KYUNGPOOK Math. J. 45(2005), 13-19

Some Applications of Generalized Closed Sets in Fuzzy Topological Spaces

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Abstract. In this paper we introduce the notions of g-regularity and g-normality in fuzzy topological spaces. We obtain some characterizations and several preservation theorems of such spaces.

1. Introduction

Levine ([3]) introduced the concept of generalized closed sets in toplogical spaces and a class of topological spaces called $T_{\frac{1}{2}}$-spaces. In 1997, Balasubramanian and Sundaram ([1]) introduced the concept of generalized closed sets in fuzzy topological spaces. In this paper, we introduce the notions of fuzzy generalized regular spaces and fuzzy generalized normal spaces. We obtain some characterizations and several preservation theorems of such spaces.

2. Preliminaries

Throughout of the present paper, (X, τ) and (Y, Δ) (or simply X, Y) always mean fuzzy topological spaces on which no separation axioms are assumed unless explicitly stated.

Let X be a set of points and I be the unit interval [0,1]. A fuzzy set μ in X is a mapping from X into I. The class of all fuzzy sets on X is denoted by I^X. For $x \in X$ and $\alpha \in (0,1]$, a fuzzy set x_α is called a fuzzy point in X iff

$$x_\alpha(y) = \begin{cases} \alpha & : \quad y = x \\ 0 & : \quad y \neq x. \end{cases}$$

The class of all fuzzy points of X is denoted by $FP(X)$.

Let μ be a fuzzy subset of a fuzzy topological space (fts, for short) X, the closure, the interior and the complement of μ are denoted by $\text{cl}(\mu)$, $\text{int}(\mu)$ and $1-\mu$, respectively. For $\mu, \lambda \in I^X$, μ is called quasi–coincident with λ, denoted by $\mu q \lambda$, if $\mu(x) + \lambda(x) > 1$ for some $x \in X$, otherwise we write $\mu \bar{q} \lambda$. A fuzzy subset μ is

Received March 3, 2003.
2000 Mathematics Subject Classification: 54A40, 54A99, 54D10.
Key words and phrases: g-closed fuzzy sets, fuzzy g-continuous maps, fuzzy g-regular and g-normal spaces.

said to be regular open (resp. regular closed) if $\mu = \text{int}(\text{cl}(\mu))$ (resp. $\mu = \text{cl}(\text{int}(\mu))$). The family of all regular open (resp. all regular closed) fuzzy subsets of a fts X is denoted by $RO(X)$ (resp. $RC(X)$).

Lemma 2.1 ([2]). *Let (X, τ) be a fts, $x_t \in FP(X)$ and $\mu \in I^X$. Then:*

(i) *$x_t q$ $\text{cl}(\mu)$ iff $Uq\mu$ for each open fuzzy set U containing x_t.*

(ii) *If $\mu \bar{q} U$, then $\text{cl}(\mu)\bar{q}U$ for each $U \in \tau$.*

Definition 2.2 ([1]). A fuzzy subset μ of a fts (X, τ) is said to be

(i) generalized closed (briefly, g-closed) if $\text{cl}(\mu) \leq \lambda$ whenever $\mu \leq \lambda$ and λ is open fuzzy set.

(ii) generalized open (briefly, g-open) if $1 - \mu$ is g-closed.

Definition 2.3 ([1]). A fts (X, τ) is said to be $FT_{\frac{1}{2}}$-space iff every g-closed fuzzy set in X is closed fuzzy set.

Definition 2.4 ([2]). A fts (X, τ) is said to be

(i) FT_1-space iff x_t is closed fuzzy set for every fuzzy point $x_t \in FP(X)$.

(ii) FT_2-space iff $x_t \bar{q} y_r$ implies there exist open fuzzy sets U and V such that $x_t \in U$, $y_r \in V$ and $U\bar{q}V$.

(iii) $FT_{2\frac{1}{2}}$-space iff $x_t \bar{q} y_r$ implies there exist open fuzzy sets U and V such that $x_t \in U$, $y_r \in V$ and $\text{cl}(U)\bar{q}\text{cl}(V)$.

(iv) FR_2–space (or F–regular) iff $x_t \bar{q} F$, F is closed fuzzy set implies that there exist open fuzzy sets U and V such that $x_t \in U$, $F \leq V$ and $U\bar{q}V$.

(v) FR_3-space (or F-normal) iff for each closed fuzzy sets F_1, F_2 with $F_1 \bar{q} F_2$, there exist open fuzzy sets U and V such that $F_1 \leq U$, $F_2 \leq V$ and $U\bar{q}V$.

(vi) FT_3-space iff it is FR_2 and FT_1.

(vii) FT_4-space iff it is FR_2 and FT_1.

Definition 2.5 ([1]). A map $f : (X, \tau) \to (Y, \Delta)$ is called:

(i) Fg-continuous if the inverse image of every closed fuzzy set in Y is g-closed fuzzy set in X.

(ii) Fgc-irresolute if the inverse image of every g-closed fuzzy set in Y is g-closed fuzzy set in X.

Evidently, the Fgc-irresolute map is Fg-continuous.

3. Fg-regular spaces

Definition 3.1. A fts (X,τ) is said to be Fg-regular (or FGR_2) if for every g-closed fuzzy set F and a fuzzy point x_t such that $x_t\bar{q}F$, there exist open fuzzy sets U and V such that $F \leq U$, $x_t \in V$ and $U\bar{q}V$.

Evidently, every FGR_2-space is FR_2. The following example shows that the converse is not true in general.

Example 3.2. Let $X = \{x, y\}$ and $\tau = \{1_X, 0_X, x_{0.2} \vee y_{0.5}, x_{0.8} \vee y_{0.5}\}$. Then (X,τ) is a fts. It is easy to verify that (X,τ) is FR_2 but not FGR_2.

Theorem 3.3. *A fts (X,τ) is FGR_2 iff it is FR_2 and $FT_{\frac{1}{2}}$.*

Proof. Obvious. □

Theorem 3.4. *Let (X,τ) be a fts. Then the following statements are equivalent:*

(i) *(X,τ) is FGR_2-space.*

(ii) *For each fuzzy point x_t in X and each g-open fuzzy set U containing x_t there exists an open fuzzy set V containing x_t such that $\mathrm{cl}(V) \leq U$.*

Proof. Easy. □

Theorem 3.5. *A fts (X,τ) is FGR_2 iff for each g-closed fuzzy set F in X and each fuzzy point x_t with $x_t\bar{q}F$, there exist open fuzzy sets U and V such that $x_t \in U$, $F \leq V$ and $\mathrm{cl}(U)\bar{q}\mathrm{cl}(V)$.*

Proof. "Necessity." Let F be a g-closed fuzzy set in X and $x_t\bar{q}F$. There exist open fuzzy sets W and V in X such that $x_t \in W, F \leq V$ and $W\bar{q}V$; hence $W\bar{q}\mathrm{cl}(V)$. Again, since X is FGR_2, there exist open fuzzy sets G and H of X such that $x_t \in G$, $\mathrm{cl}(V) \leq H$ and $G\bar{q}H$; hence $\mathrm{cl}(G)\bar{q}H$. Now, put $U = W \cap G$, then U and V are open fuzzy sets of X such that $x_t \in U, F \leq V$ and $\mathrm{cl}(U)\bar{q}\mathrm{cl}(V)$.

"Sufficiency." This is obvious. □

Definition 3.6. A fts (X,τ) is said to be F-symmetric iff $x_t\bar{q}\mathrm{cl}(y_r)$ implies that $y_r\bar{q}\mathrm{cl}(x_t)$ for any fuzzy points $x_t, y_r \in FP(X)$.

Theorem 3.7. *A fts (X,τ) is F-symmetric iff $\mathrm{cl}(x_t)\bar{q}F$ whenever $x_t\bar{q}F$ for any closed fuzzy set F in X.*

Proof. "Necessity." Suppose that F is a closed fuzzy set in X with $x_t\bar{q}F$. Then $\mathrm{cl}(y_r) \leq F$ for all $y_r \in F$ and hence $x_t\bar{q}\mathrm{cl}(y_r)$. Since X is F-symmetric, then $y_r\bar{q}\mathrm{cl}(x_t)$ for all $y_r \in F$ and hence for all $y_r \in F$ there exists an open fuzzy set U_{y_r} in X containing y_r such that $x_t\bar{q}U_{y_r}$. Let $V = \bigcup_{y_r \in F} \{U_{y_r} : x_t\bar{q}U_{y_r}\}$. Then V is an open fuzzy set in X containing F and $x_t\bar{q}V$. Therefore, $x_t \in 1 - V$ and hence $\mathrm{cl}(x_t) \leq 1 - V$. It follows that $\mathrm{cl}(x_t)\bar{q}V$ and hence $\mathrm{cl}(x_t)\bar{q}F$.

"Sufficiency." This is obvious. □

Corollary 3.8. *A fts (X,τ) is F-symmetric iff x_t is g-closed for each fuzzy point x_t in X.*

Evidently, every FT_1-space is F-symmetric space. The following example shows that the converse is not true in general.

Example 3.9. Let $X = \{x\}$ and $\tau = \{1_X, 0_X, x_{0.5}\}$. Then (X,τ) is F-symmetric space but not FT_1-space. Moreover, the fts (X,τ) is not $FT_{\frac{1}{2}}$.

Definition 3.10. An FGR_2 which is F-symmetric space, is called FG_3-space

Theorem 3.11. *If a fts (X,τ) is FG_2-space, then it is $FT_{2\frac{1}{2}}$.*

Proof. Let x_t and y_r be any fuzzy points in X such that $x_t\bar{q}y_r$. Since X is F-symmetric, then x_t is g-closed fuzzy set and by Theorem 3.5 there exist open fuzzy sets U and V such that $x_t \in U, y_r \in V$ and $\text{cl}(U)\bar{q}\text{cl}(V)$. □

Corollary 3.12. *If a fts (X,τ) is FG_3-space, then it is FT_2.*

The following example shows that the converse of Corollary 3.12 is not true in general.

Example 3.13. Let X be an infinite set. For $x, y \in X$, $x \neq y$, let $U_{x,y}$ be a fuzzy set in X defined by :

$$U_{x,y}(z) = \begin{cases} 1 & : \quad z = x \\ 0 & : \quad z = y \\ \frac{1}{2} & : \quad z \neq x, z \neq y \end{cases}$$

for each $z \in X$. Now, consider the fuzzy topology τ on X generated by the family $\{U_{x,y} : x, y \in X, x \neq y\}$. Then a fts (X,τ) is FT_2 but not FGR_2 and hence not FG_3.

Theorem 3.14. *A fts (X,τ) is FG_3 iff it is FT_3.*

Proof. Let X be an FG_3-space. Therefore it is FGR_2-space and F-symmetric. Now, every FGR_2-space is FR_2 and every FG_3-space is FT_2. Hence X is FR_2 and FT_2. Hence X is FT_3. Conversely, let X be an FT_3-space. Therefore it is FR_2 and FT_1. Then it is $FT_{\frac{1}{2}}$ and F-symmetric. Therefore X is FR_2 and $FT_{\frac{1}{2}}$ which implies that X is FGR_2. As it is F-symmetric too, it is FG_3. □

4. Fg-normal spaces

Definition 4.1. A fts (X,τ) is said to be Fg-normal (or FGR_3) if for every g-closed fuzzy sets F_1 and F_2 such that $F_1\bar{q}F_2$, there exist open fuzzy sets U and V such that $F_1 \leq U$, $F_2 \leq V$ and $U\bar{q}V$.

Evidently, every FGR_3-space is FR_3. Also, a fts is FGR_3 iff it is FR_3 and $FT_{\frac{1}{2}}$.

Theorem 4.2. *Let (X,τ) be a fts. Then the following statements are equivalent:*

(i) (X, τ) *is* FGR_3*-space.*

(ii) *For each* g*-closed fuzzy set* F *and each* g*-open fuzzy set* U *containing* F*, there exists an open fuzzy set* V *such that* $F \leq V \leq \mathrm{cl}(V) \leq U$*.*

Proof. Easy. □

Theorem 4.3. *A fts* (X, τ) *is* FGR_3 *iff for every* g*-closed fuzzy sets* F_1 *and* F_2 *such that* $F_1 \bar{q} F_2$*, there exist open fuzzy sets* U *and* V *such that* $F_1 \leq U$*,* $F_2 \leq V$ *and* $\mathrm{cl}(U) \bar{q} \mathrm{cl}(V)$*.*

Proof. "Necessity." Let F_1 and F_2 be any g-closed fuzzy sets in X with $F_1 \bar{q} F_2$. There exist open fuzzy sets W and V in X such that $F_1 \leq W$, $F_2 \leq V$ and $W \bar{q} V$; hence $W \bar{q} \mathrm{cl}(V)$. Since X is FGR_3, there exist open fuzzy sets G and H in X such that $F_1 \leq G$, $\mathrm{cl}(V) \leq H$ and $G \bar{q} H$; hence $\mathrm{cl}(G) \bar{q} H$. Now, put $U = W \cap G$, then U and V are open fuzzy sets in X such that $F_1 \leq U$, $F_2 \leq V$ and $\mathrm{cl}(U) \bar{q} \mathrm{cl}(V)$.

"Sufficiency." This is obvious. □

Definition 4.4. An FGR_3 and F-symmetric fts is called an FG_4-space.

Theorem 4.5. *Every* FG_4*-space is also* FG_3*-space.*

Proof. Let (X, τ) be an FGR_3 and F-symmetric space. Let F be g-closed fuzzy set in X and $x_t \bar{q} F$. Then x_t is g-closed fuzzy set, since X is F-symmetric. Then there exist two open fuzzy sets U and V such that $F \leq U$, $x_t \in V$ and $U \bar{q} V$, since X is FGR_3. Then X is FGR_2 and hence X is FG_3. □

Theorem 4.6. *A fts* (X, τ) *is* FG_4 *iff it is* FT_4*.*

Proof. Easy. □

5. Some applications

We shall investigate some preservation theorems of FGR_2 and FGR_3-spaces. For this purpose, we introduce some definitions of mappings used in the sequel.

Definition 5.1. A map $f : X \to Y$ is said to be

(a) Fg-closed if $f(F)$ is g-closed in Y for every closed fuzzy set F in X.

(b) F-almost open if $f(U)$ is open in Y for every $U \in RO(X)$.

(c) F-almost closed if $f(F)$ is closed in Y for every $F \in RC(X)$.

Evidently every F-open (F-closed) map is F-almost open (F-almost closed) map.

Lemma 5.2. *If* $f : X \to Y$ *is an* F*-open,* Fg*-continuous bijection map, then* f *is* Fgc*-irresolute.*

Proof. Let F be any g-closed fuzzy set in Y and $f^{-1}(F) \leq U$, where U is an open fuzzy set in X. Then $F \leq f(U)$. Since f is F-open, then $f(U)$ is open fuzzy set in Y. Since F is g-closed fuzzy set in Y, then $\mathrm{cl}(F) \leq f(U)$. Hence $f^{-1}(\mathrm{cl}(F)) \leq U$

(f is injective). Since f is Fg-continuous, then $f^{-1}(\mathrm{cl}(F))$ is g-closed fuzzy set in X and hence $\mathrm{cl}(f^{-1}(F)) \leq \mathrm{cl}(f^{-1}(\mathrm{cl}(F))) \leq U$. Thus $f^{-1}(F)$ is g-closed fuzzy set in X. □

Theorem 5.3. *If $f : X \to Y$ is an F-open, Fg-continuous bijection map and X is FGR_2, then Y is FGR_2.*

Proof. Let F be any g-closed fuzzy set in Y and $y_r \bar{q} F$. Since f is F-open, Fg-continuous bijective, then by Lemma 5.2, f is Fgc-irresolute and hence $f^{-1}(F)$ is g-closed. Put $f(x_r) = y_r$, then $x_r \bar{q} f^{-1}(F)$. Since X is FGR_2, then there exist F-open fuzzy sets U and V such that $x_r \in U$, $f^{-1}(F) \leq V$ and $U \bar{q} V$. Since f is F-open and bijective, we obtain $y_r \in f(U)$, $F \leq f(V)$ and $f(U) \bar{q} f(V)$. This shows that Y is FGR_2. □

Theorem 5.4. *If $f : X \to Y$ is an F-continuous, Fg-closed injection and Y is FGR_2, then X is FGR_2.*

Proof. Let F be any g-closed fuzzy set in X and $x_t \bar{q} F$. Let us note that F-continuity and Fg-closedness imply that $f(F)$ is g-closed in Y. Indeed, if $f(F) \leq U$ and U is open fuzzy set in Y, then $F \leq f^{-1}(U))$, and hence $\mathrm{cl}(F) \leq f^{-1}(U)$. Then $f(F) \leq f(\mathrm{cl}(F)) \leq f f^{-1}(U) \leq U$. Hence $\mathrm{cl}(f(F)) \leq U$. Thus $f(F)$ is g-closed. Since f is injective, then $f(x_t) \bar{q} f(F)$. Since Y is FGR_2, then there exist open fuzzy sets U and V such that $f(x_t) \in U$, $f(F) \leq V$ and $U \bar{q} V$. Thus, we obtain $x_t \in f^{-1}(U)$, $F \leq f^{-1}(V)$ and $f^{-1}(U) \bar{q} f^{-1}(V)$. Since f is F-continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are open fuzzy sets in X. Thus X is FGR_2. □

Theorem 5.5. *If $f : X \to Y$ is an F-almost open, Fgc-irresolute, F-almost closed surjection and X is FGR_2-space, then Y is FGR_2.*

Proof. Let V be any $g$open fuzzy set in Y and $y_r \in V$. Take a fuzzy point $x_t \in f^{-1}(y_r)$, then we have $x_t \in f^{-1}(V)$ and $f^{-1}(V)$ is g-open fuzzy set in X. Since X is FGR_2, then by Theorem 3.4, there exists an open fuzzy set U in X such that $x_t \in U \leq \mathrm{int}(\mathrm{cl}(U)) \leq \mathrm{cl}(U) \leq f^{-1}(V)$. Then we have $y_r \in f(U) \leq f(\mathrm{int}(\mathrm{cl}(U))) \leq f(\mathrm{cl}(U)) \leq V$. Since f is F-almost open, F-closed map, then $f(\mathrm{int}(\mathrm{cl}(U))$ is open fuzzy set in Y and $f(\mathrm{cl}(U))$ is closed fuzzy set in Y. Therefore, we obtain, $y_r \in f(\mathrm{int}(\mathrm{cl}(U)) \leq \mathrm{cl}(f(\mathrm{int}(\mathrm{cl}(U))) \leq f(\mathrm{cl}(U)) \leq V$. Thus by Theorem 3.4, Y is FGR_2. □

Theorem 5.6. *If $f : X \to Y$ is an F-open, Fg-continuous bijection and X is FGR_3, then Y is FGR_3.*

Proof. Let F_1 and F_2 be any g-closed fuzzy sets in Y such that $F_1 \bar{q} F_2$. By Lemma 5.2, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are g-closed fuzzy sets in X and $f^{-1}(F_1) \bar{q} f^{-1}(F_2)$. Since X is FGR_3, then there exist open fuzzy sets U and V such that $f^{-1}(F_1) \leq U$ and $f^{-1}(F_2) \leq V$ and $U \bar{q} V$. Since f is F-open and bijective, we obtain $F_1 \leq f(U)$, $F_2 \leq f(V)$ and $f(U) \bar{q} f(V)$ and also $f(U)$ and $f(V)$ are open fuzzy sets in Y. This shows that Y is FGR_3. □

Theorem 5.7. *If $f : X \to Y$ is an Fcontinuous, Fg-closed injection and Y is FGR_3, then X is FGR_3.*

Proof. Let F_1 and F_2 be any g-closed fuzzy sets in X with $F_1 \bar{q} F_2$. As in Theorem 5.4, $f(F_1)$ and $f(F_2)$ are g-fuzzy sets in Y. Since f is injective, then $f(F_1)\bar{q}f(F_2)$. Since X is FGR_3, then there exist open fuzzy sets U and V such that $f(F_1) \leq U, f(F_2) \leq V$ and $U\bar{q}V$. Thus, we obtain $F_1 \leq f^{-1}(U), F_2 \leq f^{-1}(V)$ and $f^{-1}(U)\bar{q}f^{-1}(V)$. Since f is F-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open fuzzy sets in X. This complete the proof that X is FGR_3. □

Theorem 5.8. *If $f : X \to Y$ is an Fgc-irresolute, F-open surjection and X is FGR_3, then Y is FGR_3.*

Proof. Let F_1 and F_2 be g-closed fuzzy sets in Y with $F_1\bar{q}F_2$. Then $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are g-closed fuzzy sets in X and $f^{-1}(F_1)\bar{q}f^{-1}(F_2)$. Since X is FGR_3, then there exist open fuzzy sets U and V such that $f^{-1}(F_1) \leq U, f^{-1}(F_2) \leq V$ and $U\bar{q}V$. Then $F_1 \leq f(U), F_2 \leq f(V)$ and $f(U)\bar{q}f(V)$. Since f is F-open, $f(U)$ and $f(V)$ are open fuzzy sets in Y. This complete the proof that Y is FGR_3. □

Using Theorems 5.5 and 5.6 one can easily prove the following theorem.

Theorem 5.9. *The property of being FGR_2 (FGR_3) is a fuzzy topological property.*

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