KYUNGPOOK Math. J. 45(2005), 303-338

Harmonic Analysis on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$, II : Unitary Representations of the Group $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$

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Abstract. In this paper, we study unitary representations of the group $GL(n,\mathbb{R})\ltimes\mathbb{R}^{(m,n)}$.

1. Introduction

This paper is a continuation of "Harmonic Analysis on $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$, I". The aim of this paper is to study the unitary representations of the group $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ in detail.

The motivation for studying the group $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ can be explained as follows. We consider the Heisenberg group

$$H_{\mathbb{R}}^{(n,m)} = \left\{ (\lambda,\mu;\kappa) \mid \lambda,\mu \in \mathbb{R}^{(m,n)},\ \kappa \in \mathbb{R}^{(m,m)},\ \kappa + \mu\, {}^t\lambda \text{ symmetric } \right\}$$

endowed with the following multiplication law

$$(\lambda,\mu;\kappa) \circ (\lambda',\mu';\kappa') = (\lambda+\lambda', \mu+\mu', \kappa+\kappa' + \lambda\, {}^t\mu' - \mu\, {}^t\lambda').$$

We define the semidirect product of $Sp(n,\mathbb{R})$ and $H_{\mathbb{R}}^{(n,m)}$

$$Sp_{n,m} = Sp(n,\mathbb{R}) \ltimes H_{\mathbb{R}}^{(n,m)}$$

endowed with the following multiplication law

$$\begin{aligned} & (M,(\lambda,\mu;\kappa)) \cdot (M',(\lambda',\mu';\kappa')) \\ = \ & (MM', (\tilde{\lambda}+\lambda', \tilde{\mu}+\mu', ;\kappa+\kappa' + \tilde{\lambda}\, {}^t\mu' - \tilde{\mu}\, {}^t\lambda')), \end{aligned}$$

where $M,\ M' \in Sp(n,\mathbb{R})$ and $(\tilde{\lambda},\tilde{\mu}) = (\lambda,\mu)M'$. It is easy to see that the Jacobi group $Sp_{n,m}$ acts on the homogeneous space $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ transitively by

$$(M,(\lambda,\mu;\kappa)) \cdot (Z,W) := (M\langle Z\rangle, (W+\lambda Z+\mu)(CZ+D)^{-1}), \tag{1.1}$$

Received April 19, 2004.
2000 Mathematics Subject Classification: 22E46, 22E45, 22E47.
Key words and phrases: unitary representations, automorphic forms.
This work was supported by grant R05-2000-000-00015-0 from the Basic Research Program of the Korea Science and Engineering Foundation.

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{R}),\ (\lambda,\mu;\kappa) \in H_{\mathbb{R}}^{(n,m)}$ and $(Z,W) \in \mathbb{H}_n \times \mathbb{C}^{(m,n)}$. We let

$$GL_{n,m} = GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$$

be the semidirect product of $GL(n,\mathbb{R})$ and the commutative additive group $\mathbb{R}^{(m,n)}$ equipped with the following multiplication law

$$(g,a)\cdot(h,b) = (gh, a\,{}^th^{-1} + b), \tag{1.2}$$

where $g,\ h \in GL(n,\mathbb{R})$ and $a,b \in \mathbb{R}^{(m,n)}$. Then the action (1.1) of $Sp_{n,m}$ on $\mathbb{H}_n \times \mathbb{C}^{(m,n)}$ gives a *canonical* action of $GL_{n,m}$ on the nonsymmetric homogeneous space $\mathcal{P}_n \times \mathbb{R}^{(m,n)}$ given by

$$(g,a)\cdot(Y,V) := (gY\,{}^tg,\ (V+a)\,{}^tg), \tag{1.3}$$

where $g \in GL(n,\mathbb{R}),\ a \in \mathbb{R}^{(m,n)},\ Y \in \mathcal{P}_n$ and $V \in \mathbb{R}^{(m,n)}$. In [15], we developed the theory of automorphic forms on $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ generalizing automorphic forms on $GL(n,\mathbb{R})$.

This paper is organized as follows. In Section 2, we survey the unitary representations of the general linear group $GL(n,\mathbb{R})$. The unitary dual of $GL(n,\mathbb{R})$ was completely determined by E. Stein [10], B. Speh [6]-[8], D. Vogan [14] and other people. We also review certain principal series of $GL(n,\mathbb{R})$ investigated by R. Howe and S. T. Lee [2]. In Section 3, we study the unitary representations of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$. Using the Mackey's method, we compute the unitary dual of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ explicitly in the cases of $n = 2, 3,\ m$ arbitrary. We also deal with certain unitary representations of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ (cf. (3.8)) and discuss their irreducibility.

Notations. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers, and the field of complex numbers respectively. $\mathbb{R}^\times$ denotes the multiplicative group consisting of nonzero real numbers. The symbol $\mathbb{C}_1^\times$ denotes the multiplicative group consisting of all complex numbers z with $|z| = 1$. The symbol ":=" means that the expression on the right hand side is the definition of that on the left. We denote by $\mathbb{Z}^+$ the set of all positive integers. We denote by $F^{(k,l)}$ the set of all $k \times l$ matrices with entries in a commutative ring F. For any $M \in F^{(k,l)},\ {}^tM$ denotes the transpose matrix of M. For a Lie group G, we denote by $\hat{G}$ the unitary dual of G.

2. A survey on the unitary dual of $GL(n,\mathbb{R})$

In this section, we survey the unitary dual of $GL(n,\mathbb{R})$. The references are [12]-[14], [5] and [6]-[8].

First we define the Stein's complimentary series (cf. [10]). Assume $n = 2m$ with m a positive integer. We write

$$P = LN = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \,\Big|\, A, D \in GL(m,\mathbb{R}),\ B \in \mathbb{R}^{(m,m)} \right\},$$

where

$$L = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \Bigm| A, D \in GL(m,\mathbb{R}) \right\} \cong GL(m,\mathbb{R}) \times GL(m,\mathbb{R})$$

and

$$N = \left\{ \begin{pmatrix} I_m & B \\ 0 & I_m \end{pmatrix} \Bigm| B \in \mathbb{R}^{(m,m)} \right\} \cong \mathbb{R}^{(m,m)}.$$

Then P is a maximal parabolic subgroup of $GL(2m,\mathbb{R})$ and N is the unipotent radical of P.

Let $\delta_m : GL(m,\mathbb{R}) \longrightarrow \mathbb{R}$ be the modular function. That is, $\delta_m(g) = \det g$ for $g \in GL(m,\mathbb{R})$. We *fix* a one-dimensional unitary character j of $GL(m,\mathbb{R})$ and a complex number t. We let $\phi_{2m}(j,t) : P \longrightarrow \mathbb{C}^\times$ be the (generally non-unitary) character of P defined by

$$\phi_{2m}(j,t)((g,h),n) := j(gh) \cdot [\delta_m(gh^{-1})]^t,$$

where $g,\ h \in GL(m,\mathbb{R})$ and $n \in N$. We put

$$\sigma_{2m}(j,t) = \mathrm{Ind}_P^{GL(n,\mathbb{R})} \phi_{2m}(j,t). \tag{2.1}$$

According to Stein [10], we see that $\sigma_{2m}(j,t)$ is unitary and irreducible for $t \in i\mathbb{R}$, and that $\sigma_{2m}(j,t)$ is irreducible for $|t| < \frac{1}{2}$. We call the representations $\sigma_{2m}(j,t)$ for $0 < t < \frac{1}{2}$ the *Stein complimentary series* of $GL(2m,\mathbb{R})$.

We observe that the characters of $GL(m,\mathbb{R})$ may be identified in a natural way with the characters of $GL(1,\mathbb{R})$, and hence j extends to a character of $GL(2m,\mathbb{R})$.

Now we fix a unitary character

$$j_1 : \mathbb{R}^\times \longrightarrow \mathbb{C}^\times. \tag{2.2}$$

This corresponds naturally to a family of characters

$$j_m : GL(m,\mathbb{R}) \longrightarrow \mathbb{C}^\times \tag{2.3}$$

characterized by the property that for $m \le m'$,

$$j_{m'}|_{GL(m,\mathbb{R})} = j_m.$$

We refer to the collection $\{j_m\}$ loosely as j. We recall that a representation σ of $GL(m,\mathbb{R})$ is called *spherical* if the trivial representation of $O(m)$ is contained in the restriction of σ to $O(m)$.

Definition 2.1. Let j be a family of characters as in (2.3). Define one-dimensional representations μ_m of $O(m)$ by

$$\mu_m = j_m|_{O(m)}. \tag{2.4}$$

Write μ for the collection $\{\mu_m\}$. We call μ_m a *special one dimensional representation* of $O(m)$. A representation σ of $GL(m,\mathbb{R})$ is called *almost spherical* of type μ_m if μ_m occurs in the restriction of σ to $O(m)$, in other words, if $j_m^{-1}\otimes\sigma$ is spherical.

Definition 2.2. An (ordered) partition of a positive integer n is a sequence

$$\pi=(n_1,n_2,\cdots,n_r),\quad n_i\in\mathbb{Z}^+,\ \sum_{i=1}^r n_i=n.$$

We define

$$\begin{aligned} GL(\pi) &:= GL(n_1,\mathbb{R})\times\cdots\times GL(n_r,\mathbb{R})\subset GL(n,\mathbb{R}),\\ O(\pi) &:= O(n_1)\times\cdots\times O(n_r)=O(n)\cap GL(\pi). \end{aligned}$$

We let $P(\pi)$ be the parabolic subgroup of $GL(n,\mathbb{R})$ generated by $GL(\pi)$ and the Borel subgroup B of $GL(n,\mathbb{R})$ consisting of upper triangular matrices. We let $N(\pi)$ the unipotent radical of $P(\pi)$.

We fix $\mu=\{\mu_m\}$ as in Definition 2.1. The data are a partition $\pi=(n_i)$ of n, and a collection

$$\tau=(\tau_i),\quad \tau_i\in\widehat{GL(n_i,\mathbb{R})},$$

such that

(a) τ_i is almost spherical of type μ_{n_i}, and

(b) τ_i is either a unitary character or a Stein complimentary series.

We call the following induced representation

$$\sigma_\pi(\tau):=\mathrm{Ind}_{P(\pi)}^{GL(n,\mathbb{R})}\otimes\tau_i$$

a *basic almost spherical representation* of type μ.

Theorem 2.3.

(1) *$\sigma_\pi(\tau)$ and $\sigma_{\pi'}(\tau')$ are equivalent if and only if (π',τ') is a permutation of (π,τ).*

(2) *The basic almost spherical representations are unitary.*

(3) *The basic almost spherical representations are irreducible.*

(4) *Any irreducible unitary almost spherical representation of $GL(n,\mathbb{R})$ is basic.*

The outline of proof can be found in [14], p. 455.

Definition 2.4. Let G be a real Lie group with Lie algebra $\mathfrak{g}$. Let K be a compact subgroup of G. Let V be a $\mathfrak{g}$-module that is also a module for K. We say that V is a $(\mathfrak{g},K)$-module if the following conditions (1)-(3) are satisfied:

(1) The action of $\mathfrak{g}$ on V is compatible with that of K on V. That means that $k \cdot X \cdot v = Ad(k)X \cdot k \cdot v$ for $v \in V,\ k \in K,\ X \in \mathfrak{g}$.

(2) If $v \in V$, then Kv spans a finite dimensional vector space W_v of V such that the action of K on W_v is continuous.

(3) If $Y \in \mathfrak{k}$ and if $v \in V$, then $\frac{d}{dt}\Big|_{t=0} \exp(tY)v = Yv$.

A $(\mathfrak{g}, K)$-module is said to be *finitely generated* if it is finitely generated as a $U(\mathfrak{g})$-module. V is said to be *irreducible* if V and 0 are the only $\mathfrak{g}$ and K-invariant subspaces of V.

Definition 2.5. Suppose G is a reductive Lie group with K a maximal compact subgroup of G. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{k}$, and T the corresponding Cartan subgroup. Write $2\rho_c$ for the sum of the roots of $\mathfrak{t}$ in $\mathfrak{b}$. Fix an irreducible representation μ of K of highest weight γ in $\hat{T}$. Let $\gamma_0 \in \mathfrak{t}^*$ be a weight of γ. We define the norm $\|\mu\|$ of μ by

$$\|\mu\| = \langle \mu + 2\rho_c,\ \mu + 2\rho_c \rangle. \tag{2.5}$$

If X is any $(\mathfrak{g}, K)$-module, we say that μ is a *lowest K-type* of X if

(a) μ occurs in the restriction of X to K; and

(b) $\|\mu\|$ is minimal subject to (a).

Theorem 2.6. *Let X be an irreducible $(\mathfrak{g}, K)$-module for $G = GL(n, \mathbb{R})$. Then X has a unique lowest K-type. It occurs with multiplicity one in X.*

Remark 2.7. Representations of general reductive groups may have several lowest K-types. For more detail, we refer to [12].

For a positive integer n, we let $m = [n/2]$ and $\epsilon = n - 2m$. Then $n = 2m + \epsilon$. We set

$$T_0 = SO(2) \times \cdots \times SO(2) \quad (m \text{ copies}).$$

Embedding T_0 in $O(n)$ and identifying $SO(2)$ with the circle, we obtain

$$\widehat{T_0} \cong \mathbb{Z}^m.$$

The Cartan subgroup T of $O(n)$ is given by

$$T = T_0 \rtimes \{E_n,\, r_n\},$$

where E_n denotes the identity matrix of degree n and $r_n = \mathrm{diag}(1, \cdots, 1, -1)$ is the diagonal matrix of degree n.

Proposition 2.8. *The irreducible representations of $O(n)$ are parametrized by pairs (γ, η), subject to the following conditions.*

(a) *γ is a decreasing sequence of m non-negative integers, in other words, a weight of T_0.*

(b) *If n is even and γ_m is not zero, then η is $\frac{1}{2}$; otherwise $\eta = 0$ or 1.*

Let μ be the irreducible representation of $O(n)$ of highest weight (γ, η). If $\eta = 0$ or 1, the restriction of μ to $SO(n)$ is the irreducible representation of highest weight γ. If $\eta = \frac{1}{2}$, the restriction of μ to $SO(n)$ is the sum of the irreducible representations of highest weight $\gamma = (\gamma_1, \cdots, \gamma_{m-1}, \gamma_m)$ and $(\gamma_1, \cdots, \gamma_{m-1}, -\gamma_m)$.

Let μ be an irreducible representation of $O(n)$ of highest weight (γ, η) as in Proposition 2.8. Let p be the largest integer such that γ_p is at least 2. Define

$$\lambda(\mu) = (\gamma_1 - 1, \cdots, \gamma_p - 1, 0, \cdots, 0). \tag{2.6}$$

Let $\pi = (p_1, \cdots, p_r)$ be the coarsest ordered partition of p such that γ is constant on the parts of π. Then the centralizer $L_\theta := L_\theta(\mu)$ of $\lambda(\mu)$ in $GL(n, \mathbb{R})$ is given by

$$L_\theta = GL(\pi, \mathbb{C}) \times GL(n - 2p, \mathbb{R}), \tag{2.7}$$

where

$$GL(\pi, \mathbb{C}) = \prod_{i=1}^{r} GL(p_i, \mathbb{C}).$$

We let μ_{L_θ} be the representation of $L_\theta \cap K$ of highest weight

$$((\gamma_1 - 1, \cdots, \gamma_p - 1, \gamma_{p+1}, \cdots, \gamma_m), \eta).$$

Let μ_f be the representation of $O(n-2p)$ parametrized by $((\gamma_{p+1}, \cdots, \gamma_m), \eta)$. Let $\gamma(j)$ denote the constant value of γ on the j-th part of π. Then we get

$$\mu_{L_\theta} = \left[\otimes_{j=1}^{r} \det{}^{\gamma(j)-1}\right] \otimes \mu_f. \tag{2.8}$$

We write $q = n - 2p$. The last $[n/2] - p$ terms of γ are zeros and ones; say there are q' ones. Define q_0 and q_1 as follows:

$$\text{if } \eta = 0 \text{ or } \frac{1}{2}, \text{ then } q_1 = q' \text{ and } q_0 = q - q_1; \tag{2.9}$$

and

$$\text{if } \eta = 1 \text{ or } \frac{1}{2}, \text{ then } q_0 = q' \text{ and } q_1 = q - q_0. \tag{2.10}$$

Let

$$L := GL(\pi, \mathbb{C}) \times GL(q_0, \mathbb{R}) \times GL(q_1, \mathbb{R}). \tag{2.11}$$

We define

$$\mu_L := \left[\otimes_{j=1}^{r} \det{}^{\gamma(j)-1}\right] \otimes 1 \otimes \det. \tag{2.12}$$

It is clear that μ_L is an almost spherical representation of $L \cap O(n)$.

Lemma 2.9. *Suppose* q_0 *and* q_1 *are non-negative integers, and* $q = q_0 + q_1$. *Write* $q = 2r + \epsilon$ *with* $r = [q/2]$. *Then there is a unique decreasing sequence* γ *of* r *ones and zeros, and an* η *equal to* 0, $\frac{1}{2}$ *or* 1, *with the following properties* (1) *and* (2) :

(1) η *is* $\frac{1}{2}$ *if and only if* q *is even and* $\gamma_r = 1$;

(2) (2.9) *and* (2.10) *hold, where* q' *is the number of ones in* γ.

Write $\mu_f = \mu_f(q_0, q_1)$ *for the irreducible representation of* $O(q)$ *of highest weight* (γ, η) *with* $q = q_0 + q_1$. *Then* μ_f *is the lowest* $O(q)$*-type of*

$$\mathrm{Ind}\,_{O(q_0)\times O(q_1)}^{O(q)}(1 \otimes \det).$$

D. Vogan proved the following important theorem.

Theorem 2.10 (Vogan, [14]). *Let* $G = GL(n, \mathbb{R})$ *and* $K = O(n)$. *Let* (L, μ_L) *be the one defined by* (2.11) *and* (2.12). *Then* L *is a product of various* $GL(m_i, \mathbb{R})$, *and* μ_L *is a special one dimensional representation of* $L \cap K$ *(see Definition 2.1). And there is a functor* Ω *defining a bijection from the set of irreducible unitary representations of* L, *almost spherical of type* μ_L *onto the set of irreducible unitary representations of* G *of lowest* K*-type* μ. *In particular,* Ω *has the following properties*:

(a) *If* Y *is a basic almost spherical representation of* L *of type* μ_L, *then* ΩY *is unitary and irreducible.*

(b) *If* X *is any irreducible unitary representation of* G *of lowest* K*-type* μ, *then there is a unitary almost spherical representation* Y *of* L *such that* X *is a subquotient of* ΩY.

Now we describe the functor Ω in a rough way. The main idea in the proof of Theorem 2.10 is to reduce irreducible unitary representations to the case of spherical representations. Together with Theorem 2.3, the above theorem parametrizes the unitary dual of $GL(n, \mathbb{R})$.

For brevity, we set $G = GL(n, \mathbb{R})$ and $K = O(n)$ for the time being. Fix an element μ on the unitary dual $\hat{K}$ of K. Define $\lambda := \lambda(\mu)$ as in (2.6). Then λ belongs to a fixed Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$. We may find a θ-stable parabolic subalgebra

$$\mathfrak{q}_\theta = \mathfrak{l}_\theta + \mathfrak{u}_\theta$$

of $\mathfrak{g}$, where L_θ is defined as in (2.7). $\mathfrak{u}_\theta$ is characterized by the properties:

$$\Delta(\mathfrak{u}_\theta, \mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) |\ \langle \alpha, \lambda \rangle > 0\,\}$$

and

$$\Delta(\mathfrak{l}_\theta, \mathfrak{t}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t}) |\ \langle \alpha, \lambda \rangle = 0\,\}\,.$$

We define a functor

$$\Omega_\theta = \mathcal{L}^S((\mathfrak{q}_\theta, L_\theta \cap K) \uparrow (\mathfrak{g}, K))$$

from $(\mathfrak{l}_\theta, L_\theta \cap K)$-modules to $(\mathfrak{g}, K)$-modules. The definition of $\mathcal{L}^S$ is explained in Section 5 of [12]. We define a functor

$$(\Omega^K)_\theta = (\mathcal{L}^K)^S$$

from representations of $L_\theta \cap K$ to representations of K.

Fix a real parabolic subgroup P of L_θ with Levi factor L, where L is defined as in (2.11). Let

$$P = LN$$

be the Levi decomposition of P. We define a functor

$$\Omega_{\mathbb{R}} = \mathrm{Ind}\,(L \uparrow L_\theta)$$

from $(\mathfrak{l}, L \cap K)$-modules to $(\mathfrak{l}_\theta, L_\theta \cap K)$-modules. We also define a functor

$$(\Omega^K)_{\mathbb{R}} = \mathrm{Ind}((L \cap K) \uparrow (L_\theta \cap K))$$

from representations of $L \cap K$ to representations of $L_\theta \cap K$.

We define a functor

$$\Omega = \Omega_\theta \circ \Omega_{\mathbb{R}} \tag{2.13}$$

from $(\mathfrak{l}, L \cap K)$-modules to $(\mathfrak{g}, K)$-modules. We set

$$\Omega^K = (\Omega^K)_\theta \circ (\Omega^K)_{\mathbb{R}}$$

a functor from representations of $L \cap K$ to representations of K. The functor Ω in (2.13) is nothing but the functor mentioned in Theorem 2.9. The complete description of the unitary dual of $GL(n, \mathbb{R})$ was given by V. Bargman [1] for $n = 2$, B. Speh [7] for $n = 3$, 4 and D. Vogan [14] for the general case. For the case $n = 2$, we pass from $SL(2, \mathbb{R})$ to the group $SL(2, \mathbb{R})^{\pm}$ of matrices of determinant ± 1. Then we pass from $SL(2, \mathbb{R})^{\pm}$ to $GL(2, \mathbb{R})$ pasting on a character of a group $\mathbb{R} \cdot I_2 \subset GL(2, \mathbb{R})$ (cf. [1], [3]). For the general case, first we let B be the Borel subgroup of $GL(n, \mathbb{R})$ consisting of the upper triangular matrices with nonzero determinant. We let U be the unipotent radical of B and T a split Cartan subgroup of B. Let

$$\underline{\chi} = (\chi_1, \chi_2, \cdots, \chi_n)$$

be a character of T, that is, a collection of n characters of $\mathbb{R}^\times$. We extend $\underline{\chi}$ to a character of B trivial on U. Then the induced representation

$$I(\underline{\chi}) = \mathrm{Ind}_B^{GL(n,\mathbb{R})} \underline{\chi}$$

has a unique irreducible quotient

$$J(\underline{\chi}) = I(\underline{\chi})/I(\underline{\chi})_0, \tag{2.14}$$

where $I(\underline{\chi})_0$ is the only maximal proper closed invariant subspace of $I(\underline{\chi})$. It can be shown that for a character $\underline{\chi} = (\chi_1, \chi_2, \cdots, \chi_n)$ of T such that $\mathrm{Re}\,(s_i - s_j) \in \mathbb{Z}^+$ for all $i,\ j$ with $1 \leq i < j \leq n$, the necessary and sufficient condition on the unitarity of $J(\underline{\chi})$ is that there exist a partition $n = n_1 + n_2 + \cdots + n_r$ $(r \in \mathbb{Z}^+)$ and unitary characters η_i of $\mathbb{R}^\times$ for $i = 1, \cdots, r$ such that

$$J(\underline{\chi}) \cong \mathrm{Ind}_{\prod_{i=1}^r GL(n_i,\mathbb{R})}^{GL(n,\mathbb{R})} \otimes_{i=1}^r \eta_i(\det{}_{GL(n_i,\mathbb{R})}).$$

Vogan [14] proved that the unitary dual of $GL(n,\mathbb{R})$ consists of

(UD1) unitarily induced representation;

(UD2) complimentary series;

(UD3) the one-dimensional representations;

(UD4) a family $J(\underline{\chi})$ in (2.14) which are not induced from any parabolic subgroups of $GL(n,\mathbb{R})$.

Now we discuss certain principal series of $GL(n,\mathbb{R})$. Let $\pi = (n_1, \cdots, n_r)$ be a partition of n. We recall that $P(\pi)$ is the parabolic subgroup of $GL(n,\mathbb{R})$ generated by $GL(\pi)$ and the Borel subgroup B (cf. Definition 2.2). Obviously

$$P(\pi) = \{g = (g_{ij})|\ g_{ij} \in M(n_i, n_j; \mathbb{R}),\ g_{ij} = 0\,(1 \leq j < i \leq r)\}\,. \tag{2.15}$$

If $n = r$, i.e., $n_1 = \cdots = n_r = 1$, then $P(\pi)$ is called a *minimal* parabolic subgroup of $GL(n,\mathbb{R})$. If $r = 2$, that is, if $n_1 + n_2 = n$, then $P(\pi)$ is said to be a *maximal* parabolic subgroup of $GL(n,\mathbb{R})$.

For multi-indices $\epsilon = (\epsilon_1, \cdots, \epsilon_r) \in (\mathbb{Z}/2\mathbb{Z})^r$ and $\nu = (\nu_1, \cdots, \nu_r) \in \mathbb{C}^r$, we define the character $\chi_{\epsilon,\nu}$ of $P(\pi)$ by

$$\chi_{\epsilon,\nu}(g) = \prod_{i=1}^r |\det g_{ii}|^{\nu_i}\,(\mathrm{sgn}(\det g_{ii}))^{\epsilon_i}\,, \tag{2.16}$$

where $g = (g_{ij}) \in P(\pi)$ (cf. (2.15)). It is known that for any $\epsilon = (\epsilon_1, \cdots, \epsilon_r) \in (\mathbb{Z}/2\mathbb{Z})^r$ and $\nu = (\nu_1, \cdots, \nu_r) \in (\sqrt{-1}\mathbb{R})^r$, the induced representation

$$\tau_{\epsilon,\nu}(\pi) = \mathrm{Ind}_{P(\pi)}^{GL(n,\mathbb{R})}\chi_{\epsilon,\nu} \tag{2.17}$$

is an irreducible unitary representation of $GL(n,\mathbb{R})$. If $P(\pi)$ is a minimal parabolic subgroup, $\tau_{\epsilon,\nu}(\pi)$ in (2.17) is called a unitary *principal series* of $GL(n,\mathbb{R})$. If $r < n$, that is, if one of n_j's is larger than 1, $\tau_{\epsilon,\nu}(\pi)$ in (2.17) is called a unitary *degenerate series* of $GL(n,\mathbb{R})$. If $\nu \notin (\sqrt{-1}\mathbb{R})^r$, the principal series $\tau_{\epsilon,\nu}(\pi)$ is not unitary in general.

For a positive integer k with $1 \le k \le [n/2]$, we let

$$P_k = \left\{ \begin{pmatrix} c & b \\ 0 & a \end{pmatrix} \in GL(n,\mathbb{R}) \,\middle|\, a \in GL(k,\mathbb{R}),\ c \in GL(n-k,\mathbb{R}),\ b \in M(n-k,k;\mathbb{R}) \right\}$$

be a maximal parabolic subgroup of $GL(n,\mathbb{R})$. For $\alpha \in \mathbb{C}$, we define the character $\chi_\alpha^\pm : P_k \longrightarrow \mathbb{C}$ by

$$\chi_\alpha^\pm \left(\begin{pmatrix} c & b \\ 0 & a \end{pmatrix} \right) = \begin{cases} (\det a)^\alpha & \text{if } \det a > 0, \\ \pm |\det a|^\alpha & \text{if } \det a < 0. \end{cases}$$

Howe and Lee [2] investigated the irreducibility and the unitarity of the following degenerate series $\tau_{k,\alpha}$ of $GL(n,\mathbb{R})$ defined by

$$\tau_{k,\alpha}^\pm := \mathrm{Ind}_{P_k}^{GL(n,\mathbb{R})} \chi_\alpha^\pm. \tag{2.18}$$

The representation space $\tau_{k,\alpha}^\pm$ is the space consisting of functions $f : GL(n,\mathbb{R}) \longrightarrow \mathbb{C}$ satisfying the condition

$$f(gp) = [\chi_\alpha^\pm(p)]^{-1} f(g), \quad g \in GL(n,\mathbb{R}),\ p \in P_k.$$

$GL(n,\mathbb{R})$ acts on the space $\tau_{k,\alpha}^\pm$ by left translation:

$$(g \cdot f)(h) = f(g^{-1}h), \quad g, h \in GL(n,\mathbb{R}),\ f \in \tau_{k,\alpha}^\pm.$$

Howe and Lee [2] proved the irreducibility of $\tau_{k,\alpha}^\pm$ as follows:

(a) If $\alpha \notin \mathbb{Z}$, then $\tau_{k,\alpha}^\pm$ are irreducible.

(b) If α is an even integer such that $-n/2 \le \alpha \le -2[(k+1)/2]$, then $\tau_{k,\alpha}^+$ is irreducible. If α is an even integer such that $\alpha \ge 2 - 2[(k+1)/2]$, then $\tau_{k,\alpha}^+$ is reducible.

(c) If α is an even integer such that $-n/2 \le \alpha \le -1 - 2[k/2]$, then $\tau_{k,\alpha}^-$ is irreducible. If α is an even integer such that $\alpha \ge 2 - 2[k/2]$, then $\tau_{k,\alpha}^-$ is reducible.

(d) If α is an odd integer such that $-n/2 \le \alpha \le -1 - 2[k/2]$, then $\tau_{k,\alpha}^+$ is irreducible. If α is an odd integer such that $\alpha \ge 1 - 2[k/2]$, then $\tau_{k,\alpha}^+$ is reducible.

(e) If α is an odd integer such that $-n/2 \le \alpha \le -1 - 2[(k+1)/2]$, then $\tau_{k,\alpha}^-$ is irreducible. If α is an odd integer such that $\alpha \ge 3 - 2[(k+1)/2]$, then $\tau_{k,\alpha}^-$ is reducible.

For the unitarity of $\tau^{\pm}_{k,\alpha}$, we refer to [2], pp. 306-308. We realize the degenerate series $\tau^{\pm}_{k,\alpha}$ in another way. We consider the following action σ of $GL(n,\mathbb{R})$ on $\mathbb{R}^{(n,k)}$ defined by

$$\sigma(g)(x) := {}^t g^{-1} x, \quad g \in GL(n,\mathbb{R}),\ x \in \mathbb{R}^{(n,k)}. \tag{2.19}$$

We let $M(n,k;\mathbb{R})^0$ be the set of all $n \times k$ real matrices of rank k. For $\alpha \in \mathbb{C}$, we let $\mathcal{L}^{\pm}_{k,\alpha}$ be the space consisting of functions $f : M(n,k;\mathbb{R})^0 \longrightarrow \mathbb{C}$ satisfying the following condition

$$f(xa) = \begin{cases} (\det a)^{\alpha} f(x) & \text{if } \det a > 0, \\ \pm |\det a|^{\alpha} f(x) & \text{if } \det a < 0 \end{cases}$$

for $x \in M(n,k;\mathbb{R})^0$ and $a \in GL(k,\mathbb{R})$. Then the action σ in (2.19) induces the representation $\sigma^{\pm}_{k,\alpha}$ of $GL(n,\mathbb{R})$ on $\mathcal{L}^{\pm}_{k,\alpha}$ defined by

$$\left(\sigma^{\pm}_{k,\alpha}(g) f\right)(x) = f(\sigma(g^{-1})x) = f({}^t g x), \quad g \in GL(n,\mathbb{R}),\ x \in M(n,k;\mathbb{R})^0.$$

Then we can show that $\tau^{\pm}_{k,\alpha}$ is isomorphic to $\sigma^{\pm}_{k,\alpha}$.

3. Unitary representations of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$

In this section, we find the unitary dual of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$ using the Mackey's method and deal with certain unitary representations of $GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}$.

For brevity, we put

$$A := \mathbb{R}^{(m,n)}, \quad GL_n := GL(n,\mathbb{R}) \text{ and } GL_{n,m} := GL(n,\mathbb{R}) \ltimes \mathbb{R}^{(m,n)}.$$

The multiplication on $GL_{n,m}$ is given by

$$(g,a)\cdot(h,b) = (gh,\ a\,{}^t h^{-1} + b), \quad (g,a),\,(h,b) \in GL_{n,m}. \tag{3.1}$$

We may identify A with the subgroup $\{(I_n,a) |\ a \in A\}$ of $GL_{n,m}$. It is clear that A is a commutative normal subgroup of $GL_{n,m}$ and the center of $GL_{n,m}$ consists only of the identity element $(I_n,0)$. Moreover we have the split exact sequence

$$0 \longrightarrow A \longrightarrow GL_{n,m} \longrightarrow GL_n \longrightarrow 1.$$

We see that the unitary dual $\hat{A}$ of A is isomorphic to A. Indeed, the unitary character ρ_λ of A corresponding to $\lambda \in A$ is defined by

$$\rho_\lambda(a) := e^{2\pi i \sigma({}^t\lambda a)}, \quad a \in A. \tag{3.2}$$

For the time being, we write $g_a = (g,a) \in GL_{n,m}$ for $g \in GL_n$ and $a \in A$, and we identify an element g of GL_n with an element $(g,0)$ in $GL_{n,m}$. The group

$GL_{n,m}$ acts on A by conjugation because A is a normal subgroup of $GL_{n,m}$. This induces the action of $GL_{n,m}$ on $\hat{A}$ as follows:

$$GL_{n,m} \times \hat{A} \longrightarrow \hat{A}, \quad (g_a, \rho) \mapsto \rho^{g_a}, \tag{3.3}$$

where $g_a \in GL_{n,m},\ \rho \in \hat{A}$ and the unitary character ρ^{g_a} of A is defined by

$$\rho_{g_a}(b) := \rho(g_a^{-1} b g_a), \quad b \in A.$$

Since

$$g_a^{-1} b g_a = (g,a)^{-1} b (g,a) = (I_n, b\,{}^t g^{-1}) = g^{-1} b g$$

for any $g \in GL_n$ and $a, b \in A$, we obtain

$$\rho^{g_a}(b) = \rho^g(b) = \rho(b\,{}^t g^{-1}). \tag{3.4}$$

In particular, $\rho^a = \rho$ for every element $a \in A$.

Lemma 3.1. *The action of an element $g_a = (g,a)$ on an element ρ_λ of $\hat{A}$ (cf. (3.2)) is given by*

$$\rho_\lambda^{g_a} = \rho_\lambda^g = \rho_{\lambda g^{-1}}, \quad \lambda \in A. \tag{3.5}$$

Proof. If $b \in A$, then

$$\begin{aligned} \rho_\lambda^{g_a}(b) = \rho_\lambda^g(b) &= \rho_\lambda(b\,{}^t g^{-1}) \\ &= e^{2\pi i \sigma({}^t\lambda b\,{}^t g^{-1})} \\ &= e^{2\pi i \sigma({}^t(\lambda g^{-1}) b)} \\ &= \rho_{\lambda g^{-1}}(b) \quad \text{(according to (3.2))}. \end{aligned}$$

If $\rho \in \hat{A}$, we denote by Ω_ρ the $GL_{n,m}$-orbit of ρ and let

$$GL_{n,m}(\rho) = \{ g_a \in GL_{n,m} \mid \rho^{g_a} = \rho \}$$

be the stabilizer or isotrophy subgroup of $GL_{n,m}$ at ρ. Then the mapping defined by

$$GL_{n,m}/GL_{n,m}(\rho) \longrightarrow \Omega_\rho, \quad g_a \cdot GL_{n,m}(\rho) \longrightarrow \rho^{g_a}$$

is a homeomorphism, in other words, A is regularly embedded. Obviously A is a subgroup of $GL_{n,m}(\rho)$. We define the subset $\widehat{GL_{n,m}(\rho)}_*$ of the unitary dual $\widehat{GL_{n,m}(\rho)}$ of $GL_{n,m}(\rho)$ by

$$\widehat{GL_{n,m}(\rho)}_* = \left\{ \tau \in \widehat{GL_{n,m}(\rho)} \mid \tau|_A \text{ is a multiple of } \rho \right\}.$$

According to G. Mackey [4], we obtain the following.

Theorem 3.2. *For any $\tau \in \widehat{GL_{n,m}(\rho)}_*$, the induced representation*

$$\text{Ind}\,{}_{GL_{n,m}(\rho)}^{GL_{n,m}}\tau$$

is an irreducible unitary representation of $GL_{n,m}$. And the unitary dual $\widehat{GL_{n,m}(\rho)}$ of $GL_{n,m}$ is given by

$$\widehat{GL_{n,m}} = \bigcup_{[\rho]\in GL_{n,m}\backslash \hat{A}} \left\{ \text{Ind}\,{}_{GL_{n,m}(\rho)}^{GL_{n,m}}\tau \,\middle|\, \tau \in \widehat{GL_{n,m}(\rho)}_* \right\}.$$

We deal with the special cases $n = 3,\ 4$ explicitly. The other cases $n \geq 4$ may be dealt with similarly.

Case I. $n = 2$.

(I-1) $m = 1$.

In this case, $A = \mathbb{R}^{(1,2)} \cong \mathbb{R}^2$. We identify the unitary dual $\hat{A}$ of A with $\mathbb{R}^2$. From (3.5), we see that $GL_{2,1}$-orbits in $\hat{A}$ consists of two orbits Ω_0, Ω_1 given by

$$\Omega_{[21];0} = \{(0,0)\}, \quad \Omega_{[21];1} = \mathbb{R}^2 - \{(0,0)\}.$$

We observe that $\Omega_{[21];0}$ is the $GL_{2,1}$-orbit of $(0,0)$ and $\Omega_{[22];1}$ is a $GL_{2,1}$-orbit of any element $(\lambda,\mu) \neq (0,0)$.

Now we choose the element $\delta = \rho_{(1,0)}$ of $\hat{A}$. That is, $\delta(x,y) = e^{2\pi i x}$ for $x, y \in \mathbb{R}^2$. It is easily checked that the stabilizer of $\rho_{(0,0)}$ is $GL_{2,1}$ and the stabilizer $GL_{2,1}(\delta)$ of δ is given by

$$GL_{2,1}(\delta) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix},\ \alpha \right) \in GL_{2,1} \,\middle|\, c \in \mathbb{R},\ d \in \mathbb{R}^\times,\ \alpha \in \mathbb{R}^{(1,2)} \right\}.$$

According to Theorem 3.2, we obtain

Theorem 3.3. *Let $n = 2$ and $m = 1$. Then the irreducible unitary representations of $GL_{2,1}$ are the following*:

(a) *The irreducible unitary representation π, where the restriction of π to A is trivial and the restriction of π to GL_2 is an irreducible unitary representation of GL_2.*

(b) *The representation*

$$\pi_\lambda = \text{Ind}\,{}_{GL_{2,1}(\delta)}^{GL_{2,1}}\tau_\lambda \quad (\lambda \in \mathbb{R})$$

induced from the irreducible unitary representation τ_λ of $GL_{2,1}(\delta)$ such that $\tau_\lambda|_A$ is a multiple of δ.

(I-2) $m = 2$.

In this case, $\hat{A} \cong \mathbb{R}^{(2,2)}$. From now on, we identify $\hat{A}$ with $\mathbb{R}^{(2,2)}$.

Lemma 3.4. *Let $n = 2$ and $m = 2$. Then the $GL_{2,2}$-orbits in $\hat{A}$ consist of the following orbits*

$$\begin{aligned} \Omega_{[22];0} &= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}, \\ \Omega_{[22];1} &= \left\{ \begin{pmatrix} A \\ 0 \end{pmatrix} \in \mathbb{R}^{(2,2)} \,\middle|\, A \in \mathbb{R}^{(1,2)},\ A \neq 0 \right\}, \\ \Omega_{[22];2} &= \left\{ \begin{pmatrix} 0 \\ A \end{pmatrix} \in \mathbb{R}^{(2,2)} \,\middle|\, A \in \mathbb{R}^{(1,2)},\ A \neq 0 \right\}, \\ \Omega_{[22];3}(\delta) &= \left\{ \begin{pmatrix} A \\ \delta A \end{pmatrix} \in \mathbb{R}^{(2,2)} \,\middle|\, A \in \mathbb{R}^{(1,2)},\ A \neq 0 \right\} \quad (\,\delta \in \mathbb{R}^{\times}\,) \end{aligned}$$

and

$$\Omega_{[22];4} = GL(2, \mathbb{R}).$$

$\Omega_{[22];0}$ is the $GL_{2,2}$-orbit of $\mathbf{0}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\Omega_{[22];1}$ is the $GL_{2,2}$-orbit of $\begin{pmatrix} \alpha \\ 0 \end{pmatrix}$ with $0 \neq \alpha \in \mathbb{R}^{(1,2)}$, $\Omega_{[22];2}$ is the $GL_{2,2}$-orbit of $\begin{pmatrix} 0 \\ \beta \end{pmatrix}$ with $0 \neq \beta \in \mathbb{R}^{(1,2)}$, $\Omega_{[22];3}(\delta)$ is the $GL_{2,2}$-orbit of $\begin{pmatrix} \alpha \\ \delta\alpha \end{pmatrix}$ with $0 \neq \alpha \in \mathbb{R}^{(1,2)}$ and $\Omega_{[22];4}$ is the $GL_{2,2}$-orbit of any invertible matrix $M \in GL(2, \mathbb{R})$.

Proof. Without difficulty we may prove the above lemma. We note that $\Omega_{[22];3}(\delta_1) = \Omega_{[22];3}(\delta_2)$ if and only if $\delta_1 = \delta_2$. So we leave the detail to the reader. □

We put

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Obviously $e \in \Omega_{[22];1}$ and $f \in \Omega_{[22];2}$.

Then we may prove the following lemma.

Lemma 3.5.

(a) *The stabilizer of $\mathbf{0}$ is $GL_{2,2}$.*

(b) *The stabilizer $GL_{2,2}(e)$ of e is given by*

$$GL_{2,2}(e) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \alpha \right) \,\middle|\, c \in \mathbb{R},\ d \in \mathbb{R}^{\times},\ \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

For each $x \in \Omega_{[22];1}$, the stabilizer $GL_{2,2}(x)$ of x is conjugate to $GL_{2,2}(e)$. Precisely if $x = eg_0$ with $g_0 \in GL(2, \mathbb{R})$, then $GL_{2,2}(x) = (g_0, 0)^{-1} GL_{2,2}(e)(g_0, 0)$.

(c) *The stabilizer* $GL_{2,2}(f)$ *of* f *is given by*

$$GL_{2,2}(f) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \alpha \right) \,\Big|\, c \in \mathbb{R},\ d \in \mathbb{R}^{\times},\ \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

For each $y \in \Omega_{[22];2}$, *the stabilizer* $GL_{2,2}(y)$ *of* y *is conjugate to* $GL_{2,2}(f)$.

(d) *The stabilizer* $GL_{2,2}(\delta)$ *of* $\begin{pmatrix} 1 & 0 \\ \delta & 0 \end{pmatrix}$ $(\delta \in \mathbb{R}^{\times})$ *is given by*

$$GL_{2,2}(\delta) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \alpha \right) \,\Big|\, c \in \mathbb{R},\ d \in \mathbb{R}^{\times},\ \alpha \in \mathbb{R}^{(2,2)} \right\}.$$

For each $z \in \Omega_{[22];3}(\delta)$, *the stabilizer* $GL_{2,2}(z)$ *of* z *is conjugate to* $GL_{2,2}(\delta)$.

(e) *The stabilizer* $GL_{2,2}(M)$ *of* $M \in \Omega_{[22];4}$ *is given by*

$$GL_{2,2}(M) = \left\{ (I_2, \alpha) \mid \alpha \in \mathbb{R}^{(2,2)} \right\} \cong \mathbb{R}^{(2,2)}.$$

Therefore A *is regularly embedded.*

For $\lambda \in \mathbb{R}$, *we let* χ_λ *be the unitary character of* $\mathbb{R}$ *defined by* $\chi_\lambda(a) := e^{2\pi i \lambda a}\,(a \in \mathbb{R})$ *and for* $M \in \mathbb{R}^{(2,2)}$, *we let* τ_M *be the unitary character of* $A = \mathbb{R}^{(2,2)}$ *defined by*

$$\tau_M(X) := e^{2\pi i ({}^t MX)}, \quad X \in A. \tag{3.6}$$

According to Theorem 3.2, we obtain the following

Theorem 3.6. *Let* $n = 2$ *and* $m = 2$. *Then the irreducible unitary representations of* $GL_{2,2}$ *are the following*:

(a) *The irreducible unitary representations* π, *where the restriction of* π *to* A *is trivial and the restriction of* π *to* $GL(2,\mathbb{R})$ *is an irreducible unitary representation of* $GL(2,\mathbb{R})$.

(b) *The representations* $\pi_{\lambda;e} := \mathrm{Ind}\,{}^{GL_{2,2}}_{GL_{2,2}(\mathrm{e})}\tau_{\lambda,e}\ (\lambda \in \mathbb{R})$ *induced from the irreducible unitary representation* $\tau_{\lambda,e}$ *of* $GL_{2,2}(e)$ *whose restriction to* A *is a multiple of* τ_e *(cf. (3.6)). In fact,* $\tau_{\lambda,e}$ *of the form*

$$\tau_{\lambda,e}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) \right) = e^{2\pi\alpha_1} \left(\mathrm{Ind}\,{}^{\mathbb{R}\rtimes\mathbb{R}^{\times}}_{\mathbb{R}}\, \chi_\lambda \right) \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \right),$$

where $c,\ \alpha_1,\ \cdots,\ \alpha_4 \in \mathbb{R},\ d \in \mathbb{R}^{\times}$.

(c) *The representations* $\pi_{\lambda;f} := \mathrm{Ind}\,{}^{GL_{2,2}}_{GL_{2,2}(\mathrm{f})}\theta_{\lambda,f}\ (\lambda \in \mathbb{R})$ *induced from the irreducible unitary representation* $\theta_{\lambda,f}$ *of* $GL_{2,2}(f)$ *whose restriction to* A *is a multiple of* τ_f *(cf. (3.6)). Indeed,* $\theta_{\lambda,f}$ *is of the form*

$$\theta_{\lambda,f}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \right) \right) = e^{2\pi i\alpha_3} \left(\mathrm{Ind}\,{}^{\mathbb{R}\rtimes\mathbb{R}^{\times}}_{\mathbb{R}}\, \chi_\lambda \right) \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \right),$$

where $c,\ \alpha_1,\ \cdots,\ \alpha_4 \in \mathbb{R},\ d \in \mathbb{R}^{\times}$.

(d) *The representations* $\pi_{\lambda;\delta} := \text{Ind}\,{}_{GL_{2,2}(\delta)}^{GL_{2,2}}\theta_{\lambda,\delta}\,(\lambda\in\mathbb{R},\ \delta\in\mathbb{R}^\times,\ \ r\in\mathbb{R})$ *induced from the irreducible unitary representation* $\sigma_{\lambda,\delta}$ *of* $GL_{2,2}(\delta)$*. Indeed,* $\sigma_{\lambda,\delta}$ *is of the form*

$$\sigma_{\lambda,\delta}\left(\left(\begin{pmatrix}1&0\\c&d\end{pmatrix},\begin{pmatrix}\alpha_1&\alpha_2\\\alpha_3&\alpha_4\end{pmatrix}\right)\right)=e^{2\pi i(\alpha_1+\alpha_3\delta)}\left(\text{Ind}\,{}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^\times}\chi_\lambda\right)\left(\begin{pmatrix}1&0\\c&d\end{pmatrix}\right),$$

where $c,\ \alpha_1,\ \cdots,\ \alpha_4\in\mathbb{R},\ d\in\mathbb{R}^\times$.

(e) *The representations* $\pi_M := \text{Ind}\,{}_A^{GL_{2,2}}\tau_M\,(M\in GL(2,\mathbb{R}))$ *of* $GL_{2,2}$ *induced from the unitary character* τ_M *of* A *defined by* $\tau_M(X)=e^{2\pi i\sigma(MX)},\ X\in A$.

Proof. We leave the detail of the proof to the reader. □

(I-3) $m>2$.

This case is more complicated than the above cases. Here we consider only the case $m=3$. The other case $m\ge 4$ may be dealt similarly.

Lemma 3.7. *Let* $n=2$ *and* $m=3$*. That is,* $A=\mathbb{R}^{(3,2)}$*. Then the* $GL_{2,3}$*-orbits in* $\hat{A}$ *are given by*

$$\begin{aligned}
\Omega_{[23];0} &= \left\{\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix}\right\},\\
\Omega_{[23];1} &= \left\{\begin{pmatrix}A\\0\\0\end{pmatrix}\in\mathbb{R}^{(3,2)}\,\Big|\,A\in\mathbb{R}^{(1,2)},\ A\neq 0\right\},\\
\Omega_{[22];2} &= \left\{\begin{pmatrix}0\\A\\0\end{pmatrix}\in\mathbb{R}^{(3,2)}\,\Big|\,A\in\mathbb{R}^{(1,2)},\ A\neq 0\right\},\\
\Omega_{[23];3} &= \left\{\begin{pmatrix}0\\0\\A\end{pmatrix}\in\mathbb{R}^{(3,2)}\,\Big|\,A\in\mathbb{R}^{(1,2)},\ A\neq 0\right\},
\end{aligned}$$

$$\begin{aligned}
\Omega_{[23]}(1;\delta) &= \left\{\begin{pmatrix}0\\A\\\delta A\end{pmatrix}\in\mathbb{R}^{(3,2)}\,\Big|\,A\in\mathbb{R}^{(1,2)},\ A\neq 0\right\}\quad(\delta\in\mathbb{R}^\times),\\
\Omega_{[23]}(2;\delta) &= \left\{\begin{pmatrix}A\\0\\\delta A\end{pmatrix}\in\mathbb{R}^{(3,2)}\,\Big|\,A\in\mathbb{R}^{(1,2)},\ A\neq 0\right\}\quad(\delta\in\mathbb{R}^\times),\\
\Omega_{[23]}(3;\delta) &= \left\{\begin{pmatrix}A\\\delta A\\0\end{pmatrix}\in\mathbb{R}^{(3,2)}\,\Big|\,A\in\mathbb{R}^{(1,2)},\ A\neq 0\right\}\quad(\delta\in\mathbb{R}^\times),
\end{aligned}$$

$$\Omega_{[23]}(\lambda,\mu) = \left\{ \begin{pmatrix} A \\ \lambda A \\ \mu A \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\Big|\, A \in \mathbb{R}^{(1,2)},\ A \neq 0 \right\} \quad (\lambda,\, \mu \in \mathbb{R}^{\times})$$

and

$$\Omega_{12}(\lambda,\mu) = \left\{ \begin{pmatrix} A \\ B \\ \lambda A + \mu B \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\Big|\, \begin{pmatrix} A \\ B \end{pmatrix} \in GL_2 \right\} \quad (\lambda, \mu \in \mathbb{R}),$$

$$\Omega_{13}(\lambda,\mu) = \left\{ \begin{pmatrix} A \\ \lambda A + \mu B \\ B \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\Big|\, \begin{pmatrix} A \\ B \end{pmatrix} \in GL_2 \right\} \quad (\lambda, \mu \in \mathbb{R}),$$

$$\Omega_{23}(\lambda,\mu) = \left\{ \begin{pmatrix} \lambda A + \mu B \\ A \\ B \end{pmatrix} \in \mathbb{R}^{(3,2)} \,\Big|\, \begin{pmatrix} A \\ B \end{pmatrix} \in GL_2 \right\} \quad (\lambda, \mu \in \mathbb{R}).$$

Proof. It is easy to prove the above lemma. We leave the proof to the reader. □

We put

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and for each $\delta,\ \lambda,\ \mu \in \mathbb{R}^{\times}$

$$f_{1,\delta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ \delta & 0 \end{pmatrix}, \quad f_{2,\delta} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \delta & 0 \end{pmatrix}, \quad f_{3,\delta} = \begin{pmatrix} 1 & 0 \\ \delta & 0 \\ 0 & 0 \end{pmatrix},$$

$$f_{\lambda,\mu} = \begin{pmatrix} 1 & 0 \\ \lambda & 0 \\ \mu & 0 \end{pmatrix}.$$

We also set for each $(\lambda,\mu) \in \mathbb{R}^2$,

$$h_{12}(\lambda,\mu) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \lambda & \mu \end{pmatrix}, \quad h_{13}(\lambda,\mu) = \begin{pmatrix} 1 & 0 \\ \lambda & \mu \\ 0 & 1 \end{pmatrix}$$

and

$$h_{23}(\lambda,\mu) = \begin{pmatrix} \lambda & \mu \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We note that $\mathbf{0} \in \Omega_{[23];0}$, $e_i \in \Omega_{[23];i}\,(i = 1,2,3)$, $f_{j,\delta} \in \Omega_{[23]}(j;\delta)\,(j = 1,2,3)$, $f_{\lambda,\mu} \in \Omega_{[23]}(\lambda,\mu)$, $h_{12}(\lambda,\mu) \in \Omega_{12}(\lambda,\mu)$, $h_{13}(\lambda,\mu) \in \Omega_{13}(\lambda,\mu)$, $h_{23}(\lambda,\mu) \in \Omega_{23}(\lambda,\mu)$.

Then we may prove the following lemma without difficulty.

Lemma 3.8.

(a) *The stabilizer of* $\mathbf{0}$ *is* $GL_{2,3}$.

(b) *Let* $GL_{2,3}(i)$ *be the stabilizer of* e_i $(i = 1, 2, 3)$. *Then*

$$GL_{2,3}(i) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \ \alpha \right) \mid c \in \mathbb{R}, \ d \in \mathbb{R}^{\times}, \ \alpha \in \mathbb{R}^{(3,2)} \right\}, \quad i = 1, 2, 3.$$

(c) *For* $\delta \in \mathbb{R}^{\times}$, *we let* $GL_{2,3}(i;\delta)$ *be the stabilizer of* $f_{i,\delta}$ $(i = 1, 2, 3)$. *Then*

$$GL_{2,3}(i,\delta) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \ \alpha \right) \mid c \in \mathbb{R}, \ d \in \mathbb{R}^{\times}, \ \alpha \in \mathbb{R}^{(3,2)} \right\}, \quad i = 1, 2, 3.$$

(d) *For any* $\lambda, \ \mu \in \mathbb{R}^{\times}$, *we let* $GL_{2,3}(\lambda,\mu)$ *be the stabilizer of* $f_{\lambda,\mu}$. *Then*

$$GL_{2,3}(\lambda,\mu) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \ \alpha \right) \mid c \in \mathbb{R}, \ d \in \mathbb{R}^{\times}, \ \alpha \in \mathbb{R}^{(3,2)} \right\} \quad (\lambda, \mu \in \mathbb{R}^{\times}).$$

(e) *For any* $\lambda, \ \mu \in \mathbb{R}^{\times}$, *we let* $GL_{2,3}(12;\lambda,\mu)$, $GL_{2,3}(13;\lambda,\mu)$, $GL_{2,3}(23;\lambda,\mu)$ *be the stabilizers of* $h_{12}(\lambda,\mu)$, $h_{13}(\lambda,\mu)$, $h_{23}(\lambda,\mu)$ *respectively. Then*

$$GL_{2,3}(12;\lambda,\mu) = GL_{2,3}(13;\lambda,\mu) = GL_{2,3}(23;\lambda,\mu) = \left\{ (I_2, \alpha) \mid \alpha \in \mathbb{R}^{(3,2)} \right\}.$$

Therefore we see easily that A *is regularly embedded.*

According to Theorem 3.2, we obtain the following.

Theorem 3.9. *Let* $n = 2$ *and* $m = 3$. *Then the irreducible unitary representations of* $GL_{2,3}$ *are the following*:

(a) *The irreducible unitary representations* π, *where the restriction of* π *to* A *is trivial and the restriction of* π *to* $GL(2,\mathbb{R})$ *is an irreducible unitary representation of* $GL(2,\mathbb{R})$.

(b) *The representations* $\pi_{1,\lambda} := \mathrm{Ind}\,_{GL_{2,3}(1)}^{GL_{2,3}} \tau_{1,\lambda}$ $(\lambda \in \mathbb{R})$ *induced from the unitary representation* $\tau_{1,\lambda}$ *of* $GL_{2,3}(1)$ *defined by*

$$\tau_{1,\lambda}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = e^{2\pi i \alpha_1} \left(\mathrm{Ind}_{\mathbb{R}}^{\mathbb{R} \rtimes \mathbb{R}^{\times}} \chi_\lambda \right) \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \right),$$

where $c, \alpha_1, \cdots, \alpha_6 \in \mathbb{R}$ *and* $d \in \mathbb{R}^{\times}$.

(c) *The representations* $\pi_{2,\lambda} := \mathrm{Ind}\,_{GL_{2,3}(2)}^{GL_{2,3}} \tau_{2,\lambda}$ $(\lambda \in \mathbb{R})$ *induced from the unitary representation* $\tau_{2,\lambda}$ *of* $GL_{2,3}(2)$ *defined by*

$$\tau_{2,\lambda}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix} \right) \right) = e^{2\pi i \alpha_3} \cdot \left(\mathrm{Ind}_{\mathbb{R}}^{\mathbb{R} \rtimes \mathbb{R}^{\times}} \chi_\lambda \right) \left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \right),$$

where $c, \alpha_1, \cdots, \alpha_6 \in \mathbb{R}$ *and* $d \in \mathbb{R}^{\times}$.

(d) *The representations* $\pi_{3,\lambda} := \text{Ind}\,{}_{GL_{2,3}(3)}^{GL_{2,3}} \tau_{3,\lambda}\,(\lambda \in \mathbb{R})$ *induced from the unitary representation* $\tau_{3,\lambda}$ *of* $GL_{2,3}(3)$ *defined by*

$$\tau_{3,\lambda}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix}\right)\right) = e^{2\pi i \alpha_5} \cdot \left(\text{Ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^\times} \chi_\lambda\right)\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right),$$

where $c, \alpha_1, \cdots, \alpha_6 \in \mathbb{R}$ *and* $d \in \mathbb{R}^\times$.

(e) *The representations* $\pi_{(1,\delta),\lambda} := \text{Ind}\,{}_{GL_{2,3}(1;\delta)}^{GL_{2,3}} \tau_{(1,\delta),\lambda}\,(\delta \in \mathbb{R}^\times,\ \lambda \in \mathbb{R})$ *induced from the unitary representation* $\tau_{(1,\delta),\lambda}$ *of* $GL_{2,3}(1;\delta)$ *defined by*

$$\tau_{(1,\delta),\lambda}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix}\right)\right) = e^{2\pi i (\alpha_3 + \delta\alpha_5)} \cdot \left(\text{Ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^\times} \chi_\lambda\right)\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right),$$

where $c, \alpha_1, \cdots, \alpha_6 \in \mathbb{R}$ *and* $d \in \mathbb{R}^\times$.

(f) *The representations* $\pi_{(2,\delta),\lambda} := \text{Ind}\,{}_{GL_{2,3}(2;\delta)}^{GL_{2,3}} \tau_{(2,\delta),\lambda}\,(\delta \in \mathbb{R}^\times,\ \lambda \in \mathbb{R})$ *induced from the unitary representation* $\tau_{(2,\delta),\lambda}$ *of* $GL_{2,3}(2;\delta)$ *defined by*

$$\tau_{(2,\delta),\lambda}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix}\right)\right) = e^{2\pi i (\alpha_1 + \delta\alpha_5)} \cdot \left(\text{Ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^\times} \chi_\lambda\right)\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right),$$

where $c, \alpha_1, \cdots, \alpha_6 \in \mathbb{R}$ *and* $d \in \mathbb{R}^\times$.

(g) *The representations* $\pi_{(3,\delta),\lambda} := \text{Ind}\,{}_{GL_{2,3}(3;\delta)}^{GL_{2,3}} \tau_{(3,\delta),\lambda}\,(\delta \in \mathbb{R}^\times,\ \lambda \in \mathbb{R})$ *induced from the unitary representation* $\tau_{(3,\delta),\lambda}$ *of* $GL_{2,3}(3;\delta)$ *defined by*

$$\tau_{(3,\delta),\lambda}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix}\right)\right) = e^{2\pi i (\alpha_1 + \delta\alpha_3)} \cdot \left(\text{Ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^\times} \chi_\lambda\right)\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right),$$

where $c, \alpha_1, \cdots, \alpha_6 \in \mathbb{R}$ *and* $d \in \mathbb{R}^\times$.

(h) *The representations* $\pi_{(r;\lambda,\mu)} := \text{Ind}\,{}_{GL_{2,3}(\lambda,\mu)}^{GL_{2,3}} \tau_{(\lambda,\mu),r}\,(r \in \mathbb{R},\ \lambda, \mu \in \mathbb{R}^\times)$ *induced from the unitary representation* $\tau_{(\lambda,\mu),r}$ *of* $GL_{2,3}(\lambda,\mu)$ *defined by*

$$\tau_{(\lambda,\mu),r}\left(\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{pmatrix}\right)\right) = e^{2\pi i (\alpha_1 + \lambda\alpha_3 + \mu\alpha_5)} \cdot \left(\text{Ind}_{\mathbb{R}}^{\mathbb{R}\rtimes\mathbb{R}^\times} \chi_\lambda\right)\left(\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix}\right),$$

where $c, \alpha_1, \cdots, \alpha_6 \in \mathbb{R}$ *and* $d \in \mathbb{R}^\times$.

(i) *The representations* $\pi_{(12;\lambda,\mu)} := \text{Ind}\,{}^{GL_{2,3}}_{GL_{2,3}(12;\lambda,\mu)}\tau_{(12;\lambda,\mu)}$ $(\lambda,\mu\in\mathbb{R})$ *induced from the unitary representation* $\tau_{(12;\lambda,\mu)}$ *of* $GL_{2,3}(12;\lambda,\mu)$ *defined by*

$$\tau_{(12;\lambda,\mu)}\left(\left(I_2,\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\\ \alpha_5 & \alpha_6\end{pmatrix}\right)\right)=e^{2\pi i(\alpha_1+\lambda\alpha_5+(\alpha_4+\mu\alpha_6))},\quad \alpha_1,\cdots,\alpha_6\in\mathbb{R}.$$

(j) *The representations* $\pi_{(13;\lambda,\mu)} := \text{Ind}\,{}^{GL_{2,3}}_{GL_{2,3}(13;\lambda,\mu)}\tau_{(13;\lambda,\mu)}$ $(\lambda,\mu\in\mathbb{R})$ *induced from the unitary representation* $\tau_{(13;\lambda,\mu)}$ *of* $GL_{2,3}(13;\lambda,\mu)$ *defined by*

$$\tau_{(13;\lambda,\mu)}\left(\left(I_2,\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\\ \alpha_5 & \alpha_6\end{pmatrix}\right)\right)=e^{2\pi i(\alpha_1+\lambda\alpha_3+(\alpha_6+\mu\alpha_4))},\quad \alpha_1,\cdots,\alpha_6\in\mathbb{R}.$$

(k) *The representations* $\pi_{(23;\lambda,\mu)} := \text{Ind}\,{}^{GL_{2,3}}_{GL_{2,3}(23;\lambda,\mu)}\tau_{(23;\lambda,\mu)}$ $(\lambda,\mu\in\mathbb{R})$ *induced from the unitary representation* $\tau_{(23;\lambda,\mu)}$ *of* $GL_{2,3}(23;\lambda,\mu)$ *defined by*

$$\tau_{(23;\lambda,\mu)}\left(\left(I_2,\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\\ \alpha_5 & \alpha_6\end{pmatrix}\right)\right)=e^{2\pi i(\alpha_3+\lambda\alpha_1+(\alpha_6+\mu\alpha_2))},\quad \alpha_1,\cdots,\alpha_6\in\mathbb{R}.$$

Proof. We leave the detail of the proof to the reader. □

Case II. $n=3$.

(II-1) $m=1$.

In this case, $A\cong\mathbb{R}^{(1,3)}=\mathbb{R}^3$. We identify the unitary dual $\hat{A}$ of A with $\mathbb{R}^3$. According to (3.5), we see that $GL_{3,1}$-orbits in $\hat{A}$ consists of two orbits $\Omega_{[31];0}$, $\Omega_{[31];1}$ given by

$$\Omega_0=\{(0,0,0)\},\quad \Omega_1=\mathbb{R}^3-\{(0,0,0)\}.$$

We note that Ω_0 is the $GL_{3,1}$-orbit of $(0,0,0)$ and Ω_1 is a $GL_{3,1}$-orbit of any element different from $(0,0,0)$. We put $e=(1,0,0)$. Then the stabilizer $GL_{3,1}(e)$ of e is given by

$$GL_{3,1}(e)=\left\{\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix},\alpha\right)\in GL_{3,1}\;\Big|\; a\in\mathbb{R}^{(2,1)},\ g\in GL_2,\ \alpha\in\mathbb{R}^{(1,3)}\right\}.$$

According to Theorem 3.2, we obtain the following.

Theorem 3.10. *Let* $n=3$ *and* $m=1$. *Then the irreducible unitary representations of* $GL_{3,1}$ *are the following*:

(a) *The irreducible unitary representation* π, *where the restriction of* π *to* A *is trivial and the restriction of* π *to* GL_3 *is an irreducible unitary representation of* GL_3.

(b) *The representation* $\pi_\nu := \mathrm{Ind}_{GL_{3,1}(e)}^{GL_{3,1}} \sigma_\nu$ *induced from the unitary representation* σ_ν *of* $GL_{3,1}(e)$ *defined by*

$$\sigma\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}, (\alpha_1,\alpha_2,\alpha_3)\right) = e^{2\pi i\alpha_1}\left(\mathrm{Ind}_{\mathbb{R}^2}^{\mathbb{R}^2\ltimes GL_2}\theta_\nu\right)\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}\right),$$

where $\theta_\nu\,(\nu \in \mathbb{R}^2)$ *is the unitary character of* $\mathbb{R}^2$ *defined by* $\theta_\nu(a) = e^{2\pi i({}^t\nu a)}\,(a \in \mathbb{R}^2)$. *We note that* $GL_{3,1}(e)$ *is isomorphic to the group* $\mathbb{R}^2 \ltimes GL_2 \cong GL_{2,1}$. *We already dealt with the unitary representations of* $GL_{2,1}$.

(II-2) $m = 2$.

In this case, $A \cong \mathbb{R}^{(2,3)} \cong \hat{A}$.

Lemma 3.11. *Let* $n = 3$ *and* $m = 2$. *Then the* $GL_{3,2}$*-orbits in* $\hat{A}$ *consist of the following orbits*:

$$\begin{aligned}
\Omega_{[32];0} &= \left\{\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\right\},\\
\Omega_{[32];1} &= \left\{\begin{pmatrix}A\\ 0\end{pmatrix} \in \mathbb{R}^{(2,3)} \,\Big|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0\right\},\\
\Omega_{[32];2} &= \left\{\begin{pmatrix}0\\ A\end{pmatrix} \in \mathbb{R}^{(2,3)} \,\Big|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0\right\},\\
\Omega_{[32];3}(\delta) &= \left\{\begin{pmatrix}A\\ \delta A\end{pmatrix} \in \mathbb{R}^{(2,3)} \,\Big|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0\right\} \quad (\delta \in \mathbb{R}^\times)
\end{aligned}$$

and

$$\Omega_{[32];4} = \left\{M \in \mathbb{R}^{(2,3)} \mid \mathrm{rank}\, M = 2\right\}.$$

$\Omega_{[32];0}$ *is the* $GL_{3,2}$*-orbit of* $\mathbf{0}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $\Omega_{[32];1}$ *is the* $GL_{3,2}$*-orbit of* $\begin{pmatrix}\alpha\\ 0\end{pmatrix}$ *with* $0 \neq \alpha \in \mathbb{R}^{(1,3)}$, $\Omega_{[32];2}$ *is the* $GL_{3,2}$*-orbit of* $\begin{pmatrix}0\\ \beta\end{pmatrix}$ *with* $0 \neq \beta \in \mathbb{R}^{(1,3)}$, $\Omega_{[32];3}(\delta)$ *is the* $GL_{3,2}$*-orbit of* $\begin{pmatrix}\alpha\\ \delta\alpha\end{pmatrix}$ *with* $0 \neq \alpha \in \mathbb{R}^{(1,3)}$ *and* $\Omega_{[32];4}$ *is the* $GL_{3,2}$*-orbit of any invertible matrix* $M \in \mathbb{R}^{(2.3)}$ *with* $\mathrm{rank}\, M = 2$.

Proof. Without difficulty we may prove the above lemma. We note that $\Omega_{[32];3}(\delta_1) = \Omega_{[32];3}(\delta_1)$. So we leave the detail to the reader. □

We put

$$e^* = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \text{and} \quad f^* = \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}.$$

Obviously $e^* \in \Omega_{[32];1}$ and $f^* \in \Omega_{[32];2}$.

Then we may prove the following lemma.

Lemma 3.12.

(a) *The stabilizer of* $\mathbf{0}$ *is* $GL_{3,2}$.

(b) *The stabilizer* $GL_{3,2}(e^*)$ *of* e *is given by*

$$GL_{3,2}(e^*) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ \alpha & g \end{pmatrix}, \alpha \right) \,\middle|\, a \in \mathbb{R}^{(2,1)},\ g \in GL_2,\ \alpha \in \mathbb{R}^{(2,3)} \right\}. \tag{3.7}$$

For each $x \in \Omega_{[32];1}$, *the stabilizer* $GL_{3,2}(x)$ *of* x *is conjugate to* $GL_{3,2}(e^*)$. *Precisely if* $x = e^* g_0$ *with* $g_0 \in GL_3$, *then* $GL_{3,2}(x) = (g_0, 0)^{-1} GL_{3,2}(e^*)(g_0, 0)$.

(c) *The stabilizer* $GL_{3,2}(f^*)$ *of* f^* *is given by (3.6).*

(d) *The stabilizer* $GL_{3,2}(\delta)$ *of* $\begin{pmatrix} 1 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}$ $(\delta \in \mathbb{R}^\times)$ *is given by (3.6).*

(e) *The stabilizer* $GL_{3,2}(M)$ *of* $M \in \Omega_{[32];4}$ *is given by*

$$GL_{3,2}(M) = \left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}, \alpha \right) \,\middle|\, a, b \in \mathbb{R},\ c \in \mathbb{R}^\times,\ \alpha \in \mathbb{R}^{(2,3)} \right\}.$$

Therefore A *is regularly embedded.*

According to Theorem 3.2, we obtain the following.

Theorem 3.13. *Let* $n = 3$ *and* $m = 2$. *Then the irreducible unitary representations of* $GL_{3,2}$ *are the following*:

(a) *The irreducible unitary representations* ρ, *where the restriction of* ρ *to* A *is trivial and the restriction of* ρ *to* GL_3 *is an irreducible unitary representation of* GL_3.

(b) *The representations* $\rho_{e^*} := \mathrm{Ind}_{GL_{3,2}(e^*)}^{GL_{3,2}} \tau_{e^*}$ *induced from the irreducible unitary representation* τ_{e^*} *of* $GL_{3,2}(e^*)$. *Here* τ_{e^*} *is of the form*

$$\tau_{e^*}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \end{pmatrix} \right) = e^{2\pi\alpha_1} \cdot \pi\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix} \right),$$

where π *is an irreducible unitary representation of* $\mathbb{R}^2 \rtimes GL_2$ *given by Theorem 3.3.*

(c) *The representations* $\rho_{f^*} := \mathrm{Ind}_{GL_{3,2}(f^*)}^{GL_{3,2}} \tau_{f^*}$ *induced from the irreducible unitary representation* τ_{f^*} *of* $GL_{3,2}(f^*)$. *Here* τ_{f^*} *is of the form*

$$\tau_{f^*}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \end{pmatrix} \right) = e^{2\pi\alpha_4} \cdot \pi\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix} \right),$$

where π *is an irreducible unitary representation of* $\mathbb{R}^2 \rtimes GL_2$ *given by Theorem 3.3.*

(d) *The representations* $\rho_\delta := \mathrm{Ind}_{GL_{3,2}(\delta)}^{GL_{3,2}} \tau_\delta$ *induced from the irreducible unitary representation* τ_δ *of* $GL_{3,2}(\delta)$ *defined by*

$$\tau_\delta\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \end{pmatrix}\right) = e^{2\pi\alpha_1 + \delta\alpha_4} \cdot \pi\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where π *is an irreducible unitary representation of* $\mathbb{R}^2 \rtimes GL_2$ *given by Theorem 3.3.*

(e) *The representations* $\rho_M := \mathrm{Ind}_{GL_{3,2}(M)}^{GL_{3,2}} \tau_M$ $(M \in \mathbb{R}^{(2,3)}$ *with* $\mathrm{rank}\, M = 2)$ *of* $GL_{3,2}$ *induced from the unitary character* τ_M *of* $GL_{3,2}(M)$ *defined by*

$$\tau_M(X)\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \end{pmatrix}\right) = e^{2\pi i(\alpha_1+\alpha_5)} \cdot \pi_M\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}\right),$$

where π_M *is an irreducible unitary representation of* $\mathbb{R}^2 \rtimes GL_1$.

Proof. We leave the detail of the proof to the reader. □

(II-3) $m = 3$.

In this case, $A = \mathbb{R}^{(3,3)}$.

Lemma 3.14. *Let* $n = 3$ *and* $m = 3$. *That is,* $A = \mathbb{R}^{(3,3)}$. *Then the* $GL_{3,3}$*-orbits in* $\hat{A}$ *consist of the following orbits*:

$$\begin{aligned}
\Omega_{[33];0} &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \\
\Omega_{[33];1} &= \left\{ \begin{pmatrix} A \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\}, \\
\Omega_{[33];2} &= \left\{ \begin{pmatrix} 0 \\ A \\ 0 \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\}, \\
\Omega_{[33];3} &= \left\{ \begin{pmatrix} 0 \\ 0 \\ A \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\},
\end{aligned}$$

$$\begin{aligned}
\Omega_{[33]}(1;\delta) &= \left\{ \begin{pmatrix} 0 \\ A \\ \delta A \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \quad (\delta \in \mathbb{R}^\times), \\
\Omega_{[33]}(2;\delta) &= \left\{ \begin{pmatrix} A \\ 0 \\ \delta A \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \quad (\delta \in \mathbb{R}^\times),
\end{aligned}$$

$$\Omega_{[33]}(3;\delta) = \left\{ \begin{pmatrix} A \\ \delta\, A \\ 0 \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\Big|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \quad (\delta \in \mathbb{R}^\times),$$

$$\Omega_{[33]}(\lambda,\mu) = \left\{ \begin{pmatrix} A \\ \lambda\, A \\ \mu\, A \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\Big|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \quad (\lambda,\, \mu \in \mathbb{R}^\times)$$

and

$$\Omega_{12;\lambda,\mu} = \left\{ \begin{pmatrix} A \\ B \\ \lambda A + \mu B \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\Big|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\} \quad (\lambda, \mu \in \mathbb{R}),$$

$$\Omega_{13;\lambda,\mu} = \left\{ \begin{pmatrix} A \\ \lambda A + \mu B \\ B \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\Big|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\} \quad (\lambda, \mu \in \mathbb{R}),$$

$$\Omega_{23;\lambda,\mu} = \left\{ \begin{pmatrix} \lambda A + \mu B \\ A \\ B \end{pmatrix} \in \mathbb{R}^{(3,3)} \,\Big|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\} \quad (\lambda, \mu \in \mathbb{R}),$$

$$\Omega_{[33];*} = GL_3.$$

Proof. It is easy to prove the above lemma. We leave the proof to the reader. □

We put

$$\theta_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \theta_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and for each $\delta,\ \lambda,\ \mu \in \mathbb{R}^\times$

$$\theta_{1,\delta} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}, \quad \theta_{2,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}, \quad \theta_{3,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\theta_{\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}.$$

We also set for each $\lambda,\ \mu \in \mathbb{R}^\times$,

$$\phi_{12;\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & \mu & 0 \end{pmatrix}, \quad \phi_{13;\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & \mu & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \quad \phi_{23;\lambda,\mu} = \begin{pmatrix} \lambda & \mu & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We note that $\theta_i \in \Omega_{[34];i}\,(i = 1,2,3)$, $\theta_{j,\delta} \in \Omega_{[33]}(j;\delta)\,(j = 1,2,3)$, $\theta_{\lambda,\mu} \in \Omega_{[33]}(\lambda,\mu)$, $\phi_{12;\lambda,\mu} \in \Omega_{12;\lambda,\mu}$, $\phi_{13;\lambda,\mu} \in \Omega_{13;\lambda,\mu}, \phi_{23;\lambda,\mu} \in \Omega_{23;\lambda,\mu}$.

Then by a simple calculation, we may prove the following lemma without difficulty.

Lemma 3.15.

(a) *The stabilizer of* $\mathbf{0}$ *is* $GL_{3,3}$.

(b) *The stabilizer* $GL_{3,3}(1)$ *of* θ_1 *is given by*

$$GL_{3,3}(1) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \alpha \right) \,\Big|\, a \in \mathbb{R}^{(2,1)},\ g \in GL_2,\ \alpha \in \mathbb{R}^{(3,3)} \right\}.$$

(c) *The stabilizer* $GL_{3,3}(2)$ *of* θ_2 *is given by*

$$GL_{3,3}(2) = \left\{ \left(\begin{pmatrix} * & * & * \\ 0 & 1 & 0 \\ * & * & * \end{pmatrix}, \alpha \right) \in GL_{3,3} \,\Big|\, \alpha \in \mathbb{R}^{(3,3)} \right\}.$$

(d) *The stabilizer* $GL_{3,3}(3)$ *of* θ_3 *is given by*

$$GL_{3,3}(3) = \left\{ \left(\begin{pmatrix} g & a \\ 0 & 1 \end{pmatrix}, \alpha \right) \,\Big|\, a \in \mathbb{R}^{(2,1)},\ g \in GL_2,\ \alpha \in \mathbb{R}^{(3,3)} \right\}.$$

(e) *The stabilizer* $GL_{3,3}(i;\delta)$ *of* $\theta_{i;\delta} (i = 1, 2, 3)$ *is given by*

$$GL_{3,3}(i;\delta) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \alpha \right) \,\Big|\, a \in \mathbb{R}^{(2,1)},\ g \in GL_2,\ \alpha \in \mathbb{R}^{(3,3)} \right\}.$$

(f) *The stabilizer* $GL_{3,3}(\lambda,\mu)$ *of* $\theta_{\lambda,\mu}$ *is given by*

$$GL_{3,3}(\lambda,\mu) = \left\{ \left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \alpha \right) \,\Big|\, a \in \mathbb{R}^{(2,1)},\ g \in GL_2,\ \alpha \in \mathbb{R}^{(3,3)} \right\}.$$

(g) *The stabilizers* $GL_{3,3}(12;\lambda,\mu)$, $GL_{3,3}(13;\lambda,\mu)$, $GL_{3,3}(23;\lambda,\mu)$ *of* $\phi_{12;\lambda,\mu}$, $\phi_{13;\lambda,\mu}$, $\phi_{23;\lambda,\mu}$ *respectively are given by*

$$\begin{aligned} GL_{3,3}(12;\lambda,\mu) &= GL_{3,3}(13;\lambda,\mu) = GL_{3,3}(23;\lambda,\mu) \\ &= \left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}, \alpha \right) \in GL_{3,3} \,\Big|\, a, b \in \mathbb{R},\ c \in \mathbb{R}^{\times},\ \alpha \in \mathbb{R}^{(3,3)} \right\}. \end{aligned}$$

(h) *The stabilizer of* I_3 *is* $\{(I_3, 0) |\ \alpha \in \mathbb{R}^{(3,3)}\} \cong A$.

According to Theorem 3.2, we obtain the following.

Theorem 3.16. *Let* $n = 3$ *and* $m = 3$. *Then irreducible unitary representations of* $GL_{3,3}$ *are the following.*

(a) *The irreducible unitary representations ρ, where the restriction of ρ to A is trivial and the restriction of ρ to GL_3 is an irreducible unitary representation of GL_3.*

(b) *The representation $\rho_{\theta_1} := \mathrm{Ind}\,{}_{GL_{3,3}(1)}^{GL_{3,3}} \tau_{\theta_1}$ induced from the unitary representation τ_{θ_1} of $GL_{3,3}(1)$. Here τ_{θ_1} is of the form*

$$\tau_{\theta_1}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i \alpha_1} \cdot \pi_{\theta_1}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where π_{θ_1} is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$ given by Theorem 3.3.

(c) *The representation $\rho_{\theta_2} := \mathrm{Ind}\,{}_{GL_{3,3}(2)}^{GL_{3,3}} \tau_{\theta_2}$ induced from the unitary representation τ_{θ_2} of $GL_{3,3}(2)$. Here τ_{θ_2} is of the form*

$$\tau_{\theta_2}\left(\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ 0 & 1 & 0 \\ g_{31} & g_{32} & g_{33} \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i \alpha_5} \cdot \pi_{\theta_2}\left(\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ 0 & 1 & 0 \\ g_{31} & g_{32} & g_{33} \end{pmatrix}\right),$$

where π_{θ_2} is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$ given by Theorem 3.3.

(d) *The representation $\rho_{\theta_3} := \mathrm{Ind}\,{}_{GL_{3,3}(3)}^{GL_{3,3}} \tau_{\theta_3}$ induced from the unitary representation τ_{θ_3} of $GL_{3,3}(3)$. Here τ_{θ_3} is of the form*

$$\tau_{\theta_3}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i \alpha_9} \cdot \pi_{\theta_3}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where π_{θ_3} is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$ given by Theorem 3.3.

(e) *The representation $\rho_{1,\delta} := \mathrm{Ind}\,{}_{GL_{3,3}(1;\delta)}^{GL_{3,3}} \tau_{1,\delta}$ induced from the unitary representation $\tau_{1,\delta}$ of $GL_{3,3}(1,\delta)$. Here $\tau_{1,\delta}$ is of the form*

$$\tau_{1,\delta}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i (\alpha_4 + \delta\alpha_7)} \cdot \pi_{1,\delta}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi_{1,\delta}$ is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$ given by Theorem 3.3.

(f) *The representation $\rho_{2,\delta} := \mathrm{Ind}\,{}_{GL_{3,3}(2;\delta)}^{GL_{3,3}} \tau_{2,\delta}$ induced from the unitary representation $\tau_{2,\delta}$ of $GL_{3,3}(2,\delta)$. Here $\tau_{1,\delta}$ is of the form*

$$\tau_{2,\delta}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i (\alpha_1 + \delta\alpha_7)} \cdot \pi_{2,\delta}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi_{2,\delta}$ is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$ given by Theorem 3.3.

(g) *The representation $\rho_{3,\delta} := \mathrm{Ind}\,{}^{GL_{3,3}}_{GL_{3,3}(3,\delta)}\tau_{1,\delta}$ induced from the unitary representation $\tau_{1,\delta}$ of $GL_{3,3}(3,\delta)$. Here $\tau_{3,\delta}$ is of the form*

$$\tau_{3,\delta}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i(\alpha_1+\delta\alpha_4)} \cdot \pi_{3,\delta}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi_{3,\delta}$ is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$ given by Theorem 3.3.

(h) *The representation $\rho_{\lambda,\mu} := \mathrm{Ind}\,{}^{GL_{3,3}}_{GL_{3,3}(\ \mu)}\tau_{\lambda,\mu}$ induced from the unitary representation $\tau_{\lambda,\mu}$ of $GL_{3,3}(\lambda,\mu)$. Here $\tau_{\lambda,\mu}$ is of the form*

$$\tau_{\lambda,\mu}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i(\alpha_1+\lambda\alpha_4+\mu\alpha_7)} \cdot \pi_{\lambda,\mu}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi_{\lambda,\mu}$ is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$ given by Theorem 3.3.

(i) *The representation $\rho_{12;\lambda,\mu} := \mathrm{Ind}\,{}^{GL_{3,3}}_{GL_{3,3}(12;\lambda,\mu)}\tau_{12;\lambda,\mu}$ induced from the unitary representation $\tau_{12;\lambda,\mu}$ of $GL_{3,3}(12;\lambda,\mu)$. Here $\tau_{12;\lambda,\mu}$ is of the form*

$$\begin{aligned} &\tau_{12;\lambda,\mu}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) \\ = \ & e^{2\pi i(\alpha_1+\alpha_5+\lambda\alpha_7+\mu\alpha_8)} \cdot \pi_{12;\lambda,\mu}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}\right), \end{aligned}$$

where $\pi_{12;\lambda,\mu}$ is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_1$ given by Theorem 3.3.

(j) *The representation $\rho_{13;\lambda,\mu} := \mathrm{Ind}\,{}^{GL_{3,3}}_{GL_{3,3}(13;\lambda,\mu)}\tau_{13;\lambda,\mu}$ induced from the unitary representation $\tau_{13;\lambda,\mu}$ of $GL_{3,3}(13;\lambda,\mu)$. Here $\tau_{13;\lambda,\mu}$ is of the form*

$$\begin{aligned} &\tau_{13;\lambda,\mu}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & b & c \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) \\ = \ & e^{2\pi i(\alpha_1+\alpha_8+\lambda\alpha_4+\mu\alpha_5)} \cdot \pi_{13;\lambda,\mu}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & b & c \end{pmatrix}\right), \end{aligned}$$

where $\pi_{13;\lambda,\mu}$ is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_1$.

(k) *The representation* $\rho_{23;\lambda,\mu} := \text{Ind}\,_{GL_{3,3}(23;\lambda,\mu)}^{GL_{3,3}} \tau_{23;\lambda,\mu}$ *induced from the unitary representation* $\tau_{23;\lambda,\mu}$ *of* $GL_{3,3}(23;\lambda,\mu)$*. Here* $\tau_{23;\lambda,\mu}$ *is of the form*

$$\begin{aligned} &\tau_{23;\lambda,\mu}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}, \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) \\ = \;\; & e^{2\pi i(\alpha_4+\alpha_8+\lambda\alpha_1+\mu\alpha_2)} \cdot \pi_{23;\lambda,\mu}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}\right), \end{aligned}$$

where $\pi_{23;\lambda,\mu}$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_1$*.*

(l) *The representation* $\rho_{I_3} := \text{Ind}_A^{GL_{3,3}} \chi_{I_3}$ *induced from the unitary character* χ_{I_3} *of* A *given by*

$$\chi_{I_3}\left(\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \alpha_6 \\ \alpha_7 & \alpha_8 & \alpha_9 \end{pmatrix}\right) = e^{2\pi i(\alpha_1+\alpha_5+\alpha_9)}.$$

(II-4) $m = 4$.

In this case, $A = \mathbb{R}^{(4,3)}$.

Lemma 3.17. *Let* $n = 3$ *and* $m = 4$*. Then the* $GL_{3,4}$*-orbits in* $\hat{A}$ *consists of the following orbits*:

$$\begin{aligned} \Omega_{[34];0} &= \{0\}, \\ \Omega_{[34];1} &= \left\{ \begin{pmatrix} A \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\}, \\ \Omega_{[34];2} &= \left\{ \begin{pmatrix} 0 \\ A \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\}, \\ \Omega_{[34];3} &= \left\{ \begin{pmatrix} 0 \\ 0 \\ A \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\}, \\ \Omega_{[34];4} &= \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\}, \end{aligned}$$

$$\Omega_{12;\delta} = \left\{ \begin{pmatrix} 0 \\ 0 \\ A \\ \delta A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{13;\delta} = \left\{ \begin{pmatrix} 0 \\ A \\ 0 \\ \delta A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{14;\delta} = \left\{ \begin{pmatrix} 0 \\ A \\ \delta A \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{23;\delta} = \left\{ \begin{pmatrix} A \\ 0 \\ 0 \\ \delta A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{24;\delta} = \left\{ \begin{pmatrix} A \\ 0 \\ \delta A \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{34;\delta} = \left\{ \begin{pmatrix} A \\ \delta A \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\delta \in \mathbb{R}^{\times}),$$

$$\Omega_{1;\lambda,\mu} = \left\{ \begin{pmatrix} 0 \\ A \\ \lambda A \\ \mu A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\lambda, \mu \in \mathbb{R}^{\times}),$$

$$\Omega_{2;\lambda,\mu} = \left\{ \begin{pmatrix} A \\ 0 \\ \lambda A \\ \mu A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\lambda, \mu \in \mathbb{R}^{\times}),$$

$$\Omega_{3;\lambda,\mu} = \left\{ \begin{pmatrix} A \\ \lambda A \\ 0 \\ \mu A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\lambda, \mu \in \mathbb{R}^{\times}),$$

$$\Omega_{4;\lambda,\mu} = \left\{ \begin{pmatrix} A \\ \lambda A \\ \mu A \\ 0 \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\lambda, \mu \in \mathbb{R}^{\times}),$$

$$\Omega_{\lambda,\mu,\kappa} = \left\{ \begin{pmatrix} A \\ \lambda A \\ \mu A \\ \kappa A \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, A \in \mathbb{R}^{(1,3)},\ A \neq 0 \right\} \ (\lambda, \mu, \kappa \in \mathbb{R}^{\times})$$

and for any $\lambda,\ \mu,\ \kappa,\ \delta \in \mathbb{R}$,

$$\begin{aligned}
\Omega_{12;\lambda,\mu,\kappa,\delta} &= \left\{ \begin{pmatrix} A \\ B \\ \lambda A + \mu B \\ \kappa A + \delta B \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\}, \\
\Omega_{14;\lambda,\mu,\kappa,\delta} &= \left\{ \begin{pmatrix} A \\ \lambda A + \mu B \\ \kappa A + \delta B \\ B \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\}, \\
\Omega_{23;\lambda,\mu,\kappa,\delta} &= \left\{ \begin{pmatrix} \lambda A + \mu B \\ A \\ B \\ \kappa A + \delta B \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\}, \\
\Omega_{24;\lambda,\mu,\kappa,\delta} &= \left\{ \begin{pmatrix} \lambda A + \mu B \\ A \\ \kappa A + \delta B \\ B \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\}, \\
\Omega_{34;\lambda,\mu,\kappa,\delta} &= \left\{ \begin{pmatrix} \lambda A + \mu B \\ \kappa A + \delta B \\ A \\ B \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \end{pmatrix} = 2 \right\}, \\
\Omega_{123;\lambda,\mu,\kappa} &= \left\{ \begin{pmatrix} A \\ B \\ C \\ \lambda A + \mu B + \kappa C \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 3 \right\}, \\
\Omega_{124;\lambda,\mu,\kappa} &= \left\{ \begin{pmatrix} A \\ B \\ \lambda A + \mu B + \kappa C \\ C \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 3 \right\}, \\
\Omega_{134;\lambda,\mu,\kappa} &= \left\{ \begin{pmatrix} A \\ \lambda A + \mu B + \kappa C \\ B \\ C \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 3 \right\}, \\
\Omega_{234;\lambda,\mu,\kappa} &= \left\{ \begin{pmatrix} \lambda A + \mu B + \kappa C \\ A \\ B \\ C \end{pmatrix} \in \mathbb{R}^{(4,3)} \,\middle|\, \text{rank} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = 3 \right\}.
\end{aligned}$$

We put

$$\xi_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\xi_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and for each $\delta \in \mathbb{R}^\times$, we set

$$\xi_{12;\delta} = \begin{pmatrix} 1 & 0 & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi_{13;\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\xi_{14;\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}, \quad \xi_{23;\delta} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\xi_{24;\delta} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}, \quad \xi_{34;\delta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ \delta & 0 & 0 \end{pmatrix}.$$

For any $\lambda,\ \mu,\ \kappa \in \mathbb{R}^\times$, we put

$$\xi_{1;\lambda,\mu} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \lambda & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}, \quad \xi_{2;\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix},$$

$$\xi_{3;\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \\ \mu & 0 & 0 \end{pmatrix}, \quad \xi_{4;\lambda,\mu} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 0 & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\xi_{\lambda,\mu,\kappa} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 0 & 0 \\ \mu & 0 & 0 \\ \kappa & 0 & 0 \end{pmatrix}.$$

We also put for any $\lambda,\ \mu,\ \kappa,\ \delta \in \mathbb{R}$,

$$\xi_{12;\lambda,\mu,\kappa,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & \mu & 0 \\ \kappa & \delta & 0 \end{pmatrix}, \quad \xi_{13;\lambda,\mu,\kappa,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & \mu & 0 \\ \kappa & \delta & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\xi_{14;\lambda,\mu,\kappa,\delta} = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & \mu & 0 \\ \kappa & \delta & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \xi_{23;\lambda,\mu,\kappa,\delta} = \begin{pmatrix} \lambda & \mu & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ \kappa & \delta & 0 \end{pmatrix}$$

and

$$\xi_{24;\lambda,\mu,\kappa,\delta} = \begin{pmatrix} \lambda & \mu & 0 \\ 1 & 0 & 0 \\ \kappa & \delta & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Lemma 3.18.

(a) *The stabilizer of* $\mathbf{0}$ *is* $GL_{3,4}$.

(b) *The stabilizer* $GL_{3,4}(i)$ *of* $\xi_i\ (i = 1, 2, 3, 4)$ *is given by*

$$\left\{ \left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, \alpha \right) \in GL_{3,4} \,\middle|\, a \in \mathbb{R}^{(2,1)},\ g \in GL_2,\ \alpha \in \mathbb{R}^{(4,3)} \right\}. \tag{3.8}$$

(c) *The stabilizer* $GL_{3,4}(ij;\delta)$ *of* $\xi_{ij;\delta}\ (1 \leq i \leq j \leq 4)$ *is given by* (3.7).

(d) *The stabilizer* $GL_{3,4}(i;\lambda,\mu)$ *of* $\xi_{i;\lambda,\mu}\ (1 \leq i \leq 4)$ *is given by* (3.7).

(e) *The stabilizer* $GL_{3,4}(\lambda,\mu,\kappa)$ *of* $\xi_{\lambda,\mu,\kappa}$ *is given by* (3.7).

(f) *The stabilizer* $GL_{3,4}(ij;\lambda,\mu,\kappa,\delta)$ *of* $\xi_{ij;\lambda,\mu,\kappa,\delta}\ (1 \leq i \leq j \leq 4)$ *is given by*

$$\left\{ \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{pmatrix}, \alpha \right) \in GL_{3,4} \,\middle|\, a, b, c (\neq 0) \in \mathbb{R},\ \alpha \in \mathbb{R}^{(4,3)} \right\}.$$

According to Theorem 3.2, we obtain the following.

Theorem 3.19. *Let* $n = 3$ *and* $m = 4$. *We put*

$$\alpha = (\alpha_{ij}) = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} \end{pmatrix} \in \mathbb{R}^{(4,3)}.$$

Then the irreducible unitary representations of $GL_{3,4}$ *are the following*:

(a) *The irreducible unitary representation ρ, where the restriction of ρ to A is trivial and the restriction of ρ to GL_3 is an irreducible unitary representation of GL_3.*

(b) *The representation $\rho_{\xi_i} := \mathrm{Ind}\,_{GL_{3,4}(i)}^{GL_{3,4}} \tau_{\xi_i}$ $(1 \leq i \leq 4)$ induced from the unitary irreducible representation τ_{ξ_i} of $GL_{3,4}(i)$. Here τ_{ξ_i} is of the form*

$$\tau_{\xi_i}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, (\alpha_{ij})\right) = e^{2\pi i \alpha_{31}} \cdot \pi_{\xi_i}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where π_{ξ_i} is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$.

(c) *The representation $\rho_{\xi_{ij;\delta}} := \mathrm{Ind}\,_{GL_{3,4}(ij;\delta)}^{GL_{3,4}} \tau_{\xi_{ij;\delta}}$ $(1 \leq i \leq j \leq 4)$ induced from the unitary irreducible representation $\tau_{\xi_{ij;\delta}}$ of $GL_{3,4}(ij;\delta)$. Here $\tau_{\xi_{ij;\delta}}$ is of the form*

$$\tau_{\xi_{ij;\delta}}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, (\alpha_{ij})\right) = e^{2\pi i(\alpha_{i1} + \delta\alpha_{j1})} \cdot \pi_{\xi_{34;\delta}}\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi_{\xi_{34;\delta}}$ is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$.

(d) *The representation $\rho(\xi_{1;\lambda,\mu}) := \mathrm{Ind}\,_{GL_{3,4}(1;\lambda,\mu)}^{GL_{3,4}} \tau(\xi_{1;\lambda,\mu})$ induced from the unitary irreducible representation $\tau(\xi_{1;\lambda,\mu})$ of $GL_{3,4}(1;\lambda,\mu)$. Here $\tau(\xi_{1;\lambda,\mu})$ is of the form*

$$\tau(\xi_{1;\lambda,\mu})\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, (\alpha_{ij})\right) = e^{2\pi i(\alpha_{21} + \lambda\alpha_{31} + \mu\alpha_{41})} \cdot \pi(\xi_{1;\lambda,\mu})\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi(\xi_{1;\lambda,\mu})$ is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$.

(e) *The representation $\rho(\xi_{2;\lambda,\mu}) := \mathrm{Ind}\,_{GL_{3,4}(2;\lambda,\mu)}^{GL_{3,4}} \tau(\xi_{2;\lambda,\mu})$ induced from the unitary irreducible representation $\tau(\xi_{2;\lambda,\mu})$ of $GL_{3,4}(2;\lambda,\mu)$. Here $\tau(\xi_{2;\lambda,\mu})$ is of the form*

$$\tau(\xi_{2;\lambda,\mu})\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, (\alpha_{ij})\right) = e^{2\pi i(\alpha_{11} + \lambda\alpha_{31} + \mu\alpha_{41})} \cdot \pi(\xi_{2;\lambda,\mu})\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi(\xi_{2;\lambda,\mu})$ is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$.

(f) *The representation $\rho(\xi_{3;\lambda,\mu}) := \mathrm{Ind}\,_{GL_{3,4}(3;\lambda,\mu)}^{GL_{3,4}} \tau(\xi_{3;\lambda,\mu})$ induced from the unitary irreducible representation $\tau(\xi_{3;\lambda,\mu})$ of $GL_{3,4}(3;\lambda,\mu)$. Here $\tau(\xi_{3;\lambda,\mu})$ is of the form*

$$\tau(\xi_{3;\lambda,\mu})\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}, (\alpha_{ij})\right) = e^{2\pi i(\alpha_{11} + \lambda\alpha_{21} + \mu\alpha_{41})} \cdot \pi(\xi_{3;\lambda,\mu})\left(\begin{pmatrix} 1 & 0 \\ a & g \end{pmatrix}\right),$$

where $\pi(\xi_{3;\lambda,\mu})$ is the irreducible unitary representation of $\mathbb{R}^2 \ltimes GL_2$.

(g) *The representation* $\rho(\xi_{4;\lambda,\mu}) := \mathrm{Ind}\,{}_{GL_{3,4}(4;\lambda,\mu)}^{GL_{3,4}}\tau(\xi_{4;\lambda,\mu})$ *induced from the unitary irreducible representation* $\tau(\xi_{4;\lambda,\mu})$ *of* $GL_{3,4}(4;\lambda,\mu)$*. Here* $\tau(\xi_{4;\lambda,\mu})$ *is of the form*

$$\tau(\xi_{4;\lambda,\mu})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix},(\alpha_{ij})\right)=e^{2\pi i(\alpha_{11}+\lambda\alpha_{21}+\mu\alpha_{31})}\cdot\pi(\xi_{1;\lambda,\mu})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}\right),$$

where $\pi(\xi_{1;\lambda,\mu})$ *is the irreducible unitary representation of* $\mathbb{R}^2\ltimes GL_2$*.*

(h) *The representation* $\rho(\xi_{\lambda,\mu,\kappa}) := \mathrm{Ind}\,{}_{GL_{3,4}(\lambda,\mu,\kappa)}^{GL_{3,4}}\tau(\xi_{\lambda,\mu,\kappa})$ *induced from the unitary irreducible representation* $\tau(\xi_{\lambda,\mu,\kappa})$ *of* $GL_{3,4}(\lambda,\mu,\kappa)$*. Here* $\tau(\xi_{\lambda,\mu,\kappa})$ *is of the form*

$$\tau(\xi_{\lambda,\mu,\kappa})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix},(\alpha_{ij})\right)=e^{2\pi i(\alpha_{11}+\lambda\alpha_{21}+\mu\alpha_{31}+\kappa\alpha_{41})}\cdot\pi(\xi_{\lambda,\mu,\kappa})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}\right),$$

where $\pi(\xi_{\lambda,\mu,\kappa})$ *is the irreducible unitary representation of* $\mathbb{R}^2\ltimes GL_2$*.*

(i) *The representation* $\rho(\xi_{12;\lambda,\mu,\delta}) := \mathrm{Ind}\,{}_{GL_{3,4}(12;\lambda,\mu,\kappa,\delta)}^{GL_{3,4}}\tau(\xi_{12;\lambda,\mu,\kappa,\delta})$ *induced from the unitary irreducible representation* $\tau(\xi_{12;\lambda,\mu,\kappa,\delta})$ *of* $GL_{3,4}(12;\lambda,\mu,\kappa,\delta)$*. Here* $\tau(\xi_{12;\lambda,\mu,\kappa,\delta})$ *is of the form*

$$\begin{aligned}&\tau(\xi_{12;\lambda,\mu,\kappa,\mu})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix},(\alpha_{ij})\right)\\ =\;& e^{2\pi i(\alpha_{11}+\alpha_{22}+\lambda\alpha_{31}+\mu\alpha_{32}+\kappa\alpha_{41}+\delta\alpha_{42})}\cdot\pi(\xi_{12;\lambda,\mu,\kappa,\delta})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}\right),\end{aligned}$$

where $\pi(\xi_{12;\lambda,\mu,\kappa,\delta})$ *is the irreducible unitary representation of* $\mathbb{R}^2\ltimes GL_2$*.*

(j) *The representation* $\rho(\xi_{13;\lambda,\mu,\delta}) := \mathrm{Ind}\,{}_{GL_{3,4}(13;\lambda,\mu,\kappa,\delta)}^{GL_{3,4}}\tau(\xi_{13;\lambda,\mu,\kappa,\delta})$ *induced from the unitary irreducible representation* $\tau(\xi_{13;\lambda,\mu,\kappa,\delta})$ *of* $GL_{3,4}(13;\lambda,\mu,\kappa,\delta)$*. Here* $\tau(\xi_{13;\lambda,\mu,\kappa,\delta})$ *is of the form*

$$\begin{aligned}&\tau(\xi_{13;\lambda,\mu,\kappa,\mu})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix},(\alpha_{ij})\right)\\ =\;& e^{2\pi i(\alpha_{11}+\alpha_{32}+\lambda\alpha_{21}+\mu\alpha_{22}+\kappa\alpha_{41}+\delta\alpha_{42})}\cdot\pi(\xi_{13;\lambda,\mu,\kappa,\delta})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}\right),\end{aligned}$$

where $\pi(\xi_{13;\lambda,\mu,\kappa,\delta})$ *is the irreducible unitary representation of* $\mathbb{R}^2\ltimes GL_2$*.*

(k) *The representation* $\rho(\xi_{14;\lambda,\mu,\delta}) := \mathrm{Ind}\,{}_{GL_{3,4}(14;\lambda,\mu)}^{GL_{3,4}}\tau(\xi_{14;\lambda,\mu,\kappa,\delta})$ *induced from the unitary irreducible representation* $\tau(\xi_{14;\lambda,\mu,\kappa,\delta})$ *of* $GL_{3,4}(14;\lambda,\mu,\kappa,\delta)$*. Here* $\tau(\xi_{14;\lambda,\mu,\kappa,\delta})$ *is of the form*

$$\begin{aligned}&\tau(\xi_{14;\lambda,\mu,\kappa,\mu})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix},(\alpha_{ij})\right)\\ =\;& e^{2\pi i(\alpha_{11}+\alpha_{42}+\lambda\alpha_{21}+\mu\alpha_{22}+\kappa\alpha_{31}+\delta\alpha_{32})}\cdot\pi(\xi_{14;\lambda,\mu,\kappa,\delta})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}\right),\end{aligned}$$

where $\pi(\xi_{14;\lambda,\mu,\kappa,\delta})$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$.

(l) *The representation* $\rho(\xi_{23;\lambda,\mu,\delta}) := \text{Ind}\,^{GL_{3,4}}_{GL_{3,4}(23;\lambda,\mu)}\tau(\xi_{23;\lambda,\mu,\kappa,\delta})$ *induced from the unitary irreducible representation* $\tau(\xi_{23;\lambda,\mu,\kappa,\delta})$ *of* $GL_{3,4}(23;\lambda,\mu,\kappa,\delta)$. *Here* $\tau(\xi_{23;\lambda,\mu,\kappa,\delta})$ *is of the form*

$$\begin{aligned} & \tau(\xi_{23;\lambda,\mu,\kappa,\mu})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}, (\alpha_{ij})\right) \\ = \;& e^{2\pi i(\alpha_{21}+\alpha_{32}+\lambda\alpha_{11}+\mu\alpha_{12}+\kappa\alpha_{41}+\delta\alpha_{42})}\cdot \pi(\xi_{23;\lambda,\mu,\kappa,\delta})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}\right), \end{aligned}$$

where $\pi(\xi_{23;\lambda,\mu,\kappa,\delta})$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$.

(m) *The representation* $\rho(\xi_{24;\lambda,\mu,\delta}) := \text{Ind}\,^{GL_{3,4}}_{GL_{3,4}(24;\lambda,\mu)}\tau(\xi_{24;\lambda,\mu,\kappa,\delta})$ *induced from the unitary irreducible representation* $\tau(\xi_{24;\lambda,\mu,\kappa,\delta})$ *of* $GL_{3,4}(24;\lambda,\mu,\kappa,\delta)$. *Here* $\tau(\xi_{24;\lambda,\mu,\kappa,\delta})$ *is of the form*

$$\begin{aligned} & \tau(\xi_{24;\lambda,\mu,\kappa,\mu})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}, (\alpha_{ij})\right) \\ = \;& e^{2\pi i(\alpha_{21}+\alpha_{42}+\lambda\alpha_{11}+\mu\alpha_{12}+\kappa\alpha_{31}+\delta\alpha_{32})}\cdot \pi(\xi_{14;\lambda,\mu,\kappa,\delta})\left(\begin{pmatrix}1 & 0\\ a & g\end{pmatrix}\right), \end{aligned}$$

where $\pi(\xi_{14;\lambda,\mu,\kappa,\delta})$ *is the irreducible unitary representation of* $\mathbb{R}^2 \ltimes GL_2$.

Remark 3.20. The other cases $n \geq 4$ are more complicated than the previous cases $n = 2,\ 3$ but can be dealt with in a similar way.

We note that $GL_{n,m}$ acts on $\mathbb{R}^{(m,n)}$ on the right transitively by

$$x \cdot (g,a) := x\,{}^t g^{-1} + a, \quad x,\, a \in \mathbb{R}^{(m,n)},\ g \in GL(n,\mathbb{R}).$$

For $\lambda \in \mathbb{C}$, we define the representation π_λ of $GL_{n,m}$ on $L^2(\mathbb{R}^{(m,n)})$ by

$$(\pi_\lambda((g,a))f)(x) := |\det g|^{-\lambda} f(x\cdot(g,a)), \tag{3.9}$$

where $(g,a) \in GL_{n,m},\ f \in L^2(\mathbb{R}^{(m,n)})$. Then π_λ is unitary if and only if $\lambda \in \frac{1}{2} + i\mathbb{R}$. In fact,

$$\begin{aligned} \|\pi_\lambda((g,a))f\|^2_{L^2(\mathbb{R}^{(m,n)})} &= \int_{\mathbb{R}^{(m,n)}} |\det g|^{-\lambda} f(x\,{}^t g^{-1}+a)\overline{|\det g|^{-\lambda}}\;\overline{f(x\,{}^t g^{-1}+a)}dx \\ &= \int_{\mathbb{R}^{(m,n)}} |\det g|^{1-2\mathrm{Re}\,\lambda}|f(x)|^2 dx \\ &= |\det g|^{1-2\mathrm{Re}\,\lambda}\|f\|^2_{L^2(\mathbb{R}^{(m,n)})}. \end{aligned}$$

We recall the following fact.

Theorem 3.21. *Suppose H is a subgroup of GL_n and let $H_{m,n} := H \ltimes \mathbb{R}^{(m,n)}$. Then $\big(\pi_\lambda|_{H_{m,n}}, L^2(\mathbb{R}^{(m,n)})\big)$ is irreducible if and only if the action of $H_{m,n}$ on $\mathbb{R}^{(m,n)}$ is ergodic.*

According to the above theorem, if $\lambda \in \frac{1}{2} + i\mathbb{R}$, then π_λ is irreducible because $GL_{n,m}$ acts on $\mathbb{R}^{(m,n)}$ ergodically.

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