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# INTERPOLATION PROBLEMS FOR INFINITE DIMENSIONAL TRIDIAGONAL ALGEBRA $Alg\mathcal{L}_\infty$

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## 1. Introduction

Given two vectors $x$ and $y$ in a Hilbert space $\mathcal{H}$, is there a bounded operator $T$ such that $Tx = y$? This equation is more interesting if the operator $T$ is required to lie in some fixed algebra $\mathcal{U}$, or in some ideal contained in $\mathcal{U}$. Specifically, the equation is : For which $x$ and $y$ does such a $T$ exist? This sort of equation is called an interpolation problem for the algebra $\mathcal{U}$. A variation, the "$n$-vector interpolation problem", asks for an operator $T$ such that $Tx_j = y_j$ for some fixed finite vectors $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ in a Hilbert space $\mathcal{H}$. For given fixed operators $X$ and $Y$ in $\mathcal{B}(\mathcal{H})$, the operator interpolation problem ask under what conditions there will exist an operator $A$ in some fixed algebra $\mathcal{U}$ satisfying the equation $AX = Y$. The $n$-vector interpolation problem was considered for a $C^*$-algebra $\mathcal{U}$ by Kadison [13]. Lance [14] initiated the discussion by considering a nest $\mathcal{N}$ and asking what condition on $x$ and $y$ will guarantee the existence of an operator $A$ in $Alg\mathcal{N}$ such that $Ax = y$. Hopenwasser [9] extended Lance's result to the case where the nest $\mathcal{N}$ is replaced by an arbitrary commutative subspace lattice $\mathcal{L}$. Munch [15] considered the problem of finding a Hilbert-Schmidt operator $A$ in a nest algebra $Alg\mathcal{N}$ that maps $x$ to $y$. Hopenwasser [10] once again

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extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL algebra. In case the ideal of compact operators, the Jacobson radical and the Larson's ideal are contained in a nest algebra $Alg\mathcal{L}$, the one-vector interpolation problem was solved by Anoussis, Katsoulis, Moore, Trent [1]

In this article, we investigate the interpolation problem for an infinite dimensional tridiagonal algebra $Alg\mathcal{L}_\infty$ which was introduced by F. Gilfeather and D. Larson [8]

First, we establish some notation and conventions. Let $\mathcal{H}$ be a complex Hilbert space. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, then $Alg\mathcal{L}$ denotes the algebra of all bounded operators acting on $\mathcal{H}$ that leave invariant every orthogonal projection in $\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$, containing 0 and $I$. Dually, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, the algebra consisting of all bounded operators acting on $\mathcal{H}$, then $Lat\mathcal{A}$ is the lattice of all orthogonal projections invariant for each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = AlgLat\mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a commutative subspace lattice, or CSL, if each pair of projections in $\mathcal{L}$ commutes; $Alg\mathcal{L}$ is then called a CSL-algebra. A totally ordered (and thus commutative) subspace lattice $\mathcal{N}$ is called a nest, and the associated reflexive algebra $Alg\mathcal{N}$ is a nest algebra. If $x_1, x_2, \cdots, x_n$ are vectors in $\mathcal{H}$, then $[x_1, x_2, \cdots, x_n]$ denotes the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$. If $Y$ is a subspace of $\mathcal{H}$, then the set of all vectors, in $\mathcal{H}$ and orthogonal to $Y$, is denoted by $Y^\perp$. Let $E$ be the projection from $\mathcal{H}$ onto closed subspace $Y$ and let $T$ be in $\mathcal{B}(\mathcal{H})$. When $Y$ and $Y^\perp$ are both invariant under $T$, we say that $Y$ reduces $T$; this occurs if and only if $Y$ is invariant under $T$ and $T^*$. Hence $Y$ reduces $T$ if and only if $T$ and $E$ commute.

## 2. Preliminaries and previous results

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space with an orthonormal basis $\{e_1, e_2, e_3, \cdots\}$. Let $\mathcal{L}_\infty$ be the subspace lattice of orthogonal projections generated by $\{ [e_{2i-1}], [e_{2i-1}, e_{2i}, e_{2i+1}] : i = 1, 2, 3, \cdots \}$ and let $\mathcal{A}_\infty$ be the algebra consisting of bounded operators, acting on an infinite dimensional separable complex Hilbert space $\mathcal{H}$, that are of the form

$$\begin{pmatrix} * & * & & & & & \\ & * & & & & & \\ & * & * & * & & & \\ & & & * & & & \\ & & & * & \cdot & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{pmatrix}$$

where all non-starred entries are zero and with an orthonormal basis $\{e_1, e_2, e_3, \cdots\}$. Then $Alg\mathcal{L}_\infty = \mathcal{A}_\infty$ and $Lat\mathcal{A}_\infty = \mathcal{L}_\infty$. The algebra $\mathcal{A}_\infty$ is an infinite dimensional tridiagonal algebra discovered by F. Gilfeather and D. Larson. Since every commutative subspace lattice is reflexive [1] and $\mathcal{L}_\infty$ is a commutative subspace lattice, the tridiagonal algebra $Alg\mathcal{L}_\infty$ is a CSL algebra.

**Theorem 2.1[Lance, 1969]** Let $\mathcal{L}$ be a nest and let $x$ and $y$ be vectors in $\mathcal{H}$. The following are equivalent:

(1) there exists an operator $A$ in $Alg\mathcal{L}$ such that $Ax = y$;

(2) $\sup \left\{ \frac{\|E^\perp y\|}{\|E^\perp x\|} : E \in \mathcal{L} \right\} = K < \infty$.

(We use the convention $\frac{0}{0} = 0$, when necessary.)

Moreover, if condition (2) holds, we may choose the operator $A$ so that $\|A\| = K$.

**Theorem 2.2[Hopenwasser, 1980]** Let $\mathcal{L}$ be a commutative subspace lattice and let $x$ and $y$ be vectors in $\mathcal{H}$. The following are equivalent:

(1) there exists an operator $A$ in $Alg\mathcal{L}$ such that $Ax = y$;
(2) $\sup\left\{\frac{\|E^{\perp}y\|}{\|E^{\perp}x\|} : E \in \mathcal{L}\right\} = K < \infty$.
(We use the convention $\frac{0}{0} = 0$, when necessary.)

Moreover, if condition (2) holds, we may choose the operator $A$ so that $\|A\| = K$.

More general interpolation problem is following : Let $\mathcal{L}$ be a CSL, given $X$, $Y$ in $\mathcal{B}(\mathcal{H})$, the algebra consisting of all bounded operators acting on $\mathcal{H}$, is there an operator $A$ in $Alg\mathcal{L}$ such that $AX = Y$.

**Theorem 2.3 [Hopenwasser, 1989]** Let $\mathcal{L}$ be a commutative subspace lattice and let $X$ and $Y$ be rank one operators. The following are equivalent:

(1) there exists an operator $A$ in $Alg\mathcal{L}$ such that $AX = Y$;
(2) $\sup\left\{\frac{\|E^{\perp}Yf\|}{\|E^{\perp}Xf\|} : f \in \mathcal{H} \text{ and } E \in \mathcal{L}\right\} = K < \infty$.
(We use the convention $\frac{0}{0} = 0$, when necessary.)

Moreover, if condition (2) holds, we may choose the operator $A$ so that $\|A\| = K$.

**Theorem 2.4[Douglas, 1966]** Let $X$ and $Y$ be bounded operators on a Hilbert space $\mathcal{H}$. The following statements are equivalent:

(1) range$[Y^*] \subset$ range$[X^*]$;
(2) $Y^*Y \leq \lambda^2 X^*X$ for some $\lambda \geq 0$;
(3) there exists a bounded operator $A$ on $\mathcal{H}$ so that $AX = Y$.

Moreover, if (1), (2), and (3) are valid, then there exists a unique operator $A$ so that

(a) $\|A\|^2 = \inf\{\mu : Y^*Y \leq \mu X^*X\}$;
(b) $\ker[Y^*] = \ker[A^*]$; and

(c) range$[A^*] \subset$ range$[X]^-$.

The following theorem is a direct generalization of Douglas's result.

**Theorem 2.5 [Katsoulis, Moore, Trent, 1993]** Let $X$ and $Y$ be bounded operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{L}$ be a nest. The following are equivalent:

(1) there exists an operator $A$ in $Alg\mathcal{L}$ such that $AX = Y$;

(2) $\sup\left\{\frac{\|E^\perp Yf\|}{\|E^\perp Xf\|} : f \in \mathcal{H} \text{ and } E \in \mathcal{L}\right\} = K < \infty$.

(We use the convention $\frac{0}{0} = 0$, when necessary.)

Moreover, if condition (2) holds, we may choose the operator $A$ so that $\|A\| = K$.

## 3. Interpolation problems for $Alg\mathcal{L}_\infty$

Let $Y$ be a subspace of a Hilbert space $\mathcal{H}$. The set of all vectors, in $\mathcal{H}$ and orthogonal to $Y$, is denoted by $Y^\perp$. If $\mathcal{L}$ is a subspace lattice, then the set $\{E^\perp : E \in \mathcal{L}\}$ is also a subspace lattice, we denote it by $\mathcal{L}^\perp$.

**Lemma 3.1** Let $\mathcal{L}$ be a commutative subspace lattice. Then the following are equivalent:

(1) $A \in Alg\mathcal{L}$;

(2) $AE = EAE$ for all $E \in \mathcal{L}$;

(3) $E^\perp AE = 0$ for all $E \in \mathcal{L}$;

(4) $E^\perp AE^\perp = E^\perp A$ for all $E \in \mathcal{L}$.

First we investigate the one vector interpolation problem for $Alg\mathcal{L}_\infty$.

**Theorem 3.2** Let $x$ and $y$ be two vectors in $\mathcal{H}$ such that $x \neq 0$. If

$$\sup\left\{\frac{\|\sum_{j=1}^k \alpha_j E_j y\|}{\|\sum_{j=1}^k \alpha_j E_j x\|} : k \in \mathbf{N},\ \alpha_j \in \mathbf{C} \text{ and } E_j \in \mathcal{L}_\infty\right\} < \infty,$$

then there exists an operator $A$ in $Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$ such that $Ax = y$.

**Proof.** Suppose that

$$\sup\left\{\frac{\|\sum_{j=1}^k \alpha_j E_j y\|}{\|\sum_{j=1}^k \alpha_j E_j x\|} : k \in \mathbf{N},\ \alpha_j \in \mathbf{C} \ and\ E_j \in \mathcal{L}_\infty\right\} = K < \infty.$$

Then for each $k \in \mathbf{N}$ and $a_1, a_2, \cdots, a_k \in \mathbf{C}$, and $E_1, E_2, \cdots, E_k \in \mathcal{L}_\infty$, we have

$$\left\|\sum_{j=1}^k \alpha_j E_j y\right\| \leq K \left\|\sum_{j=1}^k \alpha_j E_j x\right\|.$$

Let

$$\mathcal{M} = \left\{\sum_{j=1}^k \alpha_j E_j x : k \in \mathbf{N},\ \alpha_j \in \mathbf{C} \ and\ E_j \in \mathcal{L}_\infty\right\}.$$

Then $\mathcal{M}$ is a subspace of $\mathcal{H}$. Define $A : \mathcal{M} \to \mathcal{H}$ by

$$A\left(\sum_{j=1}^k \alpha_j E_j x\right) = \sum_{j=1}^k \alpha_j E_j y.$$

Then $A$ is well defined. For, if

$$\sum_{j=1}^{k_1} \alpha_j E_j x = \sum_{l=1}^{k_2} \beta_l F_l x$$

in $\mathcal{M}$, then

$$\sum_{j=1}^{k_1} \alpha_j E_j x + \sum_{l=1}^{k_2} (-\beta_l) F_l x = 0.$$

So

$$\left\|\sum_{j=1}^{k_1} \alpha_j E_j x + \sum_{l=1}^{k_2} (-\beta_l) F_l x\right\| = 0$$

and so

$$\left\|\sum_{j=1}^{k_1} \alpha_j E_j y + \sum_{l=1}^{k_2} (-\beta_l) F_l y\right\| = 0.$$

Thus

$$\sum_{j=1}^{k_1} \alpha_j E_j y = \sum_{l=1}^{k_2} \beta_l F_l y.$$

Hence $A$ is well defined.

Extend $A$ to $\overline{\mathcal{M}}$ by continuity, and define $A|_{\overline{\mathcal{M}}^\perp} = 0$. Then $A$ is defined on $\mathcal{H}$ and $Ax = y$ and $\|A\| \leq K$. Let $E$ be in $\mathcal{L}_\infty$. Then

$$\begin{aligned} EA\left(\sum_{j=1}^{k} \alpha_j E_j x\right) &= E\left(\sum_{j=1}^{k} \alpha_j E_j y\right) = \sum_{j=1}^{k} \alpha_j E E_j y \\ &= A\left(\sum_{j=1}^{k} \alpha_j E E_j x\right) = AE\left(\sum_{j=1}^{k} \alpha_j E_j x\right). \end{aligned}$$

Hence $EAf = AEf$ for all $f \in \mathcal{M}$ and hence $EAf = AEf$ for all $f \in \overline{\mathcal{M}}$. Let $g \in \overline{\mathcal{M}}^\perp$ and let $h = \sum_{j=1}^{k} \alpha_j E_j x \in \mathcal{M}$. Then

$$\begin{aligned} (h, Eg) &= \left(\sum_{j=1}^{k} \alpha_j E_j x, Eg\right) = \left(E\left(\sum_{j=1}^{k} \alpha_j E_j x\right), g\right) \\ &= \left(\sum_{j=1}^{k} \alpha_j E E_j x, g\right) = 0. \end{aligned}$$

Hence $Eg \in \mathcal{M}^\perp$ and hence $Eg \in \overline{\mathcal{M}}^\perp$. Thus $AEg = 0$ for all $g \in \overline{\mathcal{M}}^\perp$ and $EAg = E0 = 0$ for all $g \in \overline{\mathcal{M}}^\perp$. So $AEg = EAg$ for all $g \in \overline{\mathcal{M}}^\perp$. Hence $AE = EA$ for all $E$ in $\mathcal{L}_\infty$ and hence $E$ in $\mathcal{L}_\infty$ reduces $A$. ¿From Lemma 3.1, we have $A \in Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$.

**Theorem 3.3** Let $x$ and $y$ be two vectors in $\mathcal{H}$ such that $x \neq 0$. If there is an operator $A$ in $Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$ such that $Ax = y$, then

$$\sup\left\{\frac{\|\sum_{j=1}^{k} \alpha_j E_j y\|}{\|\sum_{j=1}^{k} \alpha_j E_j x\|} : k \in \mathbf{N},\ \alpha_j \in \mathbf{C} \text{ and } E_j \in \mathcal{L}_\infty\right\} < \infty,$$

**Proof.** Suppose that $Ax = y$ and $A \in Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$. Then from Lemma 3.1, we have $AE = EAE$ and $EA = EAE$ and so $AE = EA$ for all $E \in \mathcal{L}_\infty$. For each $k \in \mathbf{N}$, $\alpha_j \in \mathbf{C}$, and $E_j \in \mathcal{L}_\infty$,

$$A\left(\sum_{j=1}^{k} \alpha_j E_j x\right) = \sum_{j=1}^{k} \alpha_j A E_j x = \sum_{j=1}^{k} \alpha_j E_j A x = \sum_{j=1}^{k} \alpha_j E_j y.$$

Hence

$$\left\|\sum_{j=1}^{k} \alpha_j E_j y\right\| \leq \|A\| \left\|\sum_{j=1}^{k} \alpha_j E_j x\right\|.$$

If

$$\|\sum_{j=1}^{k} \alpha_j E_j x\| \neq 0,$$

then

$$\frac{\|\sum_{j=1}^{k} \alpha_j E_j y\|}{\|\sum_{j=1}^{k} \alpha_j E_j x\|} \leq \|A\|.$$

Thus

$$\sup\left\{\frac{\|\sum_{j=1}^{k} \alpha_j E_j y\|}{\|\sum_{j=1}^{k} \alpha_j E_j x\|} : k \in \mathbf{N},\ \alpha_j \in \mathbf{C} \text{ and } E_j \in \mathcal{L}_\infty\right\} < \infty.$$

If we use $\mathcal{L}_\infty^\perp$ instead of $\mathcal{L}_\infty$, from Theorems 3.2 and 3.3, we obtain the following theorem.

**Theorem 3.4** Let $x$ and $y$ be two vectors in $\mathcal{H}$ such that $x \neq 0$. Then the following are equivalent.

(1) there exists an operator $A$ in $Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$ such that $Ax = y$;

(2) $\sup\left\{\frac{\|\sum_{j=1}^{k} \alpha_j E_j^\perp y\|}{\|\sum_{j=1}^{k} \alpha_j E_j^\perp x\|} : k \in \mathbf{N},\ \alpha_j \in \mathbf{C} \text{ and } E_j \in \mathcal{L}_\infty\right\} < \infty.$

If we summarize Theorems 3.2 , 3.3 and 3.4, then we can get the following theorem.

**Theorem 3.5** Let $x$ and $y$ be two vectors in $\mathcal{H}$ such that $x \neq 0$. Then the following are equivalent.

(1) there exists an operator $A$ in $Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$ such that $Ax = y$;

(2) $\sup\left\{\frac{\|\sum_{j=1}^{k}\alpha_j E_j y\|}{\|\sum_{j=1}^{k}\alpha_j E_j x\|} : k \in \mathbf{N},\ \alpha_j \in \mathbf{C} \text{ and } E_j \in \mathcal{L}_\infty\right\} < \infty$;

(3) $\sup\left\{\frac{\|\sum_{j=1}^{k}\alpha_j E_j^\perp y\|}{\|\sum_{j=1}^{k}\alpha_j E_j^\perp x\|} : k \in \mathbf{N},\ \alpha_j \in \mathbf{C} \text{ and } E_j \in \mathcal{L}_\infty\right\} < \infty$.

In the second place we investigate the $n$-vector interpolation problem for $Alg\mathcal{L}_\infty$.

**Theorem 3.6** Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be fixed finite vectors in $\mathcal{H}$ such that $x_i \neq 0$ for all $i = 1, 2, \cdots, n$. If

$$\sup\left\{\frac{\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}y_i\|}{\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i\|} : k_i \in \mathbf{N},\ \alpha_{ij} \in \mathbf{C} \text{ and } E_{ij} \in \mathcal{L}_\infty\right\} < \infty,$$

then there exists an operator $A$ in $Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$ such that $Ax_i = y_i$ for all $i = 1, 2, \cdots, n$,

**Proof.** Suppose that

$$\sup\left\{\frac{\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}y_i\|}{\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i\|} : k_i \in \mathbf{N},\ \alpha_{ij} \in \mathbf{C} \text{ and } E_{ij} \in \mathcal{L}_\infty\right\} = K < \infty.$$

Then for each fixed $i(1 \leq i \leq n)$

$$\left\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}y_i\right\| \leq K\left\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i\right\|$$

for all $k_i \in \mathbf{N},\ \alpha_{ij} \in \mathbf{C}$, and $E_{ij} \in \mathcal{L}_\infty$, Let

$$\mathcal{M} = \left\{\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i : k_i \in \mathbf{N},\ \alpha_{ij} \in \mathbf{C}, \text{ and } E_{ij} \in \mathcal{L}_\infty\right\}$$

Then $\mathcal{M}$ is a subspace of $\mathcal{H}$. Define $A : \mathcal{M} \to \mathcal{H}$ by

$$A\left(\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}y_i.$$

Then $A$ is well defined. For, if $\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i = \sum_{i=1}^{n}\sum_{l=1}^{p_i}\beta_{il}F_{il}x_i$, then

$$\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i + \sum_{i=1}^{n}\sum_{l=1}^{p_i}(-\beta_{il})F_{il}x_i = 0.$$

So

$$\left\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i + \sum_{i=1}^{n}\sum_{l=1}^{p_i}(-\beta_{il})F_{il}x_i\right\| = 0.$$

and so

$$\left\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i + \sum_{i=1}^{n}\sum_{l=1}^{p_i}(-\beta_{il})F_{il}y_i\right\| = 0.$$

Thus

$$\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}y_i = \sum_{i=1}^{n}\sum_{l=1}^{p_i}\beta_{il}F_{il}y_i.$$

Hence $A$ is well defined.

Extend $A$ to $\overline{\mathcal{M}}$ by continuity, and define $A|_{\overline{\mathcal{M}}^\perp} = 0$. Then $A$ is defined on $\mathcal{H}$ and $Ax_i = y_i$ for all $i = 1, 2, \cdots n$ and $\|A\| \leq K$. Let $E$ be in $\mathcal{L}_\infty$. Then

$$\begin{aligned} EA\left(\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i\right) &= E\left(\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}y_i\right) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}EE_{ij}y_i \\ &= A\left(\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}EE_{ij}x_i\right) \\ &= AE\left(\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i\right). \end{aligned}$$

Hence $EAf = AEf$ for all $f \in \mathcal{M}$ and hence $EAf = AEf$ for all $f \in \overline{\mathcal{M}}$. Let $g \in \overline{\mathcal{M}}^\perp$. Then for every $h = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i$ in $\mathcal{M}$

$$
\begin{aligned}
(h, Eg) &= \left(\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i, Eg\right) = \left(E\left(\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ijl}x_i\right), g\right) \\
&= \left(\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}EE_{ij}x_i, g\right) = 0.
\end{aligned}
$$

Hence $Eg \in \mathcal{M}^\perp$ and hence $Eg \in \overline{\mathcal{M}}^\perp$ for all $g \in \overline{\mathcal{M}}^\perp$. Thus $AEg = 0$ and $EAg = E0 = 0$ for all $g \in \overline{\mathcal{M}}^\perp$. So $AEg = EAg$ for all $g \in \overline{\mathcal{M}}^\perp$. Hence $AE = EA$ for all $E$ in $\mathcal{L}_\infty$. Hence $E \in \mathcal{L}_\infty$ reduces $A$. ¿From Lemma 3.1, we have $A \in Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$.

**Theorem 3.7** Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be fixed finite vectors in $\mathcal{H}$ such that $x_i \neq 0$ for all $i = 1, 2, \cdots, n$. If there is an operator $A$ in $Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$ such that $Ax_i = y_i$ for all $i = 1, 2, \cdots, n$, then

$$
\sup\left\{\frac{\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}y_i\|}{\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i\|} : k_i \in \mathbf{N},\ \alpha_{ij} \in \mathbf{C} \text{ and } E_{ij} \in \mathcal{L}_\infty\right\} < \infty,
$$

**Proof.** Suppose that $A \in Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$ and $Ax_i = y_i$ for all $i = 1, 2, \cdots, n$. Then from Lemma 3.1, we have $AE = EAE$ and $EA = EAE$ and so $AE = EA$ for all $E \in \mathcal{L}_\infty$. Hence $AEx_i = EAx_i = Ey_i$ for all $E \in \mathcal{L}_\infty$ and all $i = 1, 2, \cdots, n$. Thus for each fixed $i (1 \leq i \leq n)$

$$
A\left(\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}y_i.
$$

for all $k_i \in \mathbf{N},\ \alpha_{ij} \in \mathbf{C}$, and $E_{ij} \in \mathcal{L}_\infty$. Hence

$$
\left\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}y_i\right\| \leq \|A\| \left\|\sum_{i=1}^{n}\sum_{j=1}^{k_i}\alpha_{ij}E_{ij}x_i\right\|.
$$

If

$$\left\| \sum_{i=1}^{n} \sum_{j=1}^{k_i} \alpha_{ij} E_{ij} x_i \right\| \neq 0,$$

then

$$\frac{\| \sum_{i=1}^{n} \sum_{j=1}^{k_i} \alpha_{ij} E_{ij} y_i \|}{\| \sum_{i=1}^{n} \sum_{j=1}^{k_i} \alpha_{ij} E_{ij} x_i \|} \leq \|A\|.$$

Thus

$$\sup \left\{ \frac{\| \sum_{i=1}^{n} \sum_{j=1}^{k_i} \alpha_{ij} E_{ij} y_i \|}{\| \sum_{i=1}^{n} \sum_{j=1}^{k_i} \alpha_{ij} E_{ij} x_i \|} : k_i \in \mathbf{N},\ \alpha_{ij} \in \mathbf{C} \text{ and } E_{ij} \in \mathcal{L}_\infty \right\} < \infty.$$

If we use $\mathcal{L}_\infty^\perp$ instead of $\mathcal{L}_\infty$, from Theorems 3.6 and 3.7, we obtain the following theorem.

**Theorem 3.8** Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be fixed finite vectors in $\mathcal{H}$ such that $x_i \neq 0$ for all $i = 1, 2, \cdots, n$. Then the following are equivalent.

(1) there exists an operator $A$ in $Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$ such that $Ax_i = y_i$ for all $i = 1, 2, \cdots, n$;

(2) $\sup \left\{ \frac{\| \sum_{i=1}^{n} \sum_{j=1}^{k_i} \alpha_{ij} E_{ij}^\perp y_i \|}{\| \sum_{i=1}^{n} \sum_{j=1}^{k_i} \alpha_{ij} E_{ij}^\perp x_i \|} : k_i \in \mathbf{N},\ \alpha_{ij} \in \mathbf{C} \text{ and } E_{ij} \in \mathcal{L}_\infty \right\} < \infty.$

If we summarize Theorems 3.6, 3.7 and 3.8, then we can get the following theorem.

**Theorem 3.9** Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be fixed finite vectors in $\mathcal{H}$ such that $x_i \neq 0$ for all $i = 1, 2, \cdots, n$. Then the following are equivalent.

(1) there exists an operator $A$ in $Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$ such that $Ax_i = y_i$ for all $i = 1, 2, \cdots, n$;

(2) $\sup \left\{ \frac{\| \sum_{i=1}^{n} \sum_{j=1}^{k_i} \alpha_{ij} E_{ij} y_i \|}{\| \sum_{i=1}^{n} \sum_{j=1}^{k_i} \alpha_{ij} E_{ij} x_i \|} : k_i \in \mathbf{N},\ \alpha_{ij} \in \mathbf{C} \text{ and } E_{ij} \in \mathcal{L}_\infty \right\} < \infty;$

(3) $\sup \left\{ \frac{\| \sum_{i=1}^{n} \sum_{j=1}^{k_i} \alpha_{ij} E_{ij}^\perp y_i \|}{\| \sum_{i=1}^{n} \sum_{j=1}^{k_i} \alpha_{ij} E_{ij}^\perp x_i \|} : k_i \in \mathbf{N},\ \alpha_{ij} \in \mathbf{C} \text{ and } E_{ij} \in \mathcal{L}_\infty \right\} < \infty.$

Finally we investigate the operator interpolation problem for $Alg\mathcal{L}_\infty$. The equations $Ax = y$ and $AX = Y$ are indistinguishable if spoken aloud, but we mean the change to capital letters to indicate that we intend to look at fixed operators $X$ and $Y$, and ask under what conditions there will exist an operator $A$ satisfying the equation $AX = Y$. If $u$ and $v$ are vectors in $\mathcal{H}$, we use the notation $u \otimes v$ for the rank-one operator defined by $(u \otimes v)f = (f, v)u$ for all $f$ in $\mathcal{H}$. Then the following lemma show that the vector interpolation problem is a special case of the operator interpolation problem.

**Lemma 3.10** Let $X = x \otimes u$ and $Y = y \otimes u$ and let $A$ be an operator on $\mathcal{H}$. Then $AX = Y$ if and only if $Ax = y$.

**Proof.** Let $X = x \otimes u$ and $Y = y \otimes u$. If $AX = Y$, then $A(x \otimes u)u = (y \otimes u)u$. Hence $(u, u)Ax = A(u, u)x = A(x \otimes u)u = (y \otimes u)u = (u, u)y$ and hence $Ax = y$. If $Ax = y$, then $A(x \otimes u)f = A(f, u)x = (f, u)Ax = (f, u)y = (y \otimes u)f$ for all $f \in \mathcal{H}$. Hence $AX = A(x \otimes u) = (y \otimes u) = Y$.

**Theorem 3.11** Let $X$ and $Y$ be in $\mathcal{B}(\mathcal{H})$. If

$$\sup \left\{ \frac{\| \sum_{j=1}^{k} E_j Y f_j \|}{\| \sum_{j=1}^{k} E_j X f_j \|} : k \in \mathbf{N},\ E_j \in\ \mathcal{L}_\infty \textit{ and } f_j \in \mathcal{H} \right\} < \infty.$$

then there exists an operator $A$ in $Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^{\perp}$ such that $AX = Y$.

**Proof.** Suppose that

$$\sup \left\{ \frac{\| \sum_{j=1}^{k} E_j Y f_j \|}{\| \sum_{j=1}^{k} E_j X f_j \|} : k \in \mathbf{N},\ E_j \in \mathcal{L}_\infty \textit{ and } f_j \in \mathcal{H} \right\} = K < \infty.$$

Then for each $k \in \mathbf{N}$, $E_j \in \mathcal{L}_\infty$ and $f_j \in \mathcal{H}$, we have

$$\left\| \sum_{j=1}^{k} E_j Y f_j \right\| \le K \left\| \sum_{j=1}^{k} E_j X f_j \right\|.$$

Let

$$\mathcal{M} = \left\{ \sum_{j=1}^{k} E_j X f_j : k \in \mathbf{N},\ E_j \in \mathcal{L}_\infty \text{ and } f_j \in \mathcal{H} \right\}.$$

Then $\mathcal{M}$ is a subspace of $\mathcal{H}$. Define $A : \mathcal{M} \to \mathcal{H}$ by

$$A\left(\sum_{j=1}^{k} E_j X f_j\right) = \sum_{j=1}^{k} E_j Y f_j.$$

Then $A$ is well defined. For, if $\sum_{j=1}^{k_1} E_j X f_j = \sum_{l=1}^{k_2} F_l X g_l$ in $\mathcal{M}$, then

$$\sum_{j=1}^{k_1} E_j X f_j + \sum_{l=1}^{k_2} F_l X(-g_l) = 0.$$

So

$$\left\| \sum_{j=1}^{k_1} E_j X f_j + \sum_{l=1}^{k_2} F_l X(-g_l) \right\| = 0$$

and so

$$\left\| \sum_{j=1}^{k_1} E_j Y f_j + \sum_{l=1}^{k_2} F_l Y(-g_l) \right\| = 0.$$

Thus $\sum_{j=1}^{k_1} E_j Y f_j = \sum_{l=1}^{k_2} F_l Y g_l$. Hence $A$ is well defined. Clearly $AX = Y$. Since

$$\left\| A\left(\sum_{j=1}^{k} E_j X f_j\right) \right\| = \left\| \sum_{j=1}^{k} E_j Y f_j \right\| \leq K \left\| \sum_{j=1}^{k} E_j X f_j \right\|,$$

we have $\|Ag\| \leq K\|g\|$ for all $g \in \mathcal{M}$. Extend $A$ to $\overline{\mathcal{M}}$ by continuity, and define $A|_{\overline{\mathcal{M}}^\perp} = 0$. Then $A$ is defined on $\mathcal{H}$ and $AX = Y$ and $\|A\| \leq K$. Let $E$ be in $\mathcal{L}_\infty$. Then

$$\begin{aligned} AE\left(\sum_{j=1}^{k} E_j X f_j\right) &= A\left(\sum_{j=1}^{k} E E_j X f_j\right) = \sum_{j=1}^{k} E E_j Y f_j \\ &= E\left(\sum_{j=1}^{k} E_j Y f_j\right) = EA\left(\sum_{j=1}^{k} E_j X f_j\right). \end{aligned}$$

So $AEg = EAg$ for all $g \in \mathcal{M}$ and hence $AEg = EAg$ for all $g \in \overline{\mathcal{M}}$. Let $g \in \overline{\mathcal{M}}^\perp$ and $h = \sum_{j=1}^k E_jXf_j \in \mathcal{M}$. Then

$$\begin{aligned}(h, Eg) &= \left(\sum_{j=1}^k E_jXf_j, Eg\right) = \left(E\left(\sum_{j=1}^k E_jXf_j\right), g\right) \\ &= \left(\sum_{j=1}^k EE_jXf_j, g\right) = 0.\end{aligned}$$

Hence $Eg \in \mathcal{M}^\perp$ and hence $Eg \in \overline{\mathcal{M}}^\perp$. Thus $AEg = 0$ and $EAg = E0 = 0$ for all $g \in \overline{\mathcal{M}}^\perp$. So $AEg = EAg$ for all $g \in \overline{\mathcal{M}}^\perp$. Hence $AE = EA$ for all $E$ in $\mathcal{L}_\infty$ and hence $E$ in $\mathcal{L}_\infty$ reduces $A$. ¿From Lemma 3.1, we have $A \in Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$.

**Theorem 3.12** Let $X$ and $Y$ be in $\mathcal{B}(\mathcal{H})$. If there is an operator $A$ in $Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$ such that $AX = Y$, then

$$\sup\left\{\frac{\|\sum_{j=1}^k E_jYf_j\|}{\|\sum_{j=1}^k E_jXf_j\|} : k \in \mathbf{N},\ E_j \in \mathcal{L}_\infty \text{ and } f_j \in \mathcal{H}\right\} < \infty.$$

**Proof.** Suppose that $AX = Y$ and $A \in Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_{infty}^\perp$. Then from Lemma 3.1, we have $AE = EAE$ and $EA = EAE$ and so $AE = EA$ for all $E \in \mathcal{L}_\infty$. For each $k \in \mathbf{N}$, $E_j \in \mathcal{L}_\infty$ and $f_j \in \mathcal{H}$, we have

$$A\left(\sum_{j=1}^k E_jXf_j\right) = \sum_{j=1}^k AE_jXf_j = \sum_{j=1}^k E_jAXf_j = \sum_{j=1}^k E_jYf_j.......................$$

Hence

$$\left\|\sum_{j=1}^k E_jYf_j\right\| \leq \|A\| \left\|\sum_{j=1}^k E_jXf_j\right\|.$$

If $\|\sum_{j=1}^k E_jXf_j\| \neq 0$, then $\frac{\|\sum_{j=1}^k \alpha_jE_jYf_j\|}{\|\sum_{j=1}^k \alpha_jE_jXf_j\|} \leq \|A\|$. Thus

$$\sup\left\{\frac{\|\sum_{j=1}^k E_jYf_j\|}{\|\sum_{j=1}^k E_jXf_j\|} : k \in \mathbf{N},\ E_j \in \mathcal{L}_\infty \text{ and } f_j \in \mathcal{H}\right\} < \infty.$$

If we use $\mathcal{L}_\infty^\perp$ instead of $\mathcal{L}_\infty$, from Theorems 3.11 and 3.12, we obtain the following theorem.

**Theorem 3.13** Let $X$ and $Y$ be in $\mathcal{B}(\mathcal{H})$. Then the following are equivalent.

(1) there exists an operator $A$ in $Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$ such that $AX = Y$;

(2) $\sup\left\{\frac{\|\sum_{j=1}^k E_j^\perp Y f_j\|}{\|\sum_{j=1}^k E_j^\perp X f_j\|} : k \in \mathbf{N},\ E_j \in \mathcal{L}_\infty \text{ and } f_j \in \mathcal{H}\right\} < \infty.$

If we summarize Theorems 3.11, 3.12 and 3.13, then we can get the following theorem.

**Theorem 3.14** Let $X$ and $Y$ be in $\mathcal{B}(\mathcal{H})$. Then the following are equivalent.

(1) there exists an operator $A$ in $Alg\mathcal{L}_\infty \cap Alg\mathcal{L}_\infty^\perp$ such that $AX = Y$;

(2) $\sup\left\{\frac{\|\sum_{j=1}^k E_j Y f_j\|}{\|\sum_{j=1}^k E_j X f_j\|} : k \in \mathbf{N},\ E_j \in \mathcal{L}_\infty \text{ and } f_j \in \mathcal{H}\right\} < \infty;$

(3) $\sup\left\{\frac{\|\sum_{j=1}^k E_j^\perp Y f_j\|}{\|\sum_{j=1}^k E_j^\perp X f_j\|} : k \in \mathbf{N},\ E_j \in \mathcal{L}_\infty \text{ and } f_j \in \mathcal{H}\right\} < \infty.$

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