^{\infty}\frac{(1-q_1^{\frac{8m}{5}})}{1-q_1^{4m}}\right)q_1^{\frac{2m(2m+3)}{5}}\}-\frac{1}{2}\sqrt{\frac{\alpha}{\pi}}\partial_1(\frac{\pi}{2},q_1^{\frac{1}{5}})J_2(\alpha)\right],
\end{aligned}
$$

which is the transformation formula for $\Psi_R(q)$.

Putting $r=2$, by (10) and (11) and using (25), we have

$$
\begin{aligned}
(31)\qquad & q^{\frac{1}{5}}\Phi_R(q)\\
= \quad & q^{\frac{1}{5}}a_2(q)+\frac{q_1^{-\frac{1}{20}}}{\psi(q_1^{10})}\left[\frac{1}{5}\sqrt{\frac{\pi}{\alpha}}\csc\frac{\pi}{5}\{\frac{\sqrt{5}}{\tau}\left(\frac{2}{5}+\sum_{m=1}^{\infty}\frac{(1-q_1^{\frac{16m}{5}})}{1-q_1^{4m}}q_1^{\frac{2m(2m+1)}{5}}\right)\right.\\
& \left.+\sqrt{5}\left(\frac{1}{5}+\sum_{m=1}^{\infty}\frac{(1-q_1^{\frac{8m}{5}})}{1-q_1^{4m}}\right)q_1^{\frac{2m(2m+3)}{5}}\}-\frac{1}{2}\sqrt{\frac{\alpha}{\pi}}\partial_1(\frac{\pi}{2},q_1^{\frac{1}{5}})J_2(\alpha)\right],
\end{aligned}
$$

where $J_1(\alpha)$ is the Mordell integral

$$J_1(\alpha)=\int_{-\infty}^{\infty}e^{-5\alpha x^2}\frac{\cos h\alpha x}{\cos h5\alpha x}dx$$

which is the transformation formula for $\Phi_R(q)$

6. Conclusion

As per definition of the order of the mock theta function given by Gordon and McIntosh [6], these tenth order mock theta functions should be called of order five.

The Mordell integrals in our transformation formula cannot be connected with the Mordell integrals come in the transformation formula of the fifth order mock theta functions, Gordon and McIntosh [6], so these mock theta functions of order ten are not connected with fifth order mock theta function originally listed in his last letter to Hardy.

In a later paper we will find the generalized Lambert series for the remaining two functions $X_R(q)$ and $\chi_R(q)$ and then their modular transformations.

References

[1] G. E. Andrews, *Mock theta functions*, Proc. Sympos. Pure Math., **49(2)**(1989), 283-298.

[2] G. E. Andrews and D. Hickerson, *Ramanujan's 'Lost'Notebook VII: "The sixth order mock theta functions"*, Adv. in Math., **89**(1991), 60-105.

[3] R. E. Bellman, A Brief Introduction to Theta Functions, Holt, Rinehart and Winston Inc., New York, 1961.

[4] Youn-Seo Choi, *Tenth order mock theta functions in Ramanujan's "Lost" Notebook*, Invent. Math., **136**(1999), 497-569.

[5] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge University Press, Cambridge, 1990.

[6] B. Gordon and R. J. McIntosh, *Modular transformations of Ramanujan's fifth and seventh order Mock theta functions*, The Ramanujan Journal, **7**(2003), 193-222.

[7] B. Gordon and R. J. McIntosh, *Some eight order Mock theta functions*, J. London Math. Soc., **62(2)**(2000), 321-335.

[8] S. Ramanujan, Collected Papers, Cambridge University Press 1927, reprinted by Chelsea, New York, 1962.

[9] S. Ramanujan, The Lost Notebook and Other Unpublished Manuscripts, Narosa, New Delhi, 1988.

[10] G. N. Watson, *The final problem: An account of the Mock theta functions*, J. London Math. Soc., **11**(1936), 55-80.

[11] G. N. Watson, *The Mock theta functions*, Proc. London Math. Soc., **42(2)**(1937), 274-304.

[12] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Fourth Edi., Cambridge University Press, 1952.