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A CHARACTERIZATION OF MANDELBROT SET OF QUADRATIC RATIONAL MAPS

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Abstract. We present some properties characterizing the Mandelbrot set of quadratic rational maps. Any quadratic rational map is conjugate to either $z^2 + c$ or $\lambda(z + 1/z) + b$. For $|\lambda| = 1$, we find the figure of the Mandelbrot set M_λ, the set of parameters b for which the Julia set of $\lambda(z + 1/z) + b$ is connected. It is seen to be the whole complex plane if $\lambda \neq 1$, but it is intricate fractal if $\lambda = 1$. This supplements the work already investigated for the case $|\lambda| > 1$.

1. Introduction

The study of the complex dynamics began by Julia and Fatou [4] in the early of 20-th century. They made research on the Julia sets of rational maps on the complex sphere and found most of the basic properties on the complex dynamics. After Sullivan [11] solved the old problem on the properties of components of Fatou set in 1982, the complex dynamics has experienced a big progress. For example, Douady and Hubbard [3] obtained essential results on the structure of the Mandelbrot set of z^2+c. Another reason the complex dynamics attracted big popularity comes from the intricate and beautiful computer images of the Mandelbrot set and the Julia sets of $z^2 + c$ [7].

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Any quadratic rational map can be seen to be conjugate to either $z^2 + c$ or $\lambda(z + \frac{1}{z}) + b$. The dynamics of $z^2 + c$ has been extensively studied (see [3, 4]). For the map $\lambda(z + 1/z) + b$ with $|\lambda| > 1$, Goldberg and Keen [5] obtained a criterion of its Julia set to be a Cantor set. He proved that the Julia set is a Cantor set if two critical points ± 1 iterate to infinity, and otherwise it is connected. Yin [12] also proved that if a quadratic rational map has an invariant stable component containing two critical points, then its Julia set is a Cantor set, and otherwise it is connected.

In this paper, some properties of the dynamics of quadratic rational maps for $|\lambda| = 1$ are presented. Unfortunately, we knew that Proposition 3.6 is the same as in [6], later. But our method of proof of the proposition is different from Milnor's. Finally, we give a criterion of the parameter b for which the Julia set of $\lambda(z + 1/z) + b$ with $|\lambda| = 1$ is connected, using iteration of critical points as in Theorem 3.10.

For clarity, in section 2 we first introduce some known results in the dynamics of rational maps from the literature. Finally in section 3, we present our analysis and results on some properties characterizing the Mandelbrot set of the quadratic rational maps.

2. Preliminaries

In this section, we introduce some basic theorems and notations of complex dynamics.

Let $\bar{\mathbb{C}}$ be the extended complex plane, i.e., $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map with degree d greater than one. For $n \in \mathbb{N}$ the n-th iteration of R is written R^n. The set

$$F(R) = \{z : \{R^n\} \text{ is a normal family in some neighborhood of } z\}$$

is called the *Fatou set* of R. The complementary set $J(R) = \bar{\mathbb{C}} - F(R)$ is called the *Julia set*.

Definition. Let $R : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map of degree $d \geq 2$, then z is called a *critical point* of R if $R'(z) = 0$. The number (counting multiplicity) of critical points of a subset E of $\bar{\mathbb{C}}$ is called a *total deficiency* $\delta_R(E)$ of R over E. We define the *connectivity* $c(E)$ of a subset E as the number of components of ∂E.

The following theorem gives the relationship between the total deficiency and the connectivity of a component of $F(R)$, and it will be used in the proof of Proposition 3.5.

Theorem 2.1. *(Riemann-Hurwitz relation, Chap. 5 in [1]) Let F_0 and F_1 be the components of the Fatou set F of a rational map R and suppose that R maps F_0 into F_1. Then, for some integer m, R is an m-fold map of F_0 onto F_1 and*

$$2 - c(F_0) + \delta_R(F_0) = m(2 - c(F_1)).$$

A subset E of $\bar{\mathbb{C}}$ is *forward invariant* under R if $R(E) = E$, *backward invariant* under R if $R^{-1}(E) = E$ and *completely invariant* under R if E is forward and backward invariant under R. The next theorem is useful to find the Julia set of the rational map.

Theorem 2.2. *(see Chap. 5 in [1]) If F_0 is a completely invariant component of $F(R)$, then $\partial F_0 = J(R)$.*

Let $\{\zeta_1, \cdots, \zeta_q\}$ be a cycle of a rational map R. Then the derivative of R^q at each point ζ_i is as follows.

$$(R^q)'(\zeta_i) = \prod_{j=1}^{q} R'(R^j(\zeta_i)) = \prod_{j=1}^{q} R'(\zeta_j).$$

The number is called the *multiplier* of the cycle.

Definition. Let $\{\zeta_1, \cdots, \zeta_q\}$ be a cycle of a rational map R. Then for any $i \in \{1, 2, \cdots, q\}$ this cycle is called by

(1) *attracting* if $|(R^q)'(\zeta_i)| < 1$;
(2) *rationally indifferent* if $(R^q)'(\zeta_i)$ is a root of unity ;
(3) *irrationally indifferent* if $|(R^q)'(\zeta_i)| = 1$, but $(R^q)'(\zeta_i)$ is not a root of unity ;
(4) *repelling* if $|(R^q)'(\zeta_i)| > 1$.

A rationally indifferent cycle is also called a *parabolic cycle.* The cycle with $|(R^q)'(\zeta_i)| \leq 1$ is called by *nonrepelling.* Shishikura [10] proved the following theorem on the number of nonrepelling cycles by using quasiconformal maps.

Theorem 2.3. *A rational map R of degree d has at most $2d-2$ nonrepelling cycles.*

Therefore any quadratic rational map has at most two nonrepelling cycles. The following theorems explain the relations between the multiplier of a cycle and the dynamics of the map near the cycle.

Theorem 2.4. *(see Chap. 6 in [1]). Let $\{\zeta_1, \cdots, \zeta_q\}$ be a cycle of R. If the cycle is attracting, then each ζ_j lies in a component, say F_j, of the Fatou set $F(R)$, and $R^{nq} \to \zeta_j$ locally uniformly on F_j as $n \to \infty$. If the cycle is rationally indifferent or repelling, then the cycle lies in the Julia set $J(R)$.*

For the irrationally indifferent cycle in Fatou set, the following theorem is known.

Theorem 2.5. *(see Chap. 6 in [1]) Let $\{\zeta_1, \cdots, \zeta_q\}$ be an irrationally indifferent cycle which lies in $F(R)$ and let ζ_j lies in a component F_j of $F(R)$. Then F_j is simply connected, and $R^q : F_j \to F_j$ is conjugate to a rotation of infinite order of the unit disc D*

Any component of the Fatou set of this type is called a *Siegel disc* after C.L. Siegel established its existence in 1941 (see [8, 9]). For example, if

$R(z) = \lambda z + z^2$, $\lambda = \exp(i\pi(\sqrt{5}-1))$, then 0 is the irrationally indifferent fixed point of R and the component containing 0 is a Siegel disc (see [7]).

Any analytic map containing a rationally indifferent fixed point at ζ can be seen to be conjugate to

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{p+2}) \tag{2.1}$$

for some integer p, in some neighborhood of the origin. For the dynamics of an analytic map near a rationally indifferent fixed point, we have the following well-known theorems.

Theorem 2.6. *(see Chap. 6 in [1]) Suppose that f is an analytic map satisfying (2.1) and let $w_1, \cdots, w_p$ be the p-th roots of unity and $\eta_1, \cdots, \eta_p$ be the p-th roots of -1. Then for sufficiently small positive numbers r_0 and θ_0, we find that $|f(z)| < |z|$ on each sector*

$$S_i = \{z | 0 < |z/w_i| < r_0, \ |\arg(z/w_i)| < \theta_0\}$$

and $|f(z)| > |z|$ on each sector

$$\Sigma_i = \{z | 0 < |z/\eta_i| < r_0, \ |\arg(z/\eta_i)| < \theta_0\}.$$

For each positive number t, each positive integer p, and each $k = 0, 1, \cdots, p-1$ in we define the sets $\Pi_k(t)$ as

$$\Pi_k(t) = \{re^{i\theta} : r^p < t(1 + \cos(p\theta)) \text{ and } |2k\pi/p - \theta| < \pi/p\}$$

These sets are called *petals* (at the origin). Note that the petals are pairwise disjoint, and that each petal subtends an angle $2\pi/p$ at the origin, so that the total angle subtended at the origin by all the petals is 2π. We call the line of symmetry of $\Pi_k(t)$ (the ray $\theta = 2\pi/p$) by the *axis* of petal $\Pi_k(t)$. The local dynamics at the parabolic fixed point is described in the following important theorem, which will be used to prove Proposition 3.4.

Theorem 2.7. *(The Petal Theorem, see Chap. 6 in [1]) Suppose that an analytic map f has a Taylor expansion*

$$f(z) = z - z^{p+1} + \mathcal{O}(z^{p+2})$$

at the origin. Then for all sufficiently small t and for $k = 1, \cdots, p$:

(1) *f maps each petal $\Pi_k(t)$ into itself ;*
(2) *$f^n \to 0$ uniformly on each petal as $n \to \infty$;*
(3) *$\arg f^n(z) \to 2k\pi/p$ locally uniformly on $\Pi_k(t)$ as $n \to \infty$;*
(4) *$|f(z)| < |z|$ on a neighborhood of the axis of each petal.*

In particular, if f is a rational map, then the Fatou set has components F_k each of which contains Π_k respectively, such that

(5) *$f^n \to 0$ uniformly on each component F_k as $n \to \infty$;*
(6) *$\arg f^n(z) \to 2k\pi/p$ locally uniformly on F_k as $n \to \infty$.*

D. Sullivan (see [7, 11]) solved the old problem of complex dynamics since Fatou and Julia. He showed that every component of the Fatou set $F(R)$ is eventually periodic. It plays an important role in the theory of complex dynamics with the following classification theorem.

Theorem 2.8. *(Sullivan's Classification Theorem [11]) Let R be a rational map and F_0 be a periodic component of the Fatou set of R with period n and let $S = R^n$. Then F_0 must be one of the following four types :*

(1) *F_0 is an attracting component if it contains a periodic point p such that $0 \leq |S'(p)| < 1$.*
(2) *F_0 is a parabolic component if there exists a periodic point p on ∂F_0 whose period divides n and $S^k(z) \to p$ as $k \to \infty$ for all $z \in F_0$.*
(3) *F_0 is a Siegel disc if F_0 is simply connected and $S|_{F_0}$ is conjugate to a rotation.*

(4) F_0 *is a Herman ring if* F_0 *is conformally equivalent to an annulus* $A = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$ *(where* r_1, $r_2 \in \mathbb{R}$, $r_1 \geq 0$, $r_2 > 0$*) and the map* $S\big|_{F_0}$ *is conjugate to a rigid rotation of the annulus.*

The union of those components of $F(R)$ whose closure contains an attracting or a parabolic cycle $\zeta_1, \cdots, \zeta_q$ is called the *immediate basin* of the cycle. Especially, we denote the immediate basin of an attracting fixed point ζ by $A(\zeta)$ which consists of only one attracting component of the Fatou set. For example, if $R(z) = z^2+c$, then ∞ is an attracting fixed point of R and $A(\infty)$ is the component containing ∞ and is completely invariant. By Theorem 2.2, $\partial A(\infty)$ is the Julia set of R. The following three theorems will be used to prove Proposition 3.7.

Theorem 2.9. *(see Chap. 9 in* [1] *) The immediate basin of an attracting cycle or a parabolic cycle of* R *contains a critical point of* R.

We denote the set of critical points of R by $C(R)$, and we use $C^+(R)$ to denote the union of forward of $C(R)$, i.e.,

$$C^+(R) = \cup_{n=0}^{\infty} R^n(C).$$

Theorem 2.10. *(see Theorem 9.3.3 in* [1] *) Let* $\Omega_1, \cdots, \Omega_q$ *be a cycle of Siegel discs or Herman rings of rational map* R. *Then the closure of* $C^+(R)$ *contains* $\cup \partial \Omega_j$.

Theorem 2.11. *(see Theorem 9.3.4 in* [1] *) Every irrationally indifferent cycle of* R *in* J *lies in the derived set of* $C^+(R)$.

Yin(see [12]) proved the following significant theorem.

Theorem 2.12. *If a quadratic rational map* $R(z)$ *has a forward invariant component of* $F(R)$ *containing two critical points, then the*

Julia set $J(R)$ is a Cantor set and R restricted to $J(R)$ is conjugate to the one-sided shift on 2-symbols.

For the other cases, $J(R)$ is connected.

3. Mandelbot set of quadratic rational map

In this section we present some properties of the Mandelbot Set of quadratic rational map. Especially, Theorem 3.10 characterizes the Mandelbrot set of the map $\lambda(z + 1/z) + b$ with $|\lambda| = 1$, i.e., the set of parameter b for which the Julia set is connected.

Lemma 3.1. *Any quadratic rational map is conjugate to one of two forms by a Möbius transformation :*

$$z^2 + c \quad \text{or} \quad \lambda(z + 1/z) + b.$$

Proof. A quadratic rational map R is at least one of the following three types :

(a) zero is a fixed point of R, that is $R(0) = 0$;
(b) non-zero fixed point of R exists in $\mathbb{C}$;
(c) R has the fixed points only at ∞.

In the case (c), using the Möbius transformation $h(z) = 1/z$, we can easily obtain the fact that $h \circ R \circ h^{-1}$ has three fixed points in $\mathbb{C}$. Thus the quadratic rational map of the type (c) is conjugate to a quadratic rational map which is one of the type (b). In the case (b), $(h \circ R \circ h^{-1})(z) = R(z + \alpha) - \alpha$ has a fixed point at zero, where $h(z) = z - \alpha$ with α a nonzero fixed point if R. Hence the quadratic rational map of the type (b) is conjugate to a quadratic rational map which is one of the type (a). Therefore, we may assume that $R(0) = 0$ and R is expressed as

$$\frac{a_2 z^2 + a_1 z}{b_2 z^2 + b_1 z + b_0}.$$

R is then conjugate to

$$\frac{b_2 + b_1 z + b_0 z^2}{a_2 + a_1 z}.$$

by the Möbius transformation $h(z) = 1/z$. If $a_1 = 0$, then R is conjugate to a quadratic polynomial. Any quadratic polynomial is conjgate to $z^2 + c$. If $a_1 \neq 0$, then there exist complex numbers λ_i $(i = 1, 2, 3)$ such that

$$\frac{b_2 + b_1 z + b_0 z^2}{a_2 + a_1 z} = \lambda_1(a_1 z + a_2) + \lambda_2 + \frac{\lambda_3}{a_1 z + a_2}.$$

Now we take $h(z) = a_1 z + a_2$ and let $R_1 = h \circ R \circ h^{-1}$. Then

$$R_1(z) = a_1 R(\frac{z - a_2}{a_1}) + a_2 = a_1 \lambda_1 z + a_1 \lambda_2 + \frac{a_1 \lambda}{z} + a_2.$$

Finally, we take $h(z) = z/r$ where $r^2 = \lambda_3/\lambda_1$ and let $R_2 = h \circ R_1 \circ h^{-1}$, $b = (a_1\lambda_1 + a_2)/r$ and $\lambda = a_1\lambda_1$. Then

$$R_2(z) = a_1 \lambda_1 z + \frac{a_1 \lambda_1}{r} + \frac{a_1 \lambda_3}{r^2 z} + \frac{a_2}{r} = \lambda z + b + \frac{\lambda}{z}.$$

□

We present some properties of Fatou set and Julia set of quadratic rational maps in the following propositions.

Proposition 3.2. *If a component F_0 of $F(R)$ is an attracting or a parabolic forward invariant component, then F_0 is completely invariant.*

Proof. By the conjugation, we may assume that R is either $z^2 + c$ or $\lambda(z+1/z)+b$. First, we let $R = \lambda(z+1/z)+b$. Then since $R(z) = R(1/z)$,

$$z \in J(R) \quad \text{iff} \quad 1/z \in J(R).$$

Let $F_1 = \{1/z : z \in F_0\}$. Then F_1 is also a component of $F(R)$. Since the forward invariant component F_0 contains a critical point 1 or -1, the component F_1 also contains the same critical point whatever F_0 has. Therefore, F_0 and F_1 must be the same component of $F(R)$, and furthermore, if $z \in F_0$ then $1/z \in F_1 = F_0$. To see that F_0 is completely invariant, let $\zeta \in F_0$. If $z_0 \in R^{-1}(\zeta)$, then $1/z_0 \in R^{-1}(\zeta)$. Since F_0 is forward invariant, one of z_0 and $1/z_0$ must be in F_0 and hence both z_0 and $1/z_0$ belong to F_0. Therefore, F_0 is completely invariant.

Now, we let $R(z) = z^2 + c$. Then since $R(z) = R(-z)$,

$$z \in J(R) \quad \text{iff} \quad -z \in J(R).$$

Let $F_1 = \{-z : z \in F_0\}$. Then F_1 is also a component of $F(R)$. Since the forward invariant component F_0 contains a critical point 0 or ∞, the component F_1 also contains the same critical point whatever F_0 has. Therefore, F_0 and F_1 must be the same component of $F(R)$, and furthermore, if $z \in F_0$ then $-z \in F_1 = F_0$. To see that F_0 is completely invariant, we let $\zeta \in F_0$. If $z_0 \in R^{-1}(\zeta)$, then $-z_0 \in R^{-1}(\zeta)$. Since F_0 is forward invariant, one of z_0 and $-z_0$ must be in F_0 and hence both z_0 and $-z_0$ belong to F_0. Therefore, F_0 is completely invariant.

□

Proposition 3.3. *There exists an attracting forward invariant component containing two critical points if and only if both critical points iterate to an attracting fixed point.*

Proof. If a component is attracting forward invariant, by the Sullivan's Classification Theorem, all points in the component iterate to an attracting fixed point.

Conversely, if both critical points iterate to an attracting fixed point ζ, then an attracting forward invariant component containing ζ is completely invariant by Proposition 3.2. Therefore, it contains two critical points. □

Proposition 3.4. *There exists a parabolic forward invariant component of R containing two critical points if and only if both critical points in $F(R)$ iterate to a parabolic fixed point ζ with $R'(\zeta) = 1$ and $R''(\zeta) \neq 0$.*

Proof. If a parabolic forward invariant component exists, then by Sullivan's Classification Theorem there exists a parabolic fixed point ζ such that all points in the component iterate to ζ and $R'(\zeta) = 1$. Suppose $R''(\zeta) = 0$. Then R has at least two petals near by the Petal Theorem. Each petal is contained in a distinct forward invariant component and each component contains at least one critical point. Since one of the components contains two critical points, the union of all components contains at least three critical points, which is impossible for the quadratic rational map.

Conversely, let both critical points in $F(R)$ iterate to a parabolic fixed point ζ with $R'(\zeta) = 1$ and $R''(\zeta) \neq 0$. Since R is analytic near ζ and $R'(\zeta) = 1$, we have

$$\begin{aligned} R(z) &= \zeta + R'(\zeta)(z-\zeta) + \frac{R'(\zeta)}{2}(z-\zeta)^2 + \cdots \\ &= z + \frac{R'(\zeta)}{2}(z-\zeta)^2 + \cdots . \end{aligned}$$

Since $R''(\zeta) \neq 0$, R has only one petal by the Petal Theorem and a component containing the petal is forward invariant. From Theorem 2.6 , we know that both critical points iterating to ζ eventually lie in the parabolic component. It follows from Proposition 3.2 that the component is completely invariant and both critical points lie in the component. □

Proposition 3.5. *If a forward invariant component of $F(R)$ contains two critical points then it is an attracting component or a parabolic component.*

Proof. Let F_0 be a forward invariant component of $F(R)$ which contains two critical points. By the Riemann-Hurwitz relation, we have

$$2 - c(F_0) + \delta_R(F_0) = m(2 - c(F_0)).$$

Since $\delta_R(F_0) = 2$ and $m = 1$ or 2 , $c(F_0) = \infty$ or 0 . If $c(F_0) = 0$ then F_0 equals to the complex sphere, which contradicts to the fact that $J(R)$ is non-empty. Hence $c(F_0)$ must be ∞. Therefore, F_0 is neither a Siegel disc nor a Herman ring. By the Sullivan's Classification Theorem, F_0 is an attracting or a parabolic component. □

Proposition 3.6. *The Julia set of a quadratic map R is a Cantor set if and only if both critical points iterate to*

(1) *an attracting fixed point, or*
(2) *a parabolic fixed point ζ with $R'(\zeta) = 1$ and $R''(\zeta) \neq 0$ in $F(R)$.*

Proof. By Theorem 2.12, $J(R)$ is a Cantor set if and only if there exists a forward invariant component containing both critical points. By Proposition 3.4, this is equivalent to the existence of an attracting or a parabolic forward invariant component containing both critical points. Hence the assertion is obtained by Propositions 3.3 - 3.4. □

Proposition 3.7. *If there exist two non-repelling cycles of R, then $J(R)$ is connected.*

Proof. Suppose that $J(R)$ is a Cantor set. By Theorem 3.5, one of the two cycles is an attracting or a parabolic fixed point, say ζ, to which both critical points in $F(R)$ iterate. Let $\{\xi_1, \cdots, \xi_q\}$ be the other cycle of R for some integer q. Then it is one of the four types :

(1) attracting cycle ;
(2) parabolic cycle ;
(3) irrationally indifferent cycle in $J(R)$;
(4) irrationally indifferent cycle in $F(R)$.

In the case (1) or (2), by Theorem 2.9, there exists a critical point and the derived set of its orbit is $\cup\{\xi_i\}$. This contradicts to the fact that both critical points iterate to ζ. In the case (3), ξ_i lie in the derived set of $C^+(R)$ by Theorem 2.11, which is a contradiction since the derived set of $C^+(R)$ is $\{\zeta\}$. In the case (4), ξ_i are the center of Siegel discs, which is impossible since $J(R)$ is totally disconnected. □

Proposition 3.8. *If there exists an indifferent cycle $\omega_1, \cdots, \omega_q$ of R such that ω_i is not a parabolic fixed point with $R'(\omega_1) = 1$ and $R''(\omega_1) \neq 0$, then $J(R)$ is connected.*

Proof. Suppose that $J(R)$ is a Cantor set. By Theorem 3.1, there exists an attracting or a parabolic fixed point ζ, which cannot be ω_i. It follows from Proposition 3.7 that $J(R)$ is connected. □

Corollary 3.9. *If an attracting cycle of R with period exists or if non-repelling cycle of R does not exist, then $J(R)$ is connected.*

Let $R_{\lambda,b}(z)$ be a quadratic rational map $\lambda(z + 1/z) + b$ with $\lambda \in \mathbb{R}$, $b \in \mathbb{C}$. We define the Mandelbrot set M_λ, for each λ, of the quadratic rational maps $R_{\lambda,b}$ as a set of parameter b for which the Julia set $J(R_{\lambda,b})$ is connected. By Propositions 3.6 and 3.8, one can easily characterize the Mandelbrot set M_λ, for $|\lambda| = 1$, as in the following theorem.

Theorem 3.10. *If $|\lambda| = 1$ and $\lambda \neq 1$, then M_λ is the whole complex plane. If $\lambda = 1$, then*

$$M_1 = \mathbb{C} - \{b \neq 0 : \lim_{n\to\infty} R^n_{1,b}(\pm 1) = \infty \text{ and } R^n_{1,b}(\pm 1) \neq 0 \text{ for any } n\}.$$

Proof. Let $R_{\lambda,b}(z)$ be conjugate to a map $f(z)$ by a Moebius transformation $h(z) = 1/z$. Then we have $f(z) = z/(\lambda(z^2+1) + bz)$ and it is clear (see P. 261, [2]) that

$$R'_{\lambda,b}(\infty) = (h \cdot R \cdot h^{-1})'(0) = f'(0) = \frac{1}{\lambda}.$$

When $|\lambda| = 1$ and $\lambda \neq 1$, ∞ is an indifferent fixed point of $R_{\lambda,b}$ with $R'_{\lambda,b}(\infty) \neq 1$. By Proposition 3.8, $J(R_{\lambda,b})$ is connected for any b. Hence we get

$$M_\lambda = \mathbb{C} \text{ for } |\lambda| = 1 \text{ and } \lambda \neq 1.$$

When $\lambda = 1$, ∞ is a parabolic fixed point of $R_{1,b}$ with $R'_{1,b}(\infty) = 1$. If $b = 0$, then $R''_{1,b}(\infty) = f''(0) = 0$ and the Julia set $J(R_{1,0})$ is connected by Proposition 3.8. Hence $b = 0 \in M_1$, If $b \neq 0$, then $R''_{1,b}(\infty) = f''(0) = -2b \neq 0$. By Theorem 3.6, $b \notin M_1$ if and only if both critical points ± 1 in $F(R_{1,b})$ iterate to ∞. Note that if one of the critical orbits lands on 0, then the critical point belongs to the Julia set and $J(R_{1,b})$ is connected. Therefore, $0 \in M_1$, and if $b \neq 0$ then $b \notin M_1$ if and only if

$$\lim_{n\to\infty} R^n_{1,b}(\pm 1) = \infty \text{ as } n \to \infty \text{ and } R^n_{1,b}(\pm 1) \neq 0 \text{ for any } n.$$

□

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