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On Upper and Lower Z-supercontinuous Multifunctions

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Abstract. In this paper, we define a multifunction $F : X \rightsquigarrow Y$ to be upper (lower) $Z-$supercontinuous if $F^+(V)$ $(F^-(V))$ is z-open in X for every open set V of Y. We obtain some characterizations and several properties concerning upper (lower) Z-supercontinuous multifunctions.

1. Introduction

Several weak and strong variants of continuity of multifunctions occur in the literature. The strong varients of continuity of multifunctions with we shall be dealing in this paper include [1], [2], [3]. Certain of these strong forms of continuity of multifunctions coincide with continuity of multifunctions if the domain / range space is suitably augmented. M. K. Singal and S. B. Niemse [4] defined z-continuous functions and investigated some properties. In 2003, J. K. Kohli [5] introduced the concept of Z-supercontinuous functions and some properties of Z-supercontinuous functions are given by him. In this paper we introduce anew strong form of continuity of multifunctions called "upper (lower) Z-supercontinuity", which coincides with upper (lower) continuity if domain or range is a completely regular space, or if range is a perfectly normal space. Characterizations and basic properties of upper (lower) Z-supercontinuous multifunctions are alaboreted in section 3. In section 4, we show that if the domain of a upper (lower) Z-supercontinuous multifunction F is retopologized in an approriate way, then F is simply a continuous multifunction.

A multifunction $F : X \rightsquigarrow Y$. is a correspondence from X to Y with $F(x)$ a nonempty subset of Y, for each $x \in X$. Let A be a subset of a topological space (X, τ). $\overset{o}{A}$ and $\overline{A}$ denote the interior and closure of A respectively. A multifunction F of a set X into Y is a correspondence such that $F(x)$ is a nonempty subset of Y for each $x \in X$. We will denote such a multifuntion by $F : X \rightsquigarrow Y$. For a multifunction F, the upper and lower inverse set of a set B of Y will be denoted by $F^+(B)$ and $F^-(B)$ respectively that is $F^+(B) = \{x \in X : F(x) \subseteq B\}$ and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. The graph $G(F)$ of the multifunction $F : X \rightsquigarrow Y$

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is strongly closed [3] if for each $(x, y) \notin G(F)$, there exist open sets U and V containing x and containing y respectively such that $(U \times \overline{V}) \cap G(F) = \emptyset$. [6] A multifunction $F : X \rightsquigarrow Y$ is said to be upper semi continuous (briefly u.s.c.) at a point $x \in X$ if for each open set V in Y with $F(x) \subseteq V$, there exists an open set U containing x such that $F(U) \subseteq V$; lower semi continuous (briefly l.s.c.) at a point $x \in X$ if for each open set V in Y with $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that $F(z) \cap V \neq \emptyset$ for every $z \in U$. A set G in a topological space X is said to be z-open if for each $x \in G$ there exists a cozero set H such that $x \in H \subset G$, or equivalently, if G is expressible as the union of cozero sets. The complement of a z-open set will be referred to as a z-closed set [5].

Throughout this paper, the spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces and $F : X \rightsquigarrow Y$ (resp. $f : X \to Y$) presents a multivalued (resp. single valued) function.

2. Preliminaries and basic properties

Definition 1. A multifunction $F : X \rightsquigarrow Y$ is said to be

(a) upper Z-supercontinuous (Briefly, u. Z-super c.) at a point $x \in X$ if for every open set V with $F(x) \subset V$, there exists a cozero set U containing x such that $F(U) = \cup\{F(u) : u \in U\} \subset V$;

(b) lower Z-supercontinuous (l. Z-super c.) at a point $x \in X$ if for every open set V with $F(x) \cap V \neq \emptyset$, there exists a cozero set U containing x such that $F(u) \cap V \neq \emptyset$ for every $u \in U$;

(c) upper Z-supercontinuous (resp. lower Z-supercontinuous) if it has this property at each point $x \in X$.

Definition 2([3]).

(a) A multifunction $F : X \rightsquigarrow Y$ is called strongly θ- upper semi continuous (s. θ-u.s.c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \subset V$ there exists an open set $U \subset X$ containing x such that $F(\overline{U}) \subset V$.

(b) A multifunction $F : X \rightsquigarrow Y$ is called strongly θ-lower semi continuous (s. θ-l.s.c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \cap V \neq \emptyset$ there exists an open set $U \subset X$ containing x such that $F(u) \cap V \neq \emptyset$ for every $x \in \overline{U}$.

Definition 3([1]).

(a) A multifunction $F : X \rightsquigarrow Y$ is called upper supercontinuous (u. super c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \subset V$ there exists an open set $U \subset X$ containing x such that $F(\overset{o}{\overline{U}}) \subset V$.

(b) A multifunction $F : X \rightsquigarrow Y$ is called lower supercontinuous (l. super c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \cap V \neq \emptyset$ there exists an open set $U \subset X$ containing x such that $F(u) \cap V \neq \emptyset$ for every $x \in \overset{o}{\overline{U}}$.

Definition 4([2]).

(a) A multifunction $F : X \rightsquigarrow Y$ is called upper D-supercontinuous (u. D-super c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \subset V$ there exists an open F_σ- set $U \subset X$ containing x such that $F(U) \subset V$.

(b) A multifunction $F : X \rightsquigarrow Y$ is called lower D-supercontinuous (l. D-super c.) at a point $x \in X$ if for any open set $V \subset Y$ such that $F(x) \cap V \neq \emptyset$ there exists an open F_σ-set $U \subset X$ containing x such that $F(u) \cap V \neq \emptyset$ for every $x \in U$.

u. Z-super c.(l. Z-super c.) $\Longrightarrow$ u. D-super c. (l. D-super c.)

$\Downarrow$

u. Strongly θ-c.(l. Strongly θ-c.)

$\Downarrow$ $\Downarrow$

u. Super c. (l. super c.) $\Longrightarrow$ u. semi c. (l. semi c.)

The diagram well illustrates the relationships that exist among u. Z-supercontinuous (l. Z-supercontinuous) and various variants of continuity of multifunctions defined above. However, none of the above implications in general is reversible as will be exhibited in the sequel.

We gave examples1 and 2 to show that a u.Strongly θ-c. (l. Strongly θ-c.) multifunction need not be u. Z-super c. (l. Z-super c.) and that u. D-super c. (l. D-super c.) multifunction need not be u. Z-super c. (l. Z-super c.).

Example 1([5]). Let $X = Y$ be the Mountain chain space due to Helderman [7] which is a regular space but not a D_δ-completely regular space [10]. Then the multifunction $F : X \rightsquigarrow X$, $F(x) = \{x\}$ for each $x \in X$.is a u. Strongly θ-continuous (l. Strongly θ-continuous) but not u. Z-supercontinuous (l. Z-supercontinuous).

Example 2. Let X denote the set of positive integers endowed with cofinite topology. Then the multifunction $F : X \rightsquigarrow X$, $F(x) = \{x\}$ for each $x \in X$.is u. D-supercontinuous (l. D-supercontinuous) but neither u. supercontinuous (l. supercontinuous) nor u. Strongly θ-continuous (l. Strongly θ-continuous) and hence not u Z-supercontinuous (l. Z-supercontinuous).

3. Characterizations

Definition 5. A set G in a topological space X is said to be z-open if for each $x \in G$ there exists a cozero set H such that $x \in H \subset G$, or equivalently, if G is expressible as the union of cozero sets. The complement of a z-open set will be referred to as a z-closed set [5].

Theorem 1. *The following statements are equivalent for a multifunction $F : X \rightsquigarrow Y$:*

(a) *F is u. Z-super c. (l. Z-super c.)*

(b) *For each open set $V \subseteq Y$, $F^+(V)$ $(F^-(V))$ is a z-open set in X.*

(c) *For each closed set $K \subseteq Y$, $F^-(K)$ $(F^+(K))$ is a z-closed set in X.*

(d) *For each x of X and for each open set V with $F(x) \subset V (F(x) \cap V \neq \emptyset)$, there is a z-open set U containing x such that the implication $y \in U \Rightarrow F(y) \subset V$ is holds $(F(y) \cap V \neq \emptyset)$.*

Proof. $(a) \Longrightarrow (b)$: Let V be an open set of Y and $x \in F^+(V)$. Then there exist a cozero set U containing x such that $F(U) \subset V$. Then $U \subset F^+(V)$. Since U is cozero, we have $x \in U \subset F^+(V)$.

$(b) \Longrightarrow (c)$: Let K be a closed set of Y. Then $Y - K$ is an open set and $F^+(Y - K) = X - F^-(K)$ is z-open. Thus $F^-(K)$ is z-closed in X.

$(c) \Longrightarrow (b)$: Obvious

$(b) \Longrightarrow (a)$: Let V be an open set of Y containing $F(x)$. Then $F^+(V)$ is z-open and $x \in F^+(V)$. Since $F^+(V)$ is a z-open set there exists a cozero set U containing x such that $U \subset F^+(V)$. Thus $F(U) \subset F(F^+(V)) \subset V$.

$(a) \Longleftrightarrow (d)$: Clear.

The proof for the case where F is l. Z-super c. is similarly proved. □

Definition 6. Let X be a topological space and let $A \subset X$. A point $x \in X$ is said to be a z-adherent point of A if every cozero set containing x intersects A. Let A_z denote the set of all z-adherent points of A. Clearly the set A is z-closed if and only if $A_z = A$. [Kohli, Z-supercontinuous Functions]

Theorem 2. *A multifunction $F : X \rightsquigarrow Y$ is l. Z-super c. if and only if $F(A_z) \subset \overline{F(A)}$ for every $A \subset X$.*

Proof. Suppose F is l. Z-super c. Since $\overline{F(A)}$ is closed in Y, by Theorem (1) $F^+(\overline{F(A)})$ is z-closed in X. Also since $A \subset F^+(\overline{F(A)})$, $A_z \subset [F^+(\overline{F(A)})]_z = F^+F(A_z)$ Thus $F(A_z) \subset F(F^+(\overline{F(A)})) \subset \overline{F(A)}$.

Conversely, suppose $F(A_z) \subset \overline{F(A)}$ for every $A \subset X$. Let K be any closed set in Y. Then $F([F^+(K)]_z) \subset \overline{F(F^+(K))}$ and $\overline{F(F^+(K))} \subset \overline{K} = K$.Hence $[F^+(K)]_z \subset F^+(K)$ which shows that F is l. Z-super c. □

Theorem 3. *A multifunction F from a space X into a space Y is l. Z-super c. if and only if $[F^+(B)]_z \subset F^+(\overline{B})$ for every $B \subset Y$.*

Proof. Suppose F is l. Z-super c. Then $F^+(\overline{B})$ is z-closed in X for every $B \subset Y$ and $F^+(\overline{B}) = [F^+(\overline{B})]_z$. Hence $[F^+(B)]_z \subset F^+(\overline{B})$.

Conversely, let K be any closed set in Y. Then $[F^+(K)]_z \subset F^+(\overline{K}) = F^+(K) \subset [F^+(K)]_z$. Thus $F^+(K) = [F^+(K)]_z$ which in turn implies that F is l. Z-super c. □

Definition 7. A filter base $\mathcal{F}$ is said to z-converge to a point x (written as $\mathcal{F} \xrightarrow{z} x$) if for every cozero set containing x contains a member of $\mathcal{F}$ [Kohli, Z-supercontinuous Functions].

Theorem 4. *A multifunction $F : X \rightsquigarrow Y$ is l. Z-super c. if and only if for each $x \in X$ and each filter base $\mathcal{F}$ that z-converges to x, y is an accumulation point of $F(\mathcal{F})$ for every $y \in F(x)$.*

Proof. Assume that F is l. Z-super c. and let $\mathcal{F} \xrightarrow{z} x$ Let W be an open set containing y, with $y \in F(x)$. Then $F(x) \cap W \neq \emptyset$, $x \in F^-(W)$ and $F^-(W)$ is z-open. Let H be an open cozero set in X such that $x \in H \subset F^-(W)$. Since $\mathcal{F} \xrightarrow{z} x$ there exists $U \in \mathcal{F}$ such that $U \subset H$. Let $F(A) \in F(\mathcal{F}$ Then for $A, U \in \mathcal{F}$ there is a set U_1 of $\mathcal{F}$ such that $U_1 \subset A \cap U$. If $x \in U_1$, then since $U_1 \subset U \subset H$, $F(x) \cap W \neq \emptyset$. On the other hand if $x \in A$, then since $F(x) \subset F(A), F(U_1) \subset F(A)$ and since $F(U_1) \cap W \neq \emptyset$, $F(A) \cap W \neq \emptyset$. Thus y is an accumulation point of $F(\mathcal{F})$.

Conversely, Let W be an open subset of Y containing $F(x)$. Now, the filter $\mathcal{F}$ generated by the filterbase $\aleph_x$ consisting cozero sets containing x, z-converges to x. If F is not l. Z-super c. at x, then there is a point $x^{'} \in U$ for every $U \in \mathcal{F}$ such that $F(x^{'}) \cap W = \emptyset$. If we define $\widetilde{U} = x' \in U \mid F(x') \cap W = \emptyset, U \in \mathcal{F}$ then $\widetilde{\mathcal{F}} = \widetilde{U} : U \in \mathcal{F}$ is a filter such that z-converges to x. Since $\widetilde{U} \subset U$, by hypothesis for each $y \in F(x)$, y is an accumulation point of $F(\mathcal{F}$. But for every $\widetilde{U} \in \widetilde{\mathcal{F}}$, $F(\widetilde{U}) \cap W = \emptyset$. This is a contradiction to hypothesis. Hence F is l. Z-super c. at x. □

Theorem 5. *If $F : X \rightsquigarrow Y$ is u. Z-super c. (l. Z-super c.) and $F(X)$ is endowed with subspace topology, then $F : X \rightsquigarrow F(X)$ is u. Z-super c. (l. Z-super c.)*

Proof. Since $F : X \rightsquigarrow Y$ is u. Z-super c. (l. Z-super c.), for every open subset V of Y, $F^+(V \cap F(X)) = F^+(V) \cap F^+(F(X)) = F^+(V)(F^-(V \cap F(X)) = F^-(V) \cap F(F(X)) = F^-(V))$ is z-open. Hence $F.X \rightsquigarrow F(X)$ is u. Z-super c. (l. Z-super c.) □

Theorem 6. *If $F : X \rightsquigarrow Y$ is u. Z-super c. (l. Z-super c.) and $G : Y \rightsquigarrow Z$ u. s. c. (l. s. c.), then $G \circ F$ is u. Z-super c (l. Z-super c.).*

Proof. Let V be an open subset of Z. Then since G is u. s. c. (l. s. c.) $G^+(V)(G^-(V))$ is open subset of Y and since F is u. Z-super c. (l. Z-super c.) $F^+(G^+(V))(F^-(G^-(V)))$ is z-open in X. Thus $G \circ F$ is u. Z-super c. (l. Z-super c.). □

Theorem 7. *Let $\{F_\alpha : X \rightsquigarrow X_\alpha, \alpha \in \Delta\}$ be a family of multifunctions and let*

$F : X \rightsquigarrow \prod_{\alpha\in\Delta} X_\alpha$ be defined by $F(x) = (F_\alpha(x))$. Then F is u. Z-super c. if and only if each $F_\alpha : X \rightsquigarrow X_\alpha$ is u. Z-super c.

Proof. Let G_{α_0} be an open set of X_{α_0}. Then

$$(P_{\alpha_0} \circ F)^+(G_{\alpha_0}) = F^+(P^+_{\alpha_0}(G_{\alpha_0})) = F^+(G_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} X_\alpha).$$

Since F is u. Z-super c. $F^+(G_{\alpha_0} \times \prod_{\alpha\neq\alpha_0} X_\alpha)$. is z-open in X. Thus $P_{\alpha_0} \circ F = F_\alpha$ is u. Z-super c. Here P_α denotes the projection of X onto α- coordinate space X_α.

Conversely, suppose that each $F_\alpha : X \rightsquigarrow X_\alpha$ is u. Z-super c. To show that multifunction F is u. Z-super c., in view of Theorem(1) it is sufficient to show that $F^+(V)$ is z-open for each open set V in the product space $\prod_{\alpha\in\Delta} X_\alpha$. Since the finite intersections and arbitary unions of z-open sets are z-open, it suffices to prove that $F^+(S)$ is z-open for every subbasic open set S in the product space $\prod_{\alpha\in\Delta} X_\alpha$. Let $U_\beta \times \prod_{\alpha\neq\beta} X_\alpha$ be a subbasic open set in $\prod_{\alpha\in\Delta} X_\alpha$. Then $F^+(U_\beta \times \prod_{\alpha\neq\beta} X_\alpha) = F^+(P^+_\beta(U_\beta)) = F^+_\beta(U_\beta)$ is z-open. Hence F is u. Z-super c. □

Theorem 8. *Let $F : X \rightsquigarrow Y$ be a multifuntion and $G : X \rightsquigarrow X \times Y$ defined by $G(x) = (x, F(x))$ for each $x \in X$ be the graph function. Then G is u. Z-super c. if and only if F is u. Z-super c. and X is completely regular.*

Proof. To prove necessity, suppose that G is Z-super c. By Theorem (7) $F = P_Y \circ G$ is Z-super c. where P_Y is the projection from $X \times Y$ onto Y. Let U be any open set in X and let $U \times Y$ be an open set containing $G(x)$. Since G is Z-super c., there exists a cozero set W containing x such that the implication $x' \in W \Rightarrow G(x') \subset U \times Y$ holds. Thus $x \in W \subset U$, which shows that U is z-open and so X is completely regular.

To prove sufficiency, let $x \in X$ and let W be an open set containing $G(x)$. There exists open sets $U \subset X$ and $V \subset Y$ such that $(x, F(x)) \subset U \times V \subset W$. Since X is completely regular, there exists a cozero set G_1in X containing x such that $x \in G_1 \subset V$. Since F is Z-super c., there exists a cozero set G_2 in X containing x such that the implication $x' \in G_2 \Rightarrow F(x') \subset V$. Let $G_1 \cap G_2 = H$. Then H is an cozero set containing x and $G(H) \subset U \times V \subset W$ which implies that G is u. Z-super c. □

Definition 8. Let $F : X \rightsquigarrow Y$ be a multifuntion.

(a) F is said to be upper Z-continuous (briefly u. Z-c.) at $x \in X$, if for each cozero set V with $F(x) \subset V$, there exists an open U set containing x such that the implication $x' \in U \Rightarrow F(x') \subset V$ is hold.

(b) F is said to be lower Z-continuous (briefly l. Z-c.) at $x \in X$, if for each cozero set V with $F(x) \cap V \neq \emptyset$, there exists an open set U containing x such that the implication $x' \in U \Rightarrow F(x') \cap V \neq \emptyset$ is hold.

(c) F is said to be Z-continuous (briefly Z-c.) at $x \in X$, if it is both u. Z-c. and l. Z-c. at $x \in X$.

(d) F is said to be u. Z-c. (l. Z-c., Z-c.) on X, if it has this property at each point $x \in X$.

Theorem 9. *For a multifunction $F : X \rightsquigarrow Y$, the following statements are equivalent:*

(a) *F is u.Z-c. (l. Z-c.)*

(b) *For every z-open set $V \subseteq Y, F^+(V)$ $(F^-(V))$ is an open set in X.*

(c) *For every z-closed set $K \subseteq Y$, $F^-(K)$ $(F^+(K))$ is a closed set in X.*

Lemma 1. *For a multifunction $F : X \rightsquigarrow Y$, the following statements are equivalent:*

(a) *F is u. Z-c.*

(b) *$F(\overline{A}) \subset [F(A)]_z$ for all $A \subseteq X$*

(c) *$\overline{F^+(B)} \subseteq F^+([B]_z)$ for all $B \subseteq X$*

(d) *For every z-closed set $K \subseteq Y$, $F^+(K)$ is closed*

(e) *For every z-open set $G \subseteq Y$, $F^+(G)$ is open*

Proof. (a)⇒(b): Let $y \in F(\overline{A})$. Choose $x \in \overline{A}$ such that $y \in F(x)$. Let V be a cozero set containing $F(x)$ so y. Since F is u. Z-c., $F^+(V)$ is an open set containing x. This gives $F^+(V) \cap A \neq \emptyset$ which in turn implies that $V \cap F(A) \neq \emptyset$ and consequently $y \in [F(A)]_z$. Hence $F(\overline{A}) \subset [F(A)]_z$.

(b)⇒(c): Let B be any subset of Y. Then $F(\overline{F^+(B)}) \subseteq [F(F^+(B))]_z \subseteq [B]_z$ and consequently $\overline{F^+(B)} \subseteq F^+([B]_z)$.

(c)⇒(d): Since a set K is z-closed if and only if $K = [K]_z$, therefore the implication (c)⇒(d) is obvious.

(d)⇒(e): Obvious.

(e)⇒(a): Since every cozero set is z-open and since a multifunction is u. Z-c. if and only if the inverse image of every cozero set is open. Hence (e)⇒(a). □

Theorem 10. *Let X, Y and Z be topological spaces and let the function $F : X \rightsquigarrow Y$ be u. Z-c. and $G : Y \rightsquigarrow Z$ be u. Z-super c. Then $G \circ F : X \rightsquigarrow Z$ is u.s.c.*

Proof. Since $(G \circ F)^+(V) = F^+(G^+(V))$, it is immediate in view of Lemma (1) and Theorem (1). □

Theorem 11. *Let $F : X \rightsquigarrow Y$ be a u. s. c. (l. s. c.) multifunction defined on a* *completely regular space. Then F is u. Z-super c. (l. Z-super c.).*

Proof. In a completely regular space, every open set is z-open. □

Theorem 12. *Let $F : X \rightsquigarrow Y$ be a u. s. c. (l. s. c.) multifunction. If Y is perfectly normal space, then F is u. Z-super c. (l. Z-super c.).*

Proof. In a perfectly normal space, every open set is a cozero set and a u. s. c. (l. s. c.) multifunction lifts cozero sets to cozero set. □

Theorem 13. *Let $F : X \rightsquigarrow Y$ be a u. s. c. (l. s. c.) multifunction defined on a completely regular space. Then F is u. Z-super c. (l. Z-super c.).*

Proof. In a completely regular space every open set is z-open and it is easily verified that a u. s. c. (l. s. c.) multifunction lifts z-open sets to z-open sets. □

Definition 9 ([8]). We may recall that a space X is quasi compact if every cover of X by cozero sets admits a finite subcover.

Theorem 14. *Let $F : X \rightsquigarrow Y$ be u. Z-super c. (l. Z-super c.) multifunction from a quasi compact space onto Y. Then Y is compact.*

Proof. Let $\wp = \{v_\alpha : \alpha \in \Delta\}$ be an open cover of Y. Then each $F^+(V_\alpha)$ is a z-open set in X and so it is a union of cozero sets. This in turn yields a cover $\hbar$ of X consisting of cozero sets. Since X is quasi compact there is a finite subcollection $\{C_1,\ C_2,\ C_3, \cdots, C_n\}$ of $\hbar$which covers X. Suppose $C_i \subset F^+(V_{\alpha_i})$ for some $\alpha_i \in \Delta$ $(i = 1, 2, \cdots, n)$. then $\{V\alpha_1,\ V\alpha_2, \cdots, V\alpha_n\}$ is a finite subcover of $\wp$. Thus X is compact. □

Definition 10 ([9]). A space X is said to be almost compact if every open covering of X has a finite subcollection the closures of whose members covers X.

Definition 11 ([10]). Let X be a topological space and let $A \subset X$. A point $x \in X$ is called a θ-limit point of A if every closed neighborhood of x intersects A. Let $cl_\theta A$ denote the set of all θ-limit points of A. The set A is called θ-closed if $A = cl_\theta A$. The complement of a θ-closed set is called a θ-open set.

Definition 12 ([10]). A space X called θ-compact if every θ-open cover of X has a finite subcover.

It is observed in [11] that every almost θ-compact space is θ-compact and every θ-compact space is quasi compact. However, none of the reverse implications hold.

The following corollaries are immediate from Theorem (14).

Corollary 1. *If $F : X \rightsquigarrow Y$ is a u. Z-super c. (l. Z-super c.) multifunction from a θ-compact space X onto Y. Then Y is compact.*

Corollary 2. *If $F : X \rightsquigarrow Y$ is a u. Z-super c. (l. Z-super c.) multifunction from an almost compact space X onto Y. Then Y is compact.*

Theorem 15. *Let $F : X \rightsquigarrow Y$ be a u. Z-c. (l. Z-c.) multifunction from a quasi* *compact space X onto a space Y. Then Y is quasi compact.*

We omit simple proof of Theorem (15).

Definition 13 ([5]). Let $f : X \to Y$ be a surjection from a topological space X onto a set$Y$. The topology on Y for which a subset $A \subset Y$ is open if and only if $f^{-1}(A)$ is z-open in X is called the z-quotient topology and the map f is called the z-quotient map.

Theorem 16. *Let F be a multifunction from a topological space $(X,\ \tau_1)$ onto a topological the space $(Y,\ \tau_2)$, where τ_2 is z-quotient topology on Y. Then F is l. Z-super c. Moreover τ_2 is the finite topology on Y which makes $F : X \rightsquigarrow Y$ l. Z-super c.*

Proof. The l. Z-super continuity of F follows from the definition of z-quotient topology. □

Theorem 17. *Let $f : X \to Y$ be a z-quotient map. Then a multifunction $F : Y \rightsquigarrow Z$ is l. s. c. if and only if $F \circ f$ is l. Z-super c.*

Proof. If U is an open set in Z and $F \circ f$ is l. Z-super c. then $(F \circ f)^+(U) = f^+(F^+(U)) = f^{-1}(F^+(U))$ which is z-open in X. Since f is z-quotient map, $F^+(U)$ is open in Y. Thus F is l. s. c. Conversely, let $F : Y \rightsquigarrow Z$ be u. s. c. Let U be an open set in Z. By l. Z-super continuity of $F \circ f$, $(F \circ f)^+(U) = f^{-1}(F^+(U))$ is z-open in X. □

4. Complete Regularization

Let $(X,\ \tau)$ be a topological space and let β denote the collection of all cozero subsets of $(X,\ \tau)$. Since the intersection of two cozero sets is a cozero set, the collection β is a base for a topology τ_z on X called the complete regularization of τ. Clearly $\tau_z \subset \tau$. The space $(X,\ \tau)$ is completely regular if and only if $\tau_z = \tau$ [5].

Throughout the section, the symbol τ_z will have the same meaning as in the above paragraph.

Theorem 18. *A multifunction $F : (X,\ \tau) \rightsquigarrow (Y,\ \sigma)$ is u. Z-super c. if and only if $F : (X,\ \tau) \rightsquigarrow (Y,\ \sigma)$ is u. s. c.*

Theorem 19. *Let $(X,\ \tau)$ be topological space.Then the following are equivalent.*

(a) *$(X,\ \tau)$ is completely regular.*

(b) *Every upper-lower semi continuous multifunction from $(X,\ \tau)$ into a space $(Y,\ \sigma)$ is upper-lower Z-super continuous.*

Proof. (a)⇒(b): Obvious

(b)⇒(a): Take $(Y,\ \sigma) = (X,\ \tau)$. Then the identity multifunction I_X on X is upper-lower semi continuous and hence upper-lower Z-super continuous. Thus by Theorem (11) $1_X : (X,\ \tau_z) \to (X,\ \tau)$ is upper-lower semi continuous. Since $U \in \tau$ implies $1_X^{-1}(U) = U \in \tau_z$, $\tau \subset \tau_z$. Therefore $\tau = \tau z$ and so $(X,\ \tau)$ is completely regular.

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