KYUNGPOOK Math. J. 45(2005), 105-114

Non-homogeneous Linear Differential Equations with Solutions of Finite Order

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Abstract. In this paper we investigate the growth of finite order solutions of the differential equation $f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z)$, where $A_0(z), \cdots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ are entire functions. We find conditions on the coefficients which will guarantees the existence of an asymptotic value for a transcendental entire solution of finite order and its derivatives. We also estimate the lower bounds of order of solutions if one of the coefficient is dominant in the sense that has larger order than any other coefficients.

1. Introduction and statement of results

For an entire function f we denote by $\sigma(f)$ the order of growth of f which is defined by

$$\sigma(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,f)}{\log r} = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r,f)}{\log r}, \tag{1.1}$$

where $T(r,\ f)$ is the Nevanlinna characteristic function of f, and $M(r,f) = \max_{|z|=r}|f(z)|$. See [4] for the notations and definitions.

For $k \geq 2$ we consider the non-homogeneous linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F, \tag{1.2}$$

where $A_0(z), \cdots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ are entire functions. It is well-known that all solutions of equation (1.2) are entire functions. It is also known that if there exists one A_s $(0 \leq s \leq k-1)$ such that A_s is transcendental with

$$\max\{\sigma(A_j)(j \neq s), \sigma(F)\} < \sigma(A_s) \leq 1/2,$$

then every transcendental solution f of (1.2) is of infinite order ([5]). Recently the growth theory of the differential equations has been an active research area, and the growth problems of the non-homogeneous linear differential equations are of

Received January 8, 2004, and, in revised form, May 18, 2004.
2000 Mathematics Subject Classification: 30D35, 34M10.
Key words and phrases: linear differential equations, growth of entire solutions.

very important aspect in this area. In [1] Belaïdi and Hamani have investigated the growth of solutions of the differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f^{'} + A_0(z)f = 0, \tag{1.3}$$

where $A_0(z), \cdots, A_{k-1}(z)$ are entire functions with $A_0(z) \not\equiv 0$ and have proved the following results:

Theorem A ([1]). *Let $A_0(z), \cdots, A_{k-1}(z)$ with $A_0(z) \not\equiv 0$ be entire functions such that for real constants α, β, θ_1 and θ_2 where $\alpha > 0, \beta > 0$ and $\theta_1 < \theta_2$, we have*

$$|A_1(z)| \geq \exp\left\{(1 + o(1))\alpha\,|z|^{\beta}\right\} \tag{1.4}$$

and

$$|A_j(z)| \leq \exp\left\{o(1)\,|z|^{\beta}\right\} \quad (j = 0, 2, \cdots, k-1) \tag{1.5}$$

as $z \to \infty$ in $\theta_1 \leq \arg z \leq \theta_2$. Let $\varepsilon > 0$ be a given small constant, and let $S(\varepsilon)$ denote the angle $\theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$. If $f \not\equiv 0$ is a solution of equation (1.3)with $\sigma(f) < +\infty$, then the following conditions hold:

(i) *There exists a constant $b \neq 0$ such that $f(z) \to b$ as $z \to \infty$ in $S(\varepsilon)$. Furthermore,*

$$|f(z) - b| \leq \exp\left\{-(1 + o(1))\alpha\,|z|^{\beta}\right\} \tag{1.6}$$

as $z \to \infty$ in $S(\varepsilon)$.

(ii) *For each integer $m \geq 1$*

$$\left|f^{(m)}(z)\right| \leq \exp\left\{-(1 + o(1))\alpha\,|z|^{\beta}\right\} \tag{1.7}$$

as $z \to \infty$ in $S(\varepsilon)$.

Theorem B ([1]). *Let $A_0(z), \cdots, A_{k-1}(z)$ be entire functions that satisfy $\max\{\sigma(A_j) : j = 0, 2, \cdots, k-1\} < \sigma(A_1)$. Then every solution $f \not\equiv 0$ of (1.3) of finite order satisfies $\sigma(f) \geq \sigma(A_1)$.*

The main aim of this paper is to extend the above results to the non-homogeneous linear differential equation (1.2) in the following theorems, in which the dominating coefficient $A_1(z)$ is replaced by $A_s(z)$.

Theorem 1.1. *Suppose that $A_0(z), \cdots, A_{k-1}(z)$ and $F \not\equiv 0$ are entire functions such that for real constants α, β, θ_1 and θ_2 where $\alpha > 0$, $\beta > 0$ and $\theta_1 < \theta_2$, we have for some $s = 1, \cdots, k-1$,*

$$|A_s(z)| \geq \exp\left\{(1 + o(1))\alpha\,|z|^{\beta}\right\} \tag{1.8}$$

and

$$\max\{|A_j(z)|,|F(z)|\} \le \exp\left\{o(1)|z|^{\beta}\right\} \tag{1.9}$$

for all $j = 0,\cdots,s-1,s+1,\cdots,k-1$ *as* $z \to \infty$ *in* $\theta_1 \le \arg z \le \theta_2$. *For given* $\varepsilon > 0$ *small enough let* $S(\varepsilon)$ *denote the angle* $\theta_1+\varepsilon \le \arg z \le \theta_2-\varepsilon$. *If* f *is a transcendental solution of equation* (1.2) *with* $\sigma(f) < +\infty$, *then the following conditions hold:*

(i) *There exists a constant* b_{s-1} *such that* $f^{(s-1)}(z) \to b_{s-1}$ *as* $z \to \infty$ *in* $S(\varepsilon)$. *Indeed,*

$$\left|f^{(s-1)}(z) - b_{s-1}\right| \le \exp\left\{-(1+o(1))\alpha|z|^{\beta}\right\} \tag{1.10}$$

as $z \to \infty$ *in* $S(\varepsilon)$.

(ii) *For each integer* $m \ge s$

$$\left|f^{(m)}(z)\right| \le \exp\left\{-(1+o(1))\alpha|z|^{\beta}\right\} \tag{1.11}$$

as $z \to \infty$ *in* $S(\varepsilon)$.

Theorem 1.2. *Let* $A_0(z),\cdots,A_{k-1}(z)$ *and* $F \not\equiv 0$ *be entire functions such that for some integer* s, $1 \le s \le k-1$, *we have* $\max\{\sigma(A_j)\ (j \ne s),\ \sigma(F)\} < \sigma(A_s)$. *Then every transcendental solution* f *of* (1.2) *of finite order satisfies* $\sigma(f) \ge \sigma(A_s)$.

2. Preliminary lemmas

Our proofs depend mainly upon the following Lemmas.

Lemma 2.1 ([3, p. 89])**.** *Let* f *be a transcendental entire function of finite order* σ, *let* $\Gamma = \{(k_1,j_1),\ (k_2,j_2),\ \cdots,\ (k_m,j_m)\}$ *denote a finite set of distinct pairs of integers that satisfy* $k_i > j_i \ge 0 (i=1,\cdots,m)$, *and let* $\varepsilon > 0$ *be a given constant. Then there exists a set* $E \subset [0,\ 2\pi)$ *that has linear measure zero, such that if* $\psi_0 \in [0,\ 2\pi) - E$, *then there is a constant* $R_0 = R_0(\psi_0) > 1$ *such that for all* z *satisfying* $\arg z = \psi_0$ *and* $|z| \ge R_0$, *and for all* $(k,j) \in \Gamma$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.1}$$

Lemma 2.2 ([2], [6]). *Let* $f(z)$ *be an entire function and suppose that* $\left|f^{(k)}(z)\right|$ *is unbounded on some ray* $\arg z = \theta$. *Then there exists an infinite sequence of points* $z_n = r_n e^{i\theta}$ $(n = 1,2,\cdots)$, *where* $r_n \to +\infty$, *such that* $f^{(k)}(z_n) \to \infty$ *and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \le \frac{1}{(k-j)!}(1+o(1))|z_n|^{k-j}\ (j=0,\cdots,k-1). \tag{2.2}$$

Lemma 2.3 ([3]). *Let $f(z)$ be a meromorphic function, let j be a positive integer, and let $\alpha > 1$ be a real constant. Then there exists a constant $R > 0$ such that for all $r \geq R$, we have*

$$T(r, f^{(j)}) \leq (j+2)T(\alpha r, f). \tag{2.3}$$

3. Proof of Theorem 1.1

Suppose that f is a transcendental solution of (1.2) with $\sigma(f) < +\infty$. Set $\rho = \sigma(f)$. Then by Lemma 2.1, there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, such that if $\psi_0 \in [0, 2\pi) - E$, then for all $k > s \geq 1$, and all $j = s+1, \cdots, k$,

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leq |z|^{(j-s)\,(\rho-1+\varepsilon)} \leq |z|^{(k-s)\rho} \ (0 < \varepsilon < 1) \tag{3.1}$$

as $z \to \infty$ along $\arg z = \psi_0$.

Now suppose that $\left|f^{(s)}(z)\right|$ is unbounded on some ray $\arg z = \phi_0$ where $\phi_0 \in [\theta_1, \theta_2] - E$. Then by Lemma 2.2, there exists an infinite sequence of points $z_n = r_n e^{i\,\phi_0}$, where $r_n \to +\infty$ such that $f^{(s)}(z_n) \to \infty$ and

$$\left|\frac{f^{(j)}(z_n)}{f^{(s)}(z_n)}\right| \leq \frac{1}{(s-j)!}(1+o(1))\,|z_n|^{s-j} \leq 2\,|z_n|^{s}\ (j = 0, \cdots, s-1) \tag{3.2}$$

as $z_n \to \infty$. By (1.2) we have

$$f^{(s)}\left[\frac{f^{(k)}}{f^{(s)}}\frac{1}{A_s} + \frac{f^{(k-1)}}{f^{(s)}}\frac{A_{k-1}}{A_s} + \cdots + \frac{f^{(s+1)}}{f^{(s)}}\frac{A_{s+1}}{A_s} + 1 + \frac{f^{(s-1)}}{f^{(s)}}\frac{A_{s-1}}{A_s} + \cdots + \frac{f}{f^{(s)}}\frac{A_0}{A_s}\right] = \frac{F}{A_s}. \tag{3.3}$$

Combining (3.1), (3.2), (1.8) and (1.9) together with (3.3) yields that $f^{(s)}(z_n) \to 0$ as $z_n \to \infty$. This contradicts that $f^{(s)}(z_n) \to \infty$ as $z_n \to \infty$. Therefore, $\left|f^{(s)}(z)\right|$ is bounded on any ray $\arg z = \phi$ where $\phi \in [\theta_1, \theta_2] - E$. It then follows from the classical Phragmén-Lindelöf theorem [7, p.214] that there exists a constant $M > 0$ such that

$$\left|f^{(s)}(z)\right| \leq M \tag{3.4}$$

for all $z \in S(\varepsilon)$.

If $\theta_0 \in [\theta_1 + \varepsilon, \theta_2 - \varepsilon] - E$, then when $\arg z = \theta_0$, we obtain for all $m < s$, by $(s-m)$-fold iterated integration along the ray under consideration,

$$f^{(m)}(z) = f^{(m)}(0) + f^{(m+1)}(0)z + \cdots + \frac{1}{(s-m-1)!}f^{(s-1)}(0)z^{s-m-1} + \int_0^z \cdots \int_0^\zeta \int_0^\xi f^{(s)}(t)dtd\xi \cdots du. \tag{3.5}$$

Therefore, by an elementary triangle inequality and (3.4), we obtain from (3.5)

$$\begin{aligned}(3.6)\qquad & \left|f^{(m)}(z)\right| \\ \le\ & \left|f^{(m)}(0)\right| + \left|f^{(m+1)}(0)\right| |z| + \cdots + \frac{1}{(s-m-1)!}\left|f^{(s-1)}(0)\right| |z|^{s-m-1} \\ & + M\int_0^z \cdots \int_0^\zeta \int_0^\xi |dt|\,|d\xi| \cdots |du| = O(|z|^{s-m}).\end{aligned}$$

We obtain from (1.2)

$$\begin{aligned}(3.7)\qquad & |A_s(z)|\left|f^{(s)}\right| \\ \le\ & |F| + \left(\left|\frac{f^{(k)}}{f^{(s)}}\right| + |A_{k-1}(z)|\left|\frac{f^{(k-1)}}{f^{(s)}}\right| + \cdots + |A_{s+1}(z)|\left|\frac{f^{(s+1)}}{f^{(s)}}\right|\right)\left|f^{(s)}\right| \\ & + |A_{s-1}(z)|\left|f^{(s-1)}\right| + \cdots + |A_1(z)|\left|f'\right| + |A_0(z)|\,|f|\,.\end{aligned}$$

Using (3.1), (3.4), (3.6), (1.8) and (1.9), we obtain from (3.7)

$$\begin{aligned}(3.8)\qquad & \exp\left\{(1+o(1))\alpha\,|z|^\beta\right\}\left|f^{(s)}\right| \\ \le\ & \exp\left\{o(1)\,|z|^\beta\right\} + |z|^{(k-s)\rho}\,(1+(k-s-1)\exp\{o(1)\,|z|^\beta\})\left|f^{(s)}\right| \\ & + (O(|z|^s) + \cdots + O(|z|))\exp\{o(1)\,|z|^\beta\} \\ \le\ & \exp\left\{o(1)\,|z|^\beta\right\} + |z|^{(k-s)\rho}\,(1+(k-s-1)\exp\left\{o(1)\,|z|^\beta\right\})M \\ & + (O(|z|^s) + \cdots + O(|z|))\exp\left\{o(1)\,|z|^\beta\right\}\end{aligned}$$

as $z \to \infty$ along $\arg z = \theta_0$. From (3.8), we conclude that

$$\begin{aligned}(3.9)\qquad & \left|f^{(s)}(z)\right| \\ \le\ & \frac{\exp\left\{o(1)\,|z|^\beta\right\} + |z|^{(k-s)\rho}\,(1+(k-s-1)\exp\left\{o(1)\,|z|^\beta\right\})M}{\exp\left\{(1+o(1))\alpha\,|z|^\beta\right\}} \\ & + \frac{(O(|z|^s) + \cdots + O(|z|))\exp\left\{o(1)\,|z|^\beta\right\}}{\exp\left\{(1+o(1))\alpha\,|z|^\beta\right\}} \\ \le\ & \exp\left\{-(1+o(1))\alpha\,|z|^\beta\right\}.\end{aligned}$$

Using an application of the Phragmén-Lindelöf theorem to (3.9), we can derive that

$$(3.10)\qquad \left|f^{(s)}(z)\right| \le \exp\left\{-(1+o(1))\alpha\,|z|^\beta\right\}$$

as $z \to \infty$ in $S(2\varepsilon)$. This proves the second assertion for $m = s$.

Now let $z \in S(3\varepsilon)$ where $|z| > 1$, let γ be a circle of radius $r = 1$ with center at z, and let $m > s$ be an integer. Then by the Cauchy integral formula and (3.10), we obtain as $z \to \infty$ in $S(3\varepsilon)$,

$$\begin{aligned} \left|f^{(m)}(z)\right| &\leq \frac{(m-s)!}{2\pi}\oint_{\gamma}\frac{\left|f^{(s)}(u)\right|}{|u-z|^{m-s+1}}\,|du| \\ &\leq \frac{(m-s)!}{2\pi}.2\pi\exp\left\{-(1+o(1))\alpha(|z|-1)^{\beta}\right\} \\ &\leq \exp\{-(1+o(1))\alpha\,|z|^{\beta}\,(1-\frac{1}{|z|})^{\beta}\} \\ &\leq \exp\left\{-(1+o(1))\alpha\,|z|^{\beta}\right\}. \end{aligned} \tag{3.11}$$

This proves the second assertion for $m > s$.

Now fix θ where $\theta_1 + \varepsilon \leq \theta \leq \theta_2 - \varepsilon$, and set

$$a_{s-1} = \int_0^{+\infty} f^{(s)}(te^{i\theta})e^{i\theta}dt. \tag{3.12}$$

By (3.10), it very easy to obtain the existence of a_{s-1} and that $a_{s-1} \in \mathbf{C}$. Indeed, integrating $f^{(s)}(u)$ along the sector boundary $0 \to R\ e^{i\psi} \to R\ e^{i\theta} \to 0$, by using (3.10) and Cauchy's theorem to conclude that the integral of $f^{(s)}(u)$ over the arc $\left[\mathrm{Re}^{i\psi}, \mathrm{Re}^{i\theta}\right]$ tends to zero as $R \to +\infty$, the independence from θ immediately follows. Let $z = |z|\, e^{i\,\psi}$ where $\theta_1 + \varepsilon \leq \psi \leq \theta_2 - \varepsilon$. Then, we obtain from (3.12)

$$\begin{aligned} & f^{(s-1)}(z) - f^{(s-1)}(0) - a_{s-1} \\ = & \int_0^z f^{(s)}(u)du - \int_0^{+\infty} f^{(s)}(te^{i\psi})e^{i\psi}dt \\ = & \int_0^z f^{(s)}(u)du - \left(\int_0^{|z|} f^{(s)}(te^{i\psi})e^{i\psi}dt + \int_{|z|}^{+\infty} f^{(s)}(te^{i\psi})e^{i\psi}dt\right) \\ = & -\int_{|z|}^{+\infty} f^{(s)}(te^{i\psi})e^{i\psi}dt. \end{aligned} \tag{3.13}$$

Then, we obtain from (3.10) and (3.13)

$$
\begin{aligned}
(3.14)\qquad \left|f^{(s-1)}(z)-f^{(s-1)}(0)-a_{s-1}\right| &= \left|\int_{|z|}^{+\infty} f^{(s)}(te^{i\psi})e^{i\psi}dt\right| \\
&\le \int_{|z|}^{+\infty} \exp\left\{-(1+o(1))\alpha t^{\beta}\right\} dt \\
&\le \frac{1}{(1+o(1))\alpha\beta\frac{|z|^{\beta-1}}{2}\exp\left\{(1+o(1))\alpha\frac{|z|^{\beta}}{2}\right\}}\int_{|z|}^{+\infty}\frac{(1+o(1))\alpha\beta\frac{t^{\beta-1}}{2}}{\exp\left\{(1+o(1))\alpha\frac{t^{\beta}}{2}\right\}}dt \\
&\le \frac{1}{(1+o(1))\alpha\beta\frac{|z|^{\beta-1}}{2}\exp\left\{(1+o(1))\alpha\frac{|z|^{\beta}}{2}\right\}}\exp\left\{-(1+o(1))\alpha\frac{|z|^{\beta}}{2}\right\} \\
&\le \exp\left\{-(1+o(1))\alpha\,|z|^{\beta}\right\}
\end{aligned}
$$

as $z \to \infty$ in $S(\varepsilon)$, where $b_{s-1} = f^{(s-1)}(0) + a_{s-1}$. We note also that $f^{(s-1)}(z) \to b_{s-1}$ as $z \to \infty$ in $S(\varepsilon)$ from (3.14). The proof of Theorem 1.1 is complete. □

Next, we give two examples that illustrate Theorem 1.1.

Example 3.1. Consider the differential equation

$$
(3.15)\qquad f''' - ze^{-z}f'' - e^{z}f' + (e^{z}+1)f = (z+1)e^{z}.
$$

In this equation, for $z = re^{i\theta}$ $(r \to +\infty)$ and $\frac{3\pi}{4} \le \theta \le \frac{5\pi}{6}$ we have

$$
\begin{aligned}
|A_2(z)| &= |-ze^{-z}| = r\exp(-r\cos\theta) \ \ge\ \exp((1+o(1))\frac{\sqrt{2}}{2}r) \\
|A_1(z)| &= |-e^{z}| \le \exp(r\cos\theta) \ \le\ \exp(o(1)r) \\
|A_0(z)| &= |e^{z}+1| \le 1+\exp(r\cos\theta) \ \le\ \exp(o(1)r) \\
|F(z)| &= |(z+1)e^{z}| = (r+1)\exp(r\cos\theta) \ \le\ \exp(o(1)r).
\end{aligned}
$$

Hence the conditions (1.8) and (1.9) of Theorem 1.1 are verified ($\alpha = \frac{\sqrt{2}}{2}, \beta = 1$), with $A_2(z) = -ze^{-z}$ is the dominating coefficient. The function $f(z) = e^{z} + z$ with $\sigma(f) = 1$ satisfies equation (3.15) and the relations (1.10), (1.11) with $b_1 = 1$.

Example 3.2. Consider the differential equation

$$
(3.16)\qquad f''' - e^{z}f'' - e^{-z}f' + e^{z}f = e^{z} - 1.
$$

In this equation, for $z = re^{i\theta}(r \to +\infty)$ and $\frac{2\pi}{3} \leq \theta \leq \frac{3\pi}{4}$ we have

$$\begin{aligned} |A_1(z)| &= |-e^{-z}| = \exp(-r\cos\theta) \geq \exp((1+o(1))\frac{r}{2}) \\ |A_0(z)| &= |e^{z}| = \exp(r\cos\theta) \leq \exp(o(1)r) \\ |A_2(z)| &= |-e^{z}| = \exp(r\cos\theta) \leq \exp(o(1)r) \\ |F(z)| &= |e^{z}-1| \leq 1+\exp(r\cos\theta) \leq \exp(o(1)r). \end{aligned}$$

Obviously, the conditions (1.8) and (1.9) of Theorem 1.1 are verified ($\alpha = \frac{1}{2}$, $\beta = 1$), with $A_1(z) = -e^{-z}$ is the dominating coefficient. The function $f(z) = e^{z}$ with $\sigma(f) = 1$ satisfies equation (3.16) and the relations (1.10), (1.11) with $b_0 = 0$.

4. Proof of Theorem 1.2

Let $\max\{\sigma(A_j)\,(\,j \neq s), \sigma(F)\} = \beta < \sigma(A_s) = \alpha$. Suppose that f is a transcendental solution of (1.2) with $\sigma(f) < +\infty$. It follows from (1.2) that

$$(4.1)\qquad \begin{aligned} A_s(z) = {} & \frac{F(z)}{f^{(s)}} - \frac{f^{(k)}}{f^{(s)}} - A_{k-1}(z)\frac{f^{(k-1)}}{f^{(s)}} - \cdots - A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}} \\ & - A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}} - \cdots - A_1(z)\frac{f'}{f^{(s)}} - A_0(z)\frac{f}{f^{(s)}}. \end{aligned}$$

Applying the lemma of the logarithmic derivative, we have

$$(4.2)\qquad m(r, \frac{f^{(j+1)}}{f^{(j)}}) = O(\log r) \quad (j = 0, \cdots, k-1), \quad (\sigma(f) < +\infty),$$

holds for all r outside a set $E \subset (0, +\infty)$ with a linear measure $m(E) = \delta < +\infty$. For $j = 0, \cdots, k-1$, and since

$$(4.3)\qquad T(r, f^{(j+1)}) \leq 2T(r, f^{(j)}) + m(r, \frac{f^{(j+1)}}{f^{(j)}}),$$

by using Lemma 2.3 and (4.2) we obtain from (4.3)

$$(4.4)\qquad T(r, f^{(j+1)}) \leq 2T(r, f^{(j)}) + O(\log r) \leq 2(j+2)T(2r, f) + O(\log r).$$

By (4.4), we can obtain from (4.1) that

$$(4.5)\qquad T(r, A_s) \leq T(r, F) + cT(2r, f) + \sum_{j \neq s} T(r, A_j) + O(\log r) \quad (r \notin E),$$

where c is a constant. Since $\sigma(A_s) = \alpha$, there exists $\left\{r_n^{'}\right\}$ $(r_n^{'} \to +\infty)$ such that

$$(4.6)\qquad \lim_{r_n^{'} \to +\infty} \frac{\log T(r_n^{'}, A_s)}{\log r_n^{'}} = \alpha.$$

Since $m(E) = \delta < +\infty$, there exists a point $r_n \in \left[r_n^{'}, r_n^{'} + \delta + 1\right] - E$. From

$$\frac{\log T(r_n, A_s)}{\log r_n} \geq \frac{\log T(r_n^{'}, A_s)}{\log(r_n^{'} + \delta + 1)} = \frac{\log T(r_n^{'}, A_s)}{\log r_n^{'} + \log(1 + (\delta + 1)/r_n^{'})} \tag{4.7}$$

we get

$$\underline{\lim}_{r_n \to +\infty} \frac{\log T(r_n, A_s)}{\log r_n} \geq \alpha. \tag{4.8}$$

So for any given $\varepsilon (0 < 2\varepsilon < \alpha - \beta)$, and for $j \neq s$

$$T(r_n, A_j) \leq r_n^{\beta+\varepsilon}, \ T(r_n, F) \leq r_n^{\beta+\varepsilon} \text{ and } \ T(r_n, A_s) \geq r_n^{\alpha-\varepsilon} \tag{4.9}$$

holds for sufficiently large r_n. By (4.5) and (4.9) we obtain for sufficiently large r_n

$$r_n^{\alpha-\varepsilon} \leq k r_n^{\beta+\varepsilon} + cT(2r_n, f) + O(\log r_n). \tag{4.10}$$

Therefore,

$$\overline{\lim}_{r_n \to +\infty} \frac{\log T(r_n, f)}{\log r_n} \geq \alpha - \varepsilon \tag{4.11}$$

and since ε is arbitrary, we get $\sigma(f) \geq \sigma(A_s) = \alpha$. This proves Theorem 1.2. □

Next, we give an example that illustrates Theorem 1.2.

Example 4.1. Consider the differential equation

$$f''' + e^{-z^2} f'' - 6zf' - (8z^3 + 12z + 7)f = (4z^2 + 4z + 3)e^z. \tag{4.12}$$

In this equation we have

$$\begin{aligned} A_2(z) &= e^{-z^2}, \quad \sigma(A_2) = 2 \\ A_0(z) &= -(8z^3 + 12z + 7), \quad \sigma(A_0) = 0 \\ A_1(z) &= -6z, \quad \sigma(A_1) = 0 \\ F(z) &= (4z^2 + 4z + 3)e^z, \quad \sigma(F) = 1. \end{aligned}$$

Hence the conditions of Theorem 1.2 are verified. The function $f(z) = e^{z^2+z}$ with $\sigma(f) = 2$ satisfies equation (4.12) and the relation $\sigma(f) \geq \sigma(A_2)$.

Acknowledgement. The author would like to thank the referee for his/her helpful remarks and suggestions.

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