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PEXIDER-EXPONENTIAL EQUATIONS IN THE SPACE OF DISTRIBUTIONS AND THEIR APPLICATIONS

SEOK RHO

Abstract. Generalizing the results in [7] that considers several functional equations in the spaces of the Schwartz tempered distributions and the Fourier hyperfunctions we consider Pexider type functional equations in the space of distributions.

1. Introduction

In the previous paper[7], several functional equations have been considered in the space $\mathcal{S}'(\mathbb{R}^n)$ of tempered distributions that is the dual space of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of infinitely differentiable functions of polynomial decay and the space $\mathcal{F}'(\mathbb{R}^n)$ of Fourier hyperfunctions that is the dual space of the Sato space $\mathcal{F}(\mathbb{R}^n)$ of analytic functions of exponential decay. We refer to [4, 5, 6, 7, 8, 9, 10, 11] for the spaces of tempered distributions and Fourier hyperfunctions.

In this paper, generalizing the results in [7] and following a similar approach as in [5, 6] we consider the Pexider equation and Pexider-exponential equation

$$f(x+y) - g(x) - h(y) = 0, \tag{1.1}$$

$$f(x+y) - g(x)\,h(y) = 0, \tag{1.2}$$

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in the space $\mathcal{D}'(\mathbb{R}^n)$ of Schwartz distributions. As special cases of the equations (1.1) and (1.2) we also consider the Cauchy equations, exponential equation, Jensen equation and Jensen-Pexider equation:

$$f(x+y) - f(x) - f(y) = 0, \tag{1.3}$$

$$f(x+y) - f(x)\,f(y) = 0, \tag{1.4}$$

$$2f(\frac{x+y}{2}) - f(x) - f(y) = 0, \tag{1.5}$$

$$2f(\frac{x+y}{2}) - g(x) - h(y) = 0. \tag{1.6}$$

Here all the functions f, g, h in the equations (1.1) $\sim$ (1.6) are regarded as functions from $\mathbb{R}^n$ to $\mathbb{C}$ and all the equations hold for all (x, y) in a subset E of $\mathbb{R}^{2n}$ with $m(E^c) = 0$.

As in the previous papers[2, 4, 5, 6, 7] we reformulate the functional equations (1.1) $\sim$ (1.6) in the space $\mathcal{D}'(\mathbb{R}^n)$ of Schwartz distributions as follows:

$$u \circ A - v \circ P_1 - w \circ P_2 = 0, \tag{1.1$'$}$$

$$u \circ A - v \otimes w = 0, \tag{1.2$'$}$$

$$u \circ A - u \circ P_1 - u \circ P_2 = 0, \tag{1.3$'$}$$

$$u \circ A - u \otimes u = 0, \tag{1.4$'$}$$

$$2u \circ \frac{A}{2} - u \circ P_1 - u \circ P_2 = 0, \tag{1.5$'$}$$

$$2u \circ \frac{A}{2} - v \circ P_1 - w \circ P_2 = 0, \tag{1.6$'$}$$

where $A(x, y) = x + y$, $B(x, y) = x - y$, $P_1(x, y) = x$, $P_2(x, y) = y$, $x, y \in \mathbb{R}^n$, and $u \circ A$, $u \circ B$, $u \circ P_1$ and $u \circ P_2$ are the pullbacks of u in $\mathcal{D}'(\mathbb{R}^n)$ by A, B, P_1 and P_2, respectively, and $\otimes$ denotes the tensor product of generalized functions[9, 10, 11].

As a matter of fact a more general type of functional equation than the above equations have been studied in the space of distribution in [3] for the case $n = 1$. However, in this paper, we follow some different approach from the methods in [3].

As results, we prove that all the solutions u, v and w in $\mathcal{D}'(\mathbb{R}^n)$ of the equations (1.1′), (1.2′), (1.5′) and (1.6′) are linear functions and that of the equations (1.2′) and (1.4′) are exponential functions. Also as simple consequences of the results we obtain that all the measurable solutions f, g and h of the equations (1.1), (1.2), (1.5) and (1.6) are linear functions almost everywhere and that of the equations (1.2), (1.4) are exponential functions almost everywhere, and we obtain the well known fact that if, in particular, the equations (1.1) $\sim$ (1.6) hold for all $x, y \in \mathbb{R}^n$ the solutions are linear functions or exponential functions.

2. Main theorems

We briefly introduce the space $\mathcal{D}'(\mathbb{R}^n)$ of distributions. Here we use the notations, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial x_j}$.

Also we denote by $C_c^\infty(\mathbb{R}^n)$ the set of all infinitely differentiable functions on $\mathbb{R}^n$ with compact supports.

Definition 2.1. A distribution u is a linear form on $C_c^\infty(\mathbb{R}^n)$ such that for every compact set $K \subset \mathbb{R}^n$ there exist constants $C > 0$ and $k \in \mathbb{N}_0$ such that

$$|\langle u, \varphi \rangle| \le C \sum_{|\alpha| \le k} \sup |\partial^\alpha \varphi|$$

for all $\varphi \in C_c^\infty(\mathbb{R}^n)$ with supports contained in K. The set of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^n)$.

We employ the function $\psi(x)$ on $\mathbb{R}^n$,

$$\psi(x) = \begin{cases} A \exp(-(1 - |x|^2)^{-1}), & |x| < 1 \\ 0, & |x| \ge 1, \end{cases}$$

where

$$A = \left(\int_{|x|<1} \exp(-(1 - |x|^2)^{-1}) dx \right)^{-1}.$$

It is easy to see that $\psi(x)$ an infinitely differentiable function with support $\{x : |x| \le 1\}$. Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $\psi_t(x) := t^{-n}\psi(x/t)$, $t > 0$. Then for each $t > 0$, $(u * \psi_t)(x) = \langle u_y, \psi_t(x-y)\rangle$ is a smooth function in $\mathbb{R}^n$ and $(u * \psi_t)(x) \to u$ as $t \to 0^+$ in the sense of distributions, that is, for every $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int (u * \psi_t)(x)\varphi(x)\,dx.$$

We first consider the Pexider equation (1.1′).

THEOREM 2.2. *Every solution u, v, $w \in \mathcal{D}'(\mathbb{R}^n)$ of the Pexider equation* (1.1′) *has the form*

$$\begin{aligned} u &= a \cdot x + c_1 + c_2, \\ v &= a \cdot x + c_1, \\ w &= a \cdot x + c_2. \end{aligned} \tag{1.1''}$$

for some $a \in \mathbb{C}^n$ and $c_1, c_2 \in \mathbb{C}$.

Proof. Convolving $\psi_t(x)\psi_s(y)$ in each side of (1.1′) we have

$$(u * \psi_t * \psi_s)(x+y) - (v * \psi_t)(x) - (w * \psi_s)(y) = 0 \tag{2.1}$$

for x, $y \in \mathbb{R}^n$, $t, s > 0$.

From (2.1) it is easy to see that for each $x \in \mathbb{R}^n$,

$$\begin{aligned} g(x) &:= \limsup_{t\to 0^+}(v * \psi_t)(x), \\ h(x) &:= \limsup_{t\to 0^+}(w * \psi_t)(x) \end{aligned}$$

exist. In (2.1), letting $y = 0$ and $s = s_n \to 0^+$ so that $(w * \psi_{s_n})(0) \to h(0)$ we have

$$(u * \psi_t)(x) - (v * \psi_t)(x) - h(0) = 0 \tag{2.2}$$

for all $x \in \mathbb{R}^n$. Similarly we have

$$(2.3) \qquad (u * \psi_t)(x) - (w * \psi_t)(x) - g(0) = 0$$

for all $x \in \mathbb{R}^n$. From (2.1), (2.2) and (2.3) we have

$$(2.4) \quad (u * \psi_t * \psi_s)(x + y) - (u * \psi_t)(x) - (u * \psi_s)(y) + g(0) + h(0) = 0,$$

for x, $y \in \mathbb{R}^n$, $t, s > 0$. From (2.4) it is easy to see that for each $x \in \mathbb{R}^n$,

$$f(x) := \limsup_{t \to 0^+} (u * \psi_t)(x)$$

exists and $f(0) = g(0) + h(0)$. Letting $y = 0$ in (2.4) we have

$$(2.5) \quad (u * \psi_t * \psi_s)(x) - (u * \psi_t)(x) - (u * \psi_s)(0) + g(0) + h(0) = 0$$

for all $x \in \mathbb{R}^n$. In (2.5), fix x and let $t = t_n \to 0^+$ so that $(u * \psi_{t_n})(x) \to f(x)$ as $n \to \infty$ to get

$$(2.6) \qquad (u * \psi_s)(x) - f(x) - (u * \psi_s)(0) + g(0) + h(0) = 0$$

for all $x \in \mathbb{R}^n$.

From the inequality (2.4), (2.5), (2.6) we have

$$(2.7) \qquad f(x + y) - f(x) - f(y) + f(0) = 0$$

for all $x, y \in \mathbb{R}^n$. Since f is a smooth function in view of (2.6) it follows that $f(x) = a \cdot x + f(0)$. Letting $s = s_n \to 0^+$ in (2.6) so that $(u * \psi_{s_n})(0) \to f(0)$ we have

$$(2.8) \qquad u = a \cdot x + f(0),$$

for some $a \in \mathbb{C}^n$. Consequently we have from (2.2) and (2.3)

$$(2.9) \qquad v = a \cdot x + g(0),$$

$$(2.10) \qquad w = a \cdot x + h(0).$$

This completes the proof. □

As direct consequences of the above result we have the followings.

COROLLARY 2.3. *Every solution $u \in \mathcal{D}'(\mathbb{R}^n)$ of the Cauchy equation (1.3′) has the form*

$$u = a \cdot x\,, \quad a \in \mathbb{C}^n. \tag{1.3''}$$

COROLLARY 2.4. *Every solution $u \in \mathcal{D}'(\mathbb{R}^n)$ of the Jensen equation (1.5′) has the form*

$$u = a \cdot x + c\,, \quad a \in \mathbb{C}^n,\, c \in \mathbb{C}. \tag{1.5''}$$

COROLLARY 2.5. *Every solution $u, v, w \in \mathcal{D}'(\mathbb{R}^n)$ of the Jensen-Pexider equation (1.6′) has the form*

$$\begin{aligned} u &= a \cdot x + c_1 + c_2, \\ v &= a \cdot x + 2c_1, \\ w &= a \cdot x + 2c_2. \end{aligned} \tag{1.6''}$$

for some $a \in \mathbb{C}^n$ and $c_1\,, c_2 \in \mathbb{C}$.

Now we consider the Pexider-exponential equation (1.2′).

THEOREM 2.6. *Every nontrivial solution u, v, $w \in \mathcal{D}'(\mathbb{R}^n)$ of the equation (1.2′) has the form*

$$\begin{aligned} u &= C_1 C_2 \exp(c \cdot x), \\ v &= C_1 \exp(c \cdot x), \\ w &= C_2 \exp(c \cdot x), \end{aligned} \tag{1.2''}$$

where $c \in \mathbb{C}^n$ and C_1, $C_2 \in \mathbb{C}$.

Proof. We consider the nontrivial case that $u \neq 0$, $v \neq 0$ and $w \neq 0$. Convolving $\psi_t(x)\psi_s(y)$ in each side of (1.2′) we have

$$(u * \psi_t * \psi_s)(x + y) - (v * \psi_t)(x)(w * \psi_s)(y) = 0 \tag{2.11}$$

for all $x,\, y \in \mathbb{R}^n$, $t, s > 0$. It follows from (2.11) that both the limits

$$g(x) := \limsup_{t \to 0^+} (v * \psi_t)(x),$$
$$h(y) := \limsup_{s \to 0^+} (w * \psi_s)(y),$$

exist. In (2.31), fix x and let $t = t_n \to 0^+$ so that $(v * \psi_{t_n})(x) \to g(x)$ as $n \to \infty$. Then we have

$$(u * \psi_s)(x + y) - g(x)\,(w * \psi_s)(y) = 0. \tag{2.12}$$

Letting $y = 0$ and $s = s_n \to 0$ so that $(w * \psi_{s_n})(0) \to h(0)$ as $n \to \infty$ in (2.12) we have

$$u - h(0)\,g(x) = 0. \tag{2.13}$$

Now it follows from (2.12) that

$$(w * \psi_s)(0)[G(x + y) - G(x)G(y)] = 0$$

for all $x, y \in \mathbb{R}^n$, $s > 0$, where $G(x) = g(0)^{-1} g(x)$. Since $(w * \psi_s)(0) \neq 0$ for some $s > 0$ we have

$$G(x + y) - G(x)G(y) = 0. \tag{2.14}$$

Since G is a smooth function in view of (2.12) the solution of the exponential equation (2.14) has the form

$$G(x) = \exp(c \cdot x)$$

for some $c \in \mathbb{C}^n$. Thus it follows from (2.13)

$$u = g(0)h(0)\exp(c \cdot x).$$

Consequently, from (2.12) we have

$$w = h(0)\exp(c \cdot x).$$

Changing the roles of v and w we have

$$v = g(0)\exp(c \cdot x).$$

This completes the proof. □

As a direct consequence of the above result we have the following.

COROLLARY 2.7. *Every nontrivial solution* $u \in \mathcal{D}'(\mathbb{R}^n)$ *of the exponential equation* (1.4′) *has the form*

$$(1.4'') \qquad u = \exp(c \cdot x)\,, \quad c \in \mathbb{C}^n.$$

REMARK. Now we return to the classical equations (1.1)∼(1.6). Note that every locally integrable function f can be regarded as a distribution via the equation

$$\langle f, \varphi\rangle = \int f(x)\varphi(x)\,dx, \quad \varphi \in C_c^\infty(\mathbb{R}^n).$$

Also it is easy to see that all the measurable solutions of the equations (1.1)∼(1.6) are locally integrable. Thus, as consequences of the above results all the measurable solutions of the equations (1.1)∼(1.6) are equal to (1.1″)∼(1.6″) almost everywhere, respectively. In particular, if the equations (1.1)∼(1.6) hold for all $x, y \in \mathbb{R}^n$ their solutions have the forms (1.1″)∼(1.6″) exactly. Indeed, for example, let f, g and h be a solution of the equation

$$(2.15) \qquad f(x+y) - g(x) - h(y) = 0, \quad x,\, y \in \mathbb{R}^n.$$

Then $f(x) = a \cdot x + c_1 + c_2, g(x) = a \cdot x + c_1, h(x) = a \cdot x + c_2$ for all x in a set E with $m(E^c) = 0$. Since for every $z \in \mathbb{R}^n$ there exist $x, y \in E$ such that $z = x + y$ we have

$$f(z) = g(x) + h(y) = a \cdot z + c_1 + c_2.$$

It follows from (2.15)

$$g(z) = f(z) - h(0) = a \cdot z + c_1,$$
$$g(z) = f(z) - g(0) = a \cdot z + c_2.$$

The above simple argument works for the other equations.

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Seok Rho
Department of Mathematics
Kunsan National University
Kunsan 573–701, Korea
Email : kn@kunsan.ac.kr