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Existence and Uniqueness Theorems for Certain Fourth-order Boundary Value Problems

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Abstract. In this paper, by using the Leray-Schauder continuation theorem and Wirtinger-type inequalities, we establish the existence and uniqueness theorems for two-point boundary value problems of a certain class of fourth-order nonlinear differential equations.

1. Introduction

Fourth-order two-point boundary value problems are essential in describing a vast class of elastic deflections, and attract close attention extensively, see the references.

In this paper, we consider the existence and uniqueness of solutions for general fourth-order nonlinear boundary value problems

$$y^{(4)} = f(x, y, y', y'', y'''), \quad 0 < x < 1, \tag{1.1}$$

with the boundary conditions

$$y(0) = y(1) = y''(0) = y''(1) = 0, \tag{1.2}$$

where $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ satisfies the Carathéodory's conditions, that is

(i) for each $(y, z, u, v) \in \mathbb{R}^4$, the function $x \in [0,1] \to f(x, y, z, u, v) \in \mathbb{R}$ is measurable on $[0,1]$;

(ii) for a.e. $x \in [0,1]$, the function $(y, z, u, v) \in \mathbb{R}^4 \to f(x, y, z, u, v) \in \mathbb{R}$ is continuous on $\mathbb{R}^4$;

(iii) for each $r > 0$, there exists $\alpha_r(x) \in L^1[0,1]$, such that $|f(x, y, z, u, v)| \le \alpha_r(x)$ for a.e. $x \in [0,1]$ and all $(y, z, u, v) \in \mathbb{R}^4$, with $\sqrt{y^2 + z^2 + u^2 + v^2} \le r$.

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In section 2, we apply the Leray-Schauder continuation theorem and Wirtinger-type inequalities to obtain the existence and uniqueness of solutions for nonlinear two point boundary value problems of equation (1.1) with boundary conditions (1.2). Our results improve and generalize the corresponding results in [1], [2], [4] and [8].

Throughout this paper, we shall use the Sobolev-space $W^{4,1}(0,1)$ defined by

$$W^{4,1}(0,1) = \{y \in C^3[0,1] : y''' \text{ is absolutely continuous}\},$$

with norm

$$||y||_{W^{4,1}} = \sum_{j=0}^{4} \int_0^1 |y^{(j)}(t)|dt.$$

2. Main results

The following Wirtinger-type inequalities are crucial in the proof of the main theorems.

Lemma 2.1 ([6]). *Let $y(x) \in C^3[0,1]$ and $y(0) = y(1) = y''(0) = y''(1) = 0$.* Then

$$||y^{(i)}||_2 \leq \frac{1}{\pi}||y^{(i+1)}||_2, \quad i = 0, 1, 2.$$

Now we are in a position to state our main results.

Theorem 2.1. *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy the Carathéodory's conditions. Assume that*

(i) *there exist positive numbers a_0, b_0, c_0, d_0 and a function $e(x) \in L^1[0,1]$ such that*

$$f(x, y, z, u, v)u \geq -a_0|yu| - b_0|zu| - c_0u^2 - d_0|vu| + e(x)|u|,$$

for a.e. $x \in [0,1]$ and all $y, z, u, v \in \mathbb{R}$;

(ii) *there exist a continuous function $\beta(x, y, z, u) : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$, a Carathéodory's function $\gamma(x, y, z, u) : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ and a number $\sigma \in [0,2]$ such that*

$$|f(x, y, z, u, v)| \leq \beta(x, y, z, u)|v|^\sigma + \gamma(x, y, z, u),$$

for a.e. $x \in [0,1]$ and all $y, z, u, v \in \mathbb{R}$.

Then BVP(1.1)(1.2) has at least one solution provided $\dfrac{a_0}{\pi^4} + \dfrac{b_0}{\pi^3} + \dfrac{c_0}{\pi^2} + \dfrac{d_0}{\pi} < 1$.

Proof. We define a linear mapping $L : D(L) \subset C^3[0,1] \to L^1[0,1]$ by setting

$$D(L) = \{y \in W^{4,1}(0,1) : y(0) = y(1) = y''(0) = y''(1) = 0\}$$

and for $y \in D(L)$,

$$Ly = y^{(4)}.$$

We also define a nonlinear mapping $N : C^3[0,1] \to L^1[0,1]$ by setting

$$(Ny)(x) = f(x, y(x), y'(x), y''(x), y'''(x)).$$

We note that N is a bounded, continuous mapping by the Lebesgue's Dominated Convergence Theorem. It is easy to see that the linear mapping $L : D(L) \subset C^3[0,1] \to L^1[0,1]$, defined above, is a one-to-one mapping. Also let $K : L^1[0,1] \to C^3[0,1]$ be the linear integral mapping defined by for $y \in L^1[0,1]$

$$(Ky)(x) = \int_0^1 G(x,t)y(t)dt,$$

where

$$G(x,t) = \begin{cases} -\dfrac{1}{6}x(1-t)(x^2-2t+t^2) + \dfrac{1}{6}(x-t)^3, & 0 \le t \le x \le 1; \\ -\dfrac{1}{6}x(1-t)(x^2-2t+t^2), & 0 \le x \le t \le 1, \end{cases}$$

is a Green's function for $y^{(4)} = 0, y(0) = y(1) = y''(0) = y''(1) = 0$. Then we see that for $y \in L^1[0,1]$, $Ky \in D(L)$ and $LKy = y$, and for $y \in D(L)$, $KLy = y$. Furthermore, it follows easily by using the Arzela-Ascoli Theorem that $KN : C^3[0,1] \to C^3[0,1]$ is a compact operator.

We next note that $y \in C^3[0,1]$ is a solution of the BVP(1.1)(1.2) if and only if y is a solution of the operator equation

$$Ly = Ny,$$

which is equivalent to the operator equation

$$y = KNy.$$

We now apply the Leray-Schauder continuation theorem to the operator equation $y = KNy$ for the existence of solutions.

To do this, it is sufficient to show that the set of all possible solutions of the family of equations

$$y^{(4)} = \lambda f(x, y, y', y'', y'''), \quad 0 < x < 1, \tag{2.1}$$

$$y(0) = y(1) = y''(0) = y''(1) = 0, \tag{2.2}$$

is, a priori, bounded in $C^3[0,1]$ independent of $\lambda \in [0,1]$.

Let $y(x)$ be a possible solution of BVP(2.1)(2.2) for some $\lambda \in [0,1]$. Then by lemma 2.1 we have

$$||y^{(i)}||_2 \le \frac{1}{\pi}||y^{(i+1)}||_2, \quad i = 0,1,2 \quad \text{and} \quad ||y''||_\infty \le ||y'''||_2.$$

Multiplying the equation (2.1) by y'' and integrating it on $[0,1]$, we have

$$\begin{aligned} 0 &= \int_0^1 y^{(4)} y'' dx - \lambda \int_0^1 f(x,y,y',y'',y''') y'' dx \\ &\le -\int_0^1 (y''')^2 dx + \lambda \int_0^1 \left[a_0|yy''| + b_0|y'y''| + c_0(y'')^2 + d_0|y'''y''| + e(x)|y''| \right] dx \\ &\le -||y'''||_2^2 + a_0||y||_2||y''||_2 + b_0||y'||_2||y''||_2 + c_0||y''||_2^2 + d_0||y'''||_2||y''||_2 + ||e||_1||y''||_\infty \\ &\le -||y'''||_2^2 + \left(\frac{a_0}{\pi^4} + \frac{b_0}{\pi^3} + \frac{c_0}{\pi^2} + \frac{d_0}{\pi} \right) ||y'''||_2^2 + ||e||_1||y'''||_2 \\ &= \left[\left(\frac{a_0}{\pi^4} + \frac{b_0}{\pi^3} + \frac{c_0}{\pi^2} + \frac{d_0}{\pi} \right) - 1 \right] ||y'''||_2^2 + ||e||_1||y'''||_2. \end{aligned}$$

Since $\frac{a_0}{\pi^4} + \frac{b_0}{\pi^3} + \frac{c_0}{\pi^2} + \frac{d_0}{\pi} < 1$, there exists a constant ρ_0 independent of $\lambda \in [0,1]$ such that

$$||y'''||_2 \le \rho_0, \tag{2.3}$$

and

$$||y||_2 \le \rho_0, \quad ||y'||_2 \le \rho_0, \quad ||y''||_2 \le \rho_0. \tag{2.4}$$

Further since $y(0) = y(1) = y''(0) = y''(1) = 0$, there exists $\xi \in [0,1]$ with $y'(\xi) = 0$, $y(x) = \int_0^x y'(t)dt$, $y'(x) = \int_\xi^x y''(t)dt$ and $y''(x) = \int_0^x y'''(t)dt$ for $x \in [0,1]$. It follows that

$$||y''||_\infty \le \rho_0, \quad ||y'||_\infty \le \rho_0 \quad \text{and} \quad ||y||_\infty \le \rho_0. \tag{2.5}$$

Noticing from Holder's inequalities that $\int_0^1 |y'''(x)|^\sigma dx \le ||y'''||_2^\sigma$, it follows from condition (ii) and equation (2.1) together with (2.5) that there is a constant ρ_1 independent of $\lambda \in [0,1]$ such that

$$||y^{(4)}||_1 \le \rho_1.$$

Since $y''(0) = y''(1) = 0$, there exists an $\eta \in [0,1]$ such that $y'''(\eta) = 0$ furthermore $y'''(x) = \int_\eta^x y^{(4)}(t)dt$. Hence

$$||y'''||_\infty \le ||y^{(4)}||_1 \le \rho_1. \tag{2.6}$$

It is now clear from (2.5) and (2.6) that there is a constant C, independent of $\lambda \in [0,1]$, such that

$$||y||_{C^3[0,1]} = \sum_{i=0}^{3} ||y^{(i)}||_\infty \le C.$$

Hence by Leray-Schauder's theorem, KN has a fixed point. This completes the proof of the theorem. □

We get the following corollaries directly from theorem 2.1.

Corollary 2.1. *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy the Carathéodory's conditions. Assume that for a.e. $x \in [0,1]$, $f(x,y,z,u,v)$ is continuously differentiable with respect to y, z, u and v. Suppose that there exist nonnegative real numbers a_0, b_0, c_0 and d_0 with $\frac{a_0}{\pi^4} + \frac{b_0}{\pi^3} + \frac{c_0}{\pi^2} + \frac{d_0}{\pi} < 1$ such that*

$$\frac{\partial f}{\partial y}(x,y,z,u,v) \geq -a_0, \quad \frac{\partial f}{\partial z}(x,0,z,u,v) \geq -b_0,$$
$$\frac{\partial f}{\partial u}(x,0,0,u,v) \geq -c_0, \quad \frac{\partial f}{\partial v}(x,0,0,0,v) \geq -d_0,$$

for a.e. $x \in [0,1]$ and all $y,z,u,v \in \mathbb{R}$. Suppose further that there exist a continuous function $\beta(x,y,z,u) : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$, a Carathéodory's function $\gamma(x,y,z,u) : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ and a number $\sigma \in [0,2]$ such that

$$|f(x,y,z,u,v)| \leq \beta(x,y,z,u)|v|^\sigma + \gamma(x,y,z,u),$$

for a.e. $x \in [0,1]$ and all $y,z,u,v \in \mathbb{R}$. Then BVP(1.1)(1.2) has at least one solution.

Corollary 2.2. *Let $f(x,y,z,u,v)$ be continuous on $[0,1] \times \mathbb{R}^4$, and there exist nonnegative real numbers a_0, b_0, c_0, d_0 and e_0 with $\frac{a_0}{\pi^4} + \frac{b_0}{\pi^3} + \frac{c_0}{\pi^2} + \frac{d_0}{\pi} < 1$ such that*

$$|f(x,y,z,u,v)| \leq a_0|y| + b_0|z| + c_0|u| + d_0|v| + e_0,$$

for all $(x,y,z,u,v) \in [0,1] \times \mathbb{R}^4$. Then BVP(1.1)(1.2) has at least one solution.

Remark 1. It is easy to see that corollary 2.2 is a generalization of theorem 1 in [8].

In condition (ii) of theorem 2.1, the continuity of β can be weaker when $\sigma \in [0,1]$ as follows.

Theorem 2.2. *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ satisfy the Carathéodory's conditions. Assume that*

(i) *there exist positive numbers $a_0,\ b_0,\ c_0,\ d_0$ and a function $e(x) \in L^1[0,1]$ such that*

$$f(x,y,z,u,v)u \geq -a_0|yu| - b_0|zu| - c_0u^2 - d_0|vu| + e(x)|u|,$$

for a.e. $x \in [0,1]$ and all $y,z,u,v \in \mathbb{R}$;

(ii) *there exist a L^2 - Carathéodory's function $\beta(x,y,z,u) : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$, a Carathéodory's function $\gamma(x,y,z,u) : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ and a number $\sigma \in [0,1]$ such that*

$$|f(x,y,z,u,v)| \leq \beta(x,y,z,u)|v|^\sigma + \gamma(x,y,z,u),$$

for a.e. $x \in [0,1]$ and all $y,z,u,v \in \mathbb{R}$.

Then BVP(1.1)(1.2) *has at least one solution provided* $\frac{a_0}{\pi^4}+\frac{b_0}{\pi^3}+\frac{c_0}{\pi^2}+\frac{d_0}{\pi}<1.$

The proof of theorem 2.2 is similar to that of theorem 2.1 and thus omitted.

Remark 2. It is easy to see that theorem 2.2 improves theorem 1 in [4] and theorem 1 in [2].

Theorem 2.3. *Let* $f:[0,1]\times\mathbb{R}^4\to\mathbb{R}$ *satisfy Carathéodory's conditions. Assume that*

(i) *there exist positive numbers* $a_0,\ b_0,\ c_0,\ d_0$ *such that*

$$\begin{aligned} & [f(x,y_1,z_1,u_1,v_1)-f(x,y_2,z_2,u_2,v_2)]\,(u_1-u_2)\\ \geq\ & -a_0|y_1-y_2||u_1-u_2|-b_0|z_1-z_2||u_1-u_2|\\ & -c_0(u_1-u_2)^2-d_0|v_1-v_2||u_1-u_2|, \end{aligned}$$

for a.e. $x\in[0,1]$ *and all* $y_i,z_i,u_i,v_i\in\mathbb{R},i=1,2;$

(ii) *there exist a continuous function* $\beta(x,y,z,u):[0,1]\times\mathbb{R}^3\to\mathbb{R}$*, a Carathéodory's function* $\gamma(x,y,z,u):[0,1]\times\mathbb{R}^3\to\mathbb{R}$ *and a number* $\sigma\in[0,2]$ *such that*

$$|f(x,y,z,u,v)|\leq\beta(x,y,z,u)|v|^\sigma+\gamma(x,y,z,u),$$

for a.e. $x\in[0,1]$ *and all* $y,z,u,v\in\mathbb{R}$.

Then BVP(1.1)(1.2) *has exactly one solution provided* $\frac{a_0}{\pi^4}+\frac{b_0}{\pi^3}+\frac{c_0}{\pi^2}+\frac{d_0}{\pi}<1.$

Proof. Note that condition (i) gives

$$f(x,y,z,u,v)u\geq a(x)|yu|+b(x)|zu|+c(x)u^2+d(x)|vu|-|f(x,0,0,0,0)||u|,$$

for all $y,z,u,v\in\mathbb{R}$ and a.e. $x\in[0,1]$. The existence of a solution for BVP(1.1)(1.2) follows from theorem 2.1 in view of our assumptions.

Now, to prove the uniqueness, suppose $y_1(x),\ y_2(x)$ are two solutions of BVP(1.1)(1.2). Setting $w(x)=y_1(x)-y_2(x)$, we get

$$w^{(4)}(x)=f(x,y_1(x),y_1'(x),y_1''(x),y_1'''(x))-f(x,y_2(x),y_2'(x),y_2''(x),y_2'''(x)), \tag{2.7}$$

$$w(0)=w(1)=w''(0)=w''(1)=0. \tag{2.8}$$

Multiplying the equation (2.7) by $w''(x)$ and integrating it on $[0,1]$, we have

$$\begin{aligned} 0 &= \int_0^1 w^{(4)}w''dx-\int_0^1[f(x,y_1,y_1',y_1'',y_1''')-f(x,y_2,y_2',y_2'',y_2''')]\,w''dx\\ &\leq -||w'''||_2^2+a_0||w||_2||w''||_2+b_0||w'||_2||w''||_2+c_0||w''||_2^2+d_0||w'''||_2||w''||_2\\ &\leq \left[\left(\frac{a_0}{\pi^4}+\frac{b_0}{\pi^3}+\frac{c_0}{\pi^2}+\frac{d_0}{\pi}\right)-1\right]||w'''||_2^2. \end{aligned}$$

Hence $||w'''||_2 = 0$, in view of $\frac{a_0}{\pi^4} + \frac{b_0}{\pi^3} + \frac{c_0}{\pi^2} + \frac{d_0}{\pi} < 1$. From the estimates $||w||_\infty \le ||w'||_2 \le \frac{1}{\pi^2}||w'''||_2$ and the continuity of $w(x)$, $w(x) \equiv 0$ on $[0,1]$. This completes the proof of the theorem. $\square$

Corollary 2.3. *Let $f(x,y,z,u,v)$ be continuous on $[0,1]\times\mathbb{R}^4$ and continuously differentiable with respect to y, z, u and v. Also suppose that there exist nonnegative real numbers a_0, b_0, c_0 and d_0 with $\frac{a_0}{\pi^4} + \frac{b_0}{\pi^3} + \frac{c_0}{\pi^2} + \frac{d_0}{\pi} < 1$ such that*

$$\frac{\partial f}{\partial y}(x,y,z,u,v) \ge -a_0, \quad \frac{\partial f}{\partial z}(x,y,z,u,v) \ge -b_0,$$

$$\frac{\partial f}{\partial u}(x,y,z,u,v) \ge -c_0, \quad \frac{\partial f}{\partial v}(x,y,z,u,v) \ge -d_0,$$

for all $(x,y,z,u,v) \in [0,1]\times\mathbb{R}^4$. Then BVP(1.1)(1.2) has exactly one solution.

Remark 3. Corollary 2.3 improves and generalizes theorem 4.1 in [1]. It is also easily seen that the condition (i) of theorem 4.1 in [1] is redundant and a condition $r + m\pi^2 < \pi^4$ should be given for the uniqueness.

Corollary 2.4. *Let $f(x,y,z,u,v)$ be continuous on $[0,1]\times\mathbb{R}^4$, and there exist nonnegative real numbers a_0, b_0, c_0 and d_0 with $\frac{a_0}{\pi^4} + \frac{b_0}{\pi^3} + \frac{c_0}{\pi^2} + \frac{d_0}{\pi} < 1$ such that*

$$|f(x,y_1,z_1,u_1,v_1) - f(x,y_2,z_2,u_2,v_2)| \le a_0|y_1-y_2| + b_0|z_1-z_2| + c_0|u_1-u_2| + d_0|v_1-v_2|,$$

for all $(x,y_i,z_i,u_i,v_i) \in [0,1]\times\mathbb{R}^4, i = 1,2$. Then BVP(1.1)(1.2) has exactly one solution.

Theorem 2.4. *Let $f : [0,1]\times\mathbb{R}^4 \to \mathbb{R}$ satisfy Carathéodory's conditions. Assume that*

(i) *there exist positive numbers a_0, b_0, c_0, d_0 such that*

$$\begin{aligned} & [f(x,y_1,z_1,u_1,v_1) - f(x,y_2,z_2,u_2,v_2)](u_1-u_2) \\ \ge\ & -a_0|y_1-y_2||u_1-u_2| - b_0|z_1-z_2||u_1-u_2| \\ & - c_0(u_1-u_2)^2 - d_0|v_1-v_2||u_1-u_2|, \end{aligned}$$

for a.e. $x \in [0,1]$ and all $y_i, z_i, u_i, v_i \in \mathbb{R}, i = 1,2$;

(ii) *there exist a Carathéodory function $\beta(x,y,z,u) : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$, a Carathéodory's function $\gamma(x,y,z,u) : [0,1]\times\mathbb{R}^3 \to \mathbb{R}$ and a number $\sigma \in [0,1]$ such that*

$$|f(x,y,z,u,v)| \le \beta(x,y,z,u)|v|^\sigma + \gamma(x,y,z,u),$$

for a.e. $x \in [0,1]$ and all $y,z,u,v \in \mathbb{R}$.

Then $BVP(1.1)(1.2)$ *has exactly one solution provided* $\frac{a_0}{\pi^4} + \frac{b_0}{\pi^3} + \frac{c_0}{\pi^2} + \frac{d_0}{\pi} < 1$.

The proof of theorem 2.4 is similar to that of theorem 2.3 and thus omitted.

Remark 4. It is easy to see that theorem 2.4 improves theorem 5 in [4] and theorem 2 in [2].

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