KYUNGPOOK Math. J. 45(2005), 1-11

A Note on Multidimensional Fractional Calculus Operators Involving Gauss Hypergeometric Functions

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Abstract. Making use of the multidimensional fractional calculus operators introduced in [20], this paper gives the images under these operators of the celebrated Fox's H-function. Special cases are briefly point out, and some of the results are also studied on general spaces of functions $M_\lambda(R_+^n)$.

1. Introduction and preliminaries

The field of fractional calculus is presently receiving keen attention by researchers. The recent monographs on the subject by Kiryakova [5], Miller and Ross [7] and Samko, Kilbas and Marichev [16] give fairly good account of the developments in fractional calculus, and consider several aspects of applications to potential problems in analysis. Most of the investigations carried out are for one-dimensional case. However, some work relating to two dimensional (and multidimensional) cases have also been considered, and some results happen to be repeated applications of the one-dimensional case (see [4], [9], [12], [13], [16] and [17]). Our motive in the present investigation is to obtain new classes of formulas giving the images under the multidimensional fractional calculus operators involving the Gauss hypergeometric function (introduced by Tuan, Raina and Saigo [20]) for the celebrated H-functions of Fox [3]. The H-function which is usually defined in terms of Mellin-Barnes type contour integral involving the quotients of gamma functions and, as already evidenced in the literature, such Mellin-Barnes contour integrals are useful in several areas of applied mathematics and statistics [6] and [17]. Some of the results are also studied on general spaces of functions $M_\lambda(R_+^n)$.
Special cases giving known and new results are also briefly mentioned.

Received December 20, 2002.
2000 Mathematics Subject Classification: 26A33, 33C40.
Key words and phrases: multidimensional operators, H-functions, Mellin transform, fractional calculus operators.

Throughout the paper, we follow the notations and conventions of [1]. Thus, R denotes the field of real numbers and C the field of complex numbers. R^n represents the set of n-tuple real numbers, with R^n_+ denoting the set of non-negative real numbers, and C^n the set of complex numbers. We reserve $a,\ b,\ c, \cdots,\ x, \cdots$ in most cases for elements of C^n, viz. $x = (x_1, \cdots, x_n)$, $a = (a_1, \cdots, a_n)$ etc. We also set $x^a = x^{a_1} \cdots x^{a_n}$ and $x.1 = x_1 + \cdots + x_n$.

By $\phi_+(x)$ we mean the positive part of a function ϕ defined by

$$\phi_+(x) = \begin{cases} \phi(x) & \phi(x) > 0, \\ 0 & \phi(x) \leq 0. \end{cases} \tag{1.1}$$

For $\mathrm{Re}(\alpha) > 0$, the multidimensional modified fractional integral of order $\alpha \in C$ of $f : R^n_+ \to C$ are defined as follows:

$$\begin{aligned} & S^{\alpha,\beta,\eta}_{+;n} f(x) \\ = & \frac{1}{\Gamma(\alpha+1)} D^1 \int_{R^n_+} \left[\min\left\{ \frac{x_1}{t_1}, \cdots, \frac{x_n}{t_n} \right\} - 1 \right]^{\alpha}_{+} \\ & \times {}_2F_1 \left(\alpha+\beta, \alpha+\eta; 1+\alpha; 1 - \min\left\{ \frac{x_1}{t_1}, \cdots, \frac{x_n}{t_n} \right\} \right) f(t) dt, \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} & S^{\alpha,\beta,\eta}_{-;n} f(x) \\ = & \frac{1}{\Gamma(\alpha+1)} D^1 \int_{R^n_+} \left[1 - \max\left\{ \frac{x_1}{t_1}, \cdots, \frac{x_n}{t_n} \right\} \right]^{\alpha}_{+} \\ & \times {}_2F_1 \left(\alpha+\beta, -\eta; 1+\alpha; 1 - \max\left\{ \frac{x_1}{t_1}, \cdots, \frac{x_n}{t_n} \right\} \right) f(t) dt. \end{aligned} \tag{1.3}$$

Definitions (1.2) and (1.3) were introduced in [20]. By suitable subdivision of the region R^n_+ ([1]) for fixed $x \in R^n_+$, the multidimensional operators (1.2) and (1.3) can be expressed in terms of finite sums of single integrals (see [19, p.255]). It may be noticed that for $n = 1$, we have the following connections:

$$S^{\alpha,\beta,\eta}_{+;1} x^{-\beta} f(x) = I^{\alpha,\beta,\eta}_{0,x} f(x), \tag{1.4}$$

and

$$S^{\alpha,\beta,\eta}_{-;1} x^{-\beta} f(x) = J^{\alpha,\beta,\eta}_{0,x} f(x), \tag{1.5}$$

where $I^{\alpha,\beta,\eta}_{0,x}$ and $J^{\alpha,\beta,\eta}_{0,x}$ are well known generalized fractional calculus operators involving the Gauss hypergeometric function due to Saigo. Also, the operators (1.2) and (1.3) contain the multidimensional modified fractional integral operators studied by Tuan and Saigo [19] (see also [1]).

A widely used class of special functions, (known as the H-function in literature) is defined as follows ([3, p.408]; see also [8, Sec-8.3] and [17, Chapter 2]):

$$H_{P,Q}^{M,N}[z] = H_{P,Q}^{M,N}\left[z\left|^{(a_j,A_j)_{1,P}}_{(b_j,B_j)_{1,Q}}\right.\right] = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\theta(\sigma)z^{-\sigma}d\sigma, \tag{1.6}$$

where

$$\theta(\sigma) = \frac{\prod_{j=1}^{M}(b_j+\sigma B_j)\prod_{j=1}^{M}(1-a_j-\sigma A_j)}{\prod_{j=M+1}^{Q}(1-b_j-\sigma B_j)\prod_{j=N+1}^{p}(b_j+\sigma A_j)}, \tag{1.7}$$

with

$$\Delta = \sum_{1}^{N}A_j - \sum_{N+1}^{P}A_j + \sum_{1}^{M}B_j - \sum_{M+1}^{Q}B_j > 0, \quad \text{or} \tag{1.8}$$

$$\Delta = 0, \quad \mathrm{Re}\left(\sum_{1}^{P}a_j - \sum_{1}^{Q}b_j\right) - \frac{(P-Q)}{2} + \lambda\left(\sum_{1}^{P}A_j - \sum_{1}^{Q}B_j\right) > 1 \tag{1.9}$$

$$\Big(\mathrm{Re}(b_j) > -\lambda B_j \ (j=1,\cdots,M), \quad \mathrm{Re}(a_j) > 1-\lambda A_j \ (j=1,\cdots,N)\Big).$$

Before stating our main results, we require the following lemma:

Lemma 1 ([14, p.157])**.** *If* $s=(s_1,\cdots,s_n)\in C^n, h=(h_1,\cdots,h_n)\in R_+^n$ *and* $g(y)^{\left(\frac{s}{h}.1\right)-1}\in L_1(R_+^n)$, *then*

$$\int_{R_+^n} x^{s-1}g(\max\{x_1^{h_1},\cdots,x_n^{h_n}\})dx = \frac{\frac{s}{h}.1}{s^1}g^*\left(\frac{s}{h}.1\right), \tag{1.10}$$
$$\Big(\mathrm{Re}(s_i)>0 \ (i=1,\cdots,n)\Big),$$

and

$$\int_{R_+^n} x^{s-1}g(\min\{x_1^{h_1},\cdots,x_n^{h_n}\})dx = \frac{(-1)^{n-1}\left(\frac{s}{h}.1\right)}{s^1}g^*\left(\frac{s}{h}.1\right), \tag{1.11}$$
$$\Big(\mathrm{Re}(s_i)<0 \ (i=1,\cdots,n)\Big),$$

where $g^*(u)$ *denotes the one-dimensional Mellin transform of* $g(u)$.

2. Main results

We first establish the following:

Theorem 1. *Let* $\mathrm{Re}(\alpha) > 0,\ \lambda_j < 0\ (j = 1, \cdots, n),\ \ \lambda.1 > \max\{\mathrm{Re}(\beta),\ \mathrm{Re}(\eta),$ $\mathrm{Re}(\alpha+\beta+\eta)\}$, *and* $h \in R_+,\ \mathrm{Re}(a_j) < 1 - \left((\lambda.1)\frac{A_j}{h}\right) (j = 1, \cdots, N),$ $\mathrm{Re}(b_j) > -\left((\lambda.1)\frac{B_j}{h}\right) (j = 1, \cdots, M)$, *and the following conditions holds true:*

(i) $Delta = \sum_1^N A_j - \sum_{N+1}^P A_j + \sum_1^M B_j - \sum_{M+1}^Q B_j > 0,$ *or*

(ii) $\Delta = 0,$ *and* $\mathrm{Re}\left(\sum_1^P a_j - \sum_1^Q b_j\right) - \frac{(P-Q)}{2} + (\lambda.1)\left(\sum_1^P A_j - \sum_1^Q B_j\right) > 1,$

then

$$S^{\alpha,\beta,\eta}_{+;n} x^{-1} H^{M,N}_{P,Q}\left[\min\{x_1^h, \cdots, x_n^h\}\right] \tag{2.1}$$
$$= x^{-1} H^{M,N+2}_{P+2,Q+2}\left[\min\{x_1^h, \cdots, x_n^h\} \,\Big|\, {(1-\beta,h),(1-\eta,h),(a_j,A_j)_{1,P} \atop (b_j,B_j)_{1,Q},(1,h),(1-\alpha-\beta-\eta,h)}\right].$$

Proof. First we recall the multidimensional Mellin inversion formula [1] given by

$$f(x) = \frac{1}{(2\pi i)^n}\int_{(\lambda)-i\infty}^{(\lambda)+i\infty} f^*(s)x^{-s}ds, \tag{2.2}$$

for the Mellin transform

$$f^*(s) = M\{f(x)\} = \int_{R_+^n} x^{s-1}f(x)dx, \tag{2.3}$$

where, we use the condensed integral notation $\int_{(\lambda)-i\infty}^{(\lambda)+i\infty}$ for the n-tuple integral $\int_{\lambda_1-i\infty}^{\lambda_1+i\infty}\cdots\int_{\lambda_n-i\infty}^{\lambda_n+i\infty}$.

In view of (2.2), and by invoking (1.11) of Lemma 1, we can express

$$x^{-1}H^{M,N}_{P,Q}\left[\min\{x_1^h, \cdots, x_n^h\}\right] = \frac{(-1)^{n-1}}{h(2\pi i)^n}\int_{(\lambda)-i\infty}^{(\lambda)+i\infty} F_h^*(s)\frac{s.1}{s^1}x^{-s-1}ds, \tag{2.4}$$

where

$$F_h^*(s) = \frac{\prod_{j=1}^M \Gamma(b_j + B_j\frac{s.1}{h})\prod_{j=1}^N \Gamma(1-a_j-A_j\frac{s.1}{h})}{\prod_{j=M+1}^Q \Gamma(1-b_j-B_j\frac{s.1}{h})\prod_{j=N+1}^P \Gamma(a_j+A_j\frac{s.1}{h})}. \tag{2.5}$$

Applying the operator $S^{\alpha,\beta,\eta}_{+;n}$ on both sides of (2.4), provided that $\mathrm{Re}(s_j) < 0$ $(j = 1, \cdots, n)$, and using the formula [20, p.147, Eq.(3.5)]:

$$S^{\alpha,\beta,\eta}_{+;n} x^{-s} = \frac{\Gamma(\beta+\eta-s.1)\Gamma(\eta+n-s.1)}{\Gamma(n-s.1)\Gamma(\alpha+\beta+\eta+n-s.1)} x^{-s}, \tag{2.6}$$

$$\Big(\mathrm{Re}(\alpha) > -1;\ \mathrm{Re}(s_j) < 1\ (j = 1, \cdots, n), \quad \mathrm{Re}(s.1) < n + \min\left[\mathrm{Re}(\beta),\ \mathrm{Re}(\eta)\right]$$

we obtain

$$\begin{aligned} &S^{\alpha,\beta,\eta}_{+;n} x^{-1} H^{M,N}_{P,Q}\left[\min\{x_{_1}^h, \cdots, x_{_n}^h\}\right] \\ &= \frac{(-1)^{n-1}}{h(2\pi i)^n} \int_{(\lambda)-i\infty}^{(\lambda)+i\infty} F_h^*(s) \frac{s.1}{s^1} \frac{\Gamma(\beta-s.1)\Gamma(\eta-s.1)}{\Gamma(-s.1)\Gamma(\alpha+\beta+\eta-s.1)} x^{-s-1} ds. \end{aligned} \tag{2.7}$$

Interpreting the R. H. S. of (2.7) by means of (2.4)(under the validity conditions stated with Theorem 1), we arrive at the desired result (2.1). □

If we put $\beta = -\alpha$, then the assertions (2.1) of Theorem 1 yields the result:

Corollary 1. *If* $\mathrm{Re}(\alpha) > -1$, $h \in R_+$, $\mathrm{Re}(\alpha) < -\lambda.1$, $\lambda = (\lambda_1, \cdots, \lambda_n) \in R^n$, *with* $\lambda_j < 0$ $(j = 1, \cdots, n)$, *and* $h \in R_+$, $\mathrm{Re}(a_j) < 1 - \left((\lambda.1)\frac{A_j}{h}\right)$ $(j = 1, \cdots, N)$, $\mathrm{Re}(b_j) > -\left((\lambda.1)\frac{B_j}{h}\right)$ $(j = 1, \cdots, M)$; *and the conditions* (i) *and* (ii) *stated with Theorem 1 are satisfied, then*

$$\begin{aligned} &X^a_{+;n} x^{-1} H^{M,N}_{P,Q}\left[\min\{x_{_1}^h, \cdots, x_{_n}^h\}\right] \\ &= x^{-1} H^{M,N+1}_{P+1,Q+1}\left[\min\{x_{_1}^h, \cdots, x_{_n}^h\}\Big|^{(1+\alpha,h)(a_j,A_j)_{1,P}}_{(b_j,B_j)_{1,Q},(1,h)}\right]. \end{aligned} \tag{2.8}$$

In an analogus manner, by using the formula [20, p.148, Eq.(3.6)]:

$$S^{\alpha,\beta,\eta}_{+;n} x^{-s} = \frac{\Gamma(1-n+s.1)\Gamma(1-\beta+\eta-n+s.1)}{\Gamma(1-\beta-n+s.1)\Gamma(1+\alpha+\eta-n+s.1)} x^{-s}, \tag{2.9}$$

$$\Big(\mathrm{Re}(\alpha) > -1; \quad \mathrm{Re}(s.1) > \max\{\mathrm{Re}(\beta-\eta),\ (n-1)\}\Big)$$

we obtain the following result involving the operator (1.3):

Theorem 2. *Let* $\mathrm{Re}(\alpha) > -1$; $\lambda_j > 0\ (j = 1, \cdots, n)$; $\lambda.1 > \max\{\mathrm{Re}(\beta),\ \mathrm{Re}(\beta-\eta)$ $\mathrm{Re}(-\alpha-\beta)\} - 1$ *and* $h \in R_+$, $\mathrm{Re}(a_j) < 1 - (\lambda.1)\frac{A_j}{h}$ $(j = 1, \cdots, N)$, $\mathrm{Re}(b_j) > -(\lambda.1)\frac{B_j}{h}$ $(j = 1, \cdots, M)$; *and the conditions* (i) *and* (ii) *stated with Theorem 1 hold true, then*

$$\begin{aligned} &S^{\alpha,\beta,\eta}_{-;\,n} x^{-1} H^{M,N}_{P,Q}\left[\max\{x_{_1}^h, \cdots, x_{_n}^h\}\right] \\ &= x^{-1} H^{M+2,N}_{P+2,Q+2}\left[\max\{x_{_1}^h, \cdots, x_{_n}^h\} \;\Big|\; {}^{(a_j,A_j)_{1,P}(1-\beta,h),(1+\alpha+\eta,h)}_{(1,h),(1-\beta+\eta,h)(b_j,B_j)_{1,Q}}\right]. \end{aligned} \tag{2.10}$$

For $\beta=-\alpha$, we obtain the following:

Corollary 2. *Let* $\mathrm{Re}(\alpha)>-1,\ \lambda=(\lambda_1,\cdots,\lambda_n)\in R^n$ *with* $\lambda_j>0\ (j=1,\cdots,n)$ *and* $h\in R_+,\ \mathrm{Re}(a_j)<1-\Big((\lambda.1)\frac{A_j}{h}\Big)\ (j=1,\cdots,N),\ \mathrm{Re}(b_j)>-\Big((\lambda.1)\frac{B_j}{h}\Big)\ (j=1,\cdots,M)$ *and the conditions* (i) *and* (ii) *stated with Theorem 1 hold true, then*

$$\begin{aligned}(2.11)\qquad & X^{a}_{-;\ n}x^{-1}H^{M,N}_{P,Q}\Big[\max\{x^h_{1},\cdots,x^h_{n}\}\Big]\\ &= x^{-1}H^{M+1,N}_{P+1,Q+1}\Big[\max\{x^h_{1},\cdots,x^h_{n}\}\ \Big|\ {}^{(a_j,A_j)_{1,P},(1+\alpha,h)}_{(1,h),(b_j,B_j)_{1,Q}}\Big].\end{aligned}$$

The results Corollaries 1 and 2 were obtained by Raina ([14]).

3. Operators on space $M_\lambda(R^n_+)$

Following [19], let $M_\lambda(R^n_+)$, where $\lambda=(\lambda_1,\cdots,\lambda_n)\in R^n$, denote the space of functions f defined on R^n_+ which are defined through the set of entire functions of exponential type. It is proved that the function $f\in M_\lambda(R^n_+)$ if and only if f can be represented as the inverse Mellin transform, viz.

$$f(x)=\frac{1}{(2\pi i)^n}\int_{(\lambda)-i\infty}^{(\lambda)+i\infty} f^*(s)x^{-s}ds,$$

of a function $f^*(s)$ which is infinitely differentiable with compact support on $\big((\lambda)-i\infty,(\lambda)+i\infty\big)$.

We now state the following known assertion pertaining to the space $M_\lambda(R^n_+)$ which are used to arrive at further new results involving the operators (1.2) and (1.3).

Lemma 2 ([20, p.148, Eq.(3.7)])**.** *If* $\mathrm{Re}(\alpha)>0,\ \lambda_j+\mathrm{Re}(d_j)<1\ (j=1,\cdots,n)$, $\alpha+\beta+\eta+n-d.1-\lambda.1>0,\quad \lambda.1+\mathrm{Re}(d.1)<n+\min[\mathrm{Re}(\beta),\ \mathrm{Re}(\eta)]$, *then,* $x^dS^{\alpha,\beta,\eta}_{+;n}x^{-d}$ *is a homeomorphism of the space* $M_\lambda\ (R^n_+)$ *onto itself, and*

$$\begin{aligned}(3.1)\qquad & x^d\ S^{\alpha,\beta,\eta}_{+;n}x^{-d}f(x)\\ &= \frac{1}{(2\pi i)^n}\int_{(\lambda)-i\infty}^{(\lambda)+i\infty} f^*(s)\frac{\Gamma(\beta+n-d.1-s.1)\Gamma(\eta+n-d.1-s.1)}{\Gamma(n-d.1-s.1)\Gamma(\alpha+\beta+\eta-n-d.1-s.1)}x^{-s}ds.\end{aligned}$$

Lemma 3 ([20, p.149, Eq.(3.8)])**.** *If* $\mathrm{Re}(\alpha)>0,\ \ \lambda_j+\mathrm{Re}(d_j)<1\ (j=1,\cdots,n)$, $-\beta-n+d.1+\lambda.1\neq-1,-2,\cdots,\ \alpha+\eta-n+d.1+\lambda.1\neq-1,-2,\cdots,\ \lambda.1+\mathrm{Re}(d.1)>n+\mathrm{Re}(\beta-\eta)-1$, *then* $x^dS^{\alpha,\beta,\eta}_{-;n}x^{-d}$ *is a homeomorphism of the space* $M_\lambda(R^n_+)$

onto itself, and

$$(3.2)\quad x^d S^{\alpha,\beta,\eta}_{-;n} x^{-d} f(x) = \frac{1}{(2\pi i)^n} \int\limits_{(\lambda)-i\infty}^{(\lambda)+i\infty} f^*(s) \frac{\Gamma(1-n+d.1+s.1)\Gamma(1-\beta+\eta-n+d.1+s.1)}{\Gamma(1-\beta-n+d.1+s.1)\Gamma(1+\alpha+\eta-n+d.1+s.1)} x^{-s} ds.$$

Using Lemma 2, we have

$$(3.3)\quad x^d S^{\alpha,\beta,\eta}_{+;n} x^{-d} f(x) H^{M,N}_{P,Q}\left[\min\{x_1^h, \cdots, x_n^h\}\right] = \frac{1}{(2\pi i)^n} \int\limits_{(\lambda)-i\infty}^{(\lambda)+i\infty} F_h^*(s) \frac{|s|}{s_1 \cdots s_n} \frac{\Gamma(\beta+n-d.1-s.1)\Gamma(\eta+n-d.1-s.1)}{\Gamma(n-d.1-s.1)\Gamma(\alpha+\beta+\eta-n-d.1-s.1)} x^{-s} ds,$$

where $F_h^*(s)$ is defined by (2.5). Interpreting the R.H.S. with the help of (2.4), we obtain the following result:

Theorem 3. *Let* $\mathrm{Re}(\alpha) > 0,\ \lambda_j + \mathrm{Re}(d_j) < 1\ (j = 1, \cdots, n)$ *and* $h \in R_+$, *such that* $\lambda.1 + \mathrm{Re}(d.1) < n + \min\Big[\mathrm{Re}(\beta),\ \mathrm{Re}(\eta),\ \mathrm{Re}(\alpha+\beta+\eta)\Big],\ \mathrm{Re}(a_j) < 1 - (\lambda.1)\frac{A_j}{h}\ (j = 1, \cdots, N),\ \mathrm{Re}(b_j) > -(\lambda.1)\frac{B_j}{h}\ (j = 1, \cdots, M)$ *and the conditions* (i) *and* (ii) *stated with Theorem 1 holds true, then*

$$(3.4)\quad x^d S^{\alpha,\beta,\eta}_{+;n} x^{-d}\ H^{M,N}_{P,Q}\left[\min\{x_1^h, \cdots, x_n^h\}\right] = H^{M,N+2}_{P+2,Q+2}\left[\min\{x_1^h, \cdots, x_n^h\}\Big|^{(1-\beta-n+d.1,h),(1-\eta-n+d.1,h),(a_j,A_j)_{1,P}}_{(b_j,B_j)_{1,Q}(1-n+d.1,h),(1-\alpha-\beta-\eta-n+d.1,h)}\right].$$

In an analogous manner, by using Lemma 3, we obtain

Theorem 4. *Let* $\mathrm{Re}(\alpha) > 0,\ \lambda_j + \mathrm{Re}(d_j) > 1\ (j = 1, \cdots, n)$ *and* $h \in R_+$, *such that* $\lambda.1 + \mathrm{Re}(d.1) > n + \mathrm{Re}(\beta - \eta) - 1,\ \mathrm{Re}(a_j) < 1 - (\lambda.1)\frac{A_j}{h}\ (j = 1, \cdots, N),\ \mathrm{Re}(b_j) > -(\lambda.1)\frac{B_j}{h}\ (j = 1, \cdots, M)$ *and the conditions* (i) *and* (ii) *stated with Theorem 1 hold true, then*

$$(3.5)\quad x^d S^{\alpha,\beta,\eta}_{-;n} x^{-d} H^{M,N}_{P,Q}\left[\max\{x_1^h, \cdots, x_n^h\}\right] = H^{M+2,N}_{P+2,Q+2}\left[\max\{x_1^h, \cdots, x_n^h\}\Big|^{(a_j,A_j)_{1,P}(1-\beta-n+d.1,h),(1+\alpha+\eta-n+d.1,h)}_{(1-n+d.1,h),(1-n-\beta+\eta+d.1,h),(b_j,B_j)_{1,Q}}\right].$$

4. Modified fractional derivatives

Following [20], we note that $S^{\alpha,\beta,\eta}_{+;n}$ is a homeomorphism of $M_\lambda(R^n_+)$ onto itself if $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\lambda_j) < 1$ $(j = 1, \cdots, n)$, $\alpha + \beta + \eta - n - \lambda.1 > 0$, $\lambda.1 < n + \min[\mathrm{Re}(\beta), \mathrm{Re}(\eta)]$. There also exists its inverse operator $(S^{\alpha,\beta,\eta}_{+;n})^{-1}$, which is a homeomorphism of $M_\lambda(R^n_+)$ onto itself (see [20, p.158]), and is given by

$$(4.1) \quad (S^{\alpha,\beta,\eta}_{+;n})^{-1} f(x) = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} f^*_h(s) \frac{\Gamma(n-s.1)\Gamma(\alpha+\beta+\eta-n-s.1)}{\Gamma(\beta+n-s.1)\Gamma(\eta+n-s.1)} x^{-s} ds.$$

Similarly, there also exists the inverse operator of $S^{\alpha,\beta,\eta}_{-;n}$ if $\mathrm{Re}(\alpha) > 0$, $\lambda_j > 1$, $-\beta - n + \lambda.1 > 0$, $\alpha + \eta - n + \lambda.1 > 0$, $\lambda.1 > \mathrm{Re}(\beta - \eta) - 1$, which is a homeomorphism of $M_\lambda(R^n_+)$ onto itself, then

$$(4.2) \quad (S^{\alpha,\beta,\eta}_{-;n})^{-1} f(x) = \frac{1}{2\pi i} \int_{\lambda-i\infty}^{\lambda+i\infty} f^*_h(s) \frac{\Gamma(1-\beta-n+s.1)\Gamma(1+\alpha+\eta-n+s.1)}{\Gamma(1-n+s.1)\Gamma(1-\beta+\eta-n+s.1)} x^{-s} ds.$$

Using (4.1) in conjunction with (1.11) of Lemma 1, we readily obtain the following:

Theorem 5. *Let* $\mathrm{Re}(\alpha) > 0$, $\lambda_j < 1$ $(j = 1, \cdots, n)$, $\alpha + \beta + \eta - \lambda.1 > 0$, $\mathrm{Re}(\lambda.1) < \min[\mathrm{Re}(\beta), \mathrm{Re}(\eta), \mathrm{Re}(\alpha+\beta+\eta)]$, $h \in R_+$, $\mathrm{Re}(a_j) < 1 - (\lambda.1)\frac{A_j}{h}$ $(j = 1, \cdots, N)$, $\mathrm{Re}(b_j) > -(\lambda.1)\frac{B_j}{h}$ $(j = 1, \cdots, M)$, *and the conditions* (i) *and* (ii) *stated with Theorem 1 hold true, then*

$$(4.3) \quad (S^{\alpha,\beta,\eta}_{+;n})^{-1} H^{M,N}_{P,Q}\left[\min\{x^h_1, \cdots, x^h_n\}\right] = H^{M,N+2}_{P+2,Q+2}\left[\min\{x^h_1, \cdots, x^h_n\} \Big|^{(1-n,h),(1-\alpha-\beta-\eta+n),(a_j,A_j)_{1,P}}_{(b_j,B_j)_{1,Q},(1-\beta-n,h),(1-\eta-n,h)}\right].$$

In an analogus manner, by using (4.2) in conjunction with (1.10) of Lemma 1, we obtain the following:

Theorem 6. *Let* $\mathrm{Re}(\alpha) > 0$, $\lambda_j > 1$ $(j = 1, \cdots, n)$, $\lambda.1 - \beta - \eta > 0$, $\alpha + \eta - n + \lambda.1 > 0$, $(\lambda.1) > [\mathrm{Re}(\beta - \eta)] - 1$, $h \in R_+$ *and conditions (i) and (ii) stated with Theorem 1 hold true, then*

$$(4.4) \quad (S^{\alpha,\beta,\eta}_{-;n})^{-1} H^{M,N}_{P,Q}\left[\max\{x^h_1, \cdots, x^h_n\}\right] = H^{M+2,N}_{P+2,Q+2}\left[\max\{x^h_1, \cdots, x^h_n\} \Big|^{(a_j,A_j)_{1,P}(1-n,h),(1-n-\beta+\eta,h)}_{(1-\beta-n,h),(1+\alpha+\eta-n,h)(b_j,B_j)_{1,Q}}\right].$$

5. Concluding remarks

The main results contained in Theorems 1-5 essentially involve the well known H-function which includes as its special cases, mathematically important functions such as the Bessel Maitland function $J_\lambda^\nu(x)$, the Wright's generalized hypergeometric functions ${}_p\Psi_q$, the Mittag-Laffler functions E_α and $E_{\alpha,\ \beta}$ ([5], [6] and [8]), and Miller-Ross functions, viz. $E_x(\nu,a), C_x(\nu,a)$ and $S_x(\nu,a)$ ([7]).

To illustrate few special cases here, we deduce below the image functions under the operators (1.2) and (1.3) from the main formulas (2.1), (2.10), (3.4) and (3.6) by assigning suitable values to the parameters such that the reduced functions are identifiable with known functions. It may also be noted that the formulas (2.8) and (2.11) mentioned in Corollaries 1 and 2 contain the one dimensional fractional derivative formulas given in [10, p.99] and [11, p.277].

(i) Let us set $M=N=P=Q=1,\ a_1=1-a,\ A_1=B_1=1,\ b_1=0,\ d=0$ in (3.4) and (3.6), we obtain

$$\begin{aligned}(5.1)\qquad & S_{+;n}^{\alpha,\beta,\eta}\left[1+\min\{x_1^h,\cdots,x_n^h\}\right]^{-a}\\ &= \frac{1}{\Gamma(a)}H_{3,3}^{1,3}\left[\min\{x_1^h,\cdots,x_n^h\}\,\Big|_{(0,1),(1-n,h),(1-\alpha-\beta-\eta-n,h)}^{(1-\beta-n,h),(1-\eta-n,h)(1-a,1)}\right],\end{aligned}$$

and

$$\begin{aligned}(5.2)\qquad & S_{-;n}^{\alpha,\beta,\eta}\left[1+\max\{x_1^h,\cdots,x_n^h\}\right]^{-a}\\ &= \frac{1}{\Gamma(a)}H_{3,3}^{3,1}\left[\max\{x_1^h,\cdots,x_n^h\}\,\Big|_{(1-n-\beta+\eta,h),(0,1),(1-n,h)}^{(1-a,1),(1-\beta-n,h)(1+\alpha+\eta-n,h)}\right].\end{aligned}$$

(ii) Next, we put $M=1,\ \ N=P=0,\ \ Q=2,\ \ b_1=0,\ \ B_1=1,\ \ b_2=-\lambda,$ $B_2=\nu,\ \ d=0$ in (2.1) and (2.10), we obtain

$$\begin{aligned}(5.3)\qquad & S_{+,n}^{\alpha,\beta,\eta}\left[x^{-1}J_\lambda^\nu\left\{\min\{x_1^h,\cdots,x_n^h\}\right\}\right]\\ &= x^{-1}H_{2,4}^{1,2}\left[\min\{x_1^h,\cdots,x_n^h\}\,\Big|_{(0,1),(1,h),(-\lambda,\nu),(1-\alpha-\beta-\eta,h)}^{(1-\beta,h),(1-\eta,h)}\right],\end{aligned}$$

and

$$\begin{aligned}(5.4)\qquad & S_{-,n}^{\alpha,\beta,\eta}\left[x^{-1}J_\lambda^\nu\left\{\max\{x_1^h,\cdots,x_n^h\}\right\}\right]\\ &= x^{-1}H_{2,4}^{3,0}\left[\max\{x_1^h,\cdots,x_n^h\}\,\Big|_{(0,1),(1,h),(1-\beta+\eta,h),(-\lambda,\nu)}^{(1-\beta,h),(1+\alpha+\eta,h)}\right],\end{aligned}$$

where $J_\lambda^\nu(x)$ is the Bessel-Maitland function defined in [8].

Acknowledgements. The work of the first-named author was supported by Council for Scientific and Industrial Research (Govt. of India).

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