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Remarks on Fixed Point Theorems of Non-Lipschitzian Self-mappings

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Abstract. In 1994, Lim-Xu asked whether the Maluta's constant $D(X) < 1$ implies the fixed point property for asymptotically nonexpansive mappings and gave a partial solution for this question under an additional assumption for T, i.e., weakly asymptotic regularity of T. In this paper, we shall prove that the result due to Lim-Xu is also satisfied for more general non-Lipschitzian mappings in reflexive Banach spaces with weak uniform normal structure. Some applications of this result are also added.

1. Introduction

Let C be a nonempty subset of a real Banach space X and let $\mathbb{N}$ be the set of natural numbers. Let $T : C \to C$ be a mapping. T is said to be *Lipschitzian* if for each $n \in \mathbb{N}$, there exists a real number k_n such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|, \quad x, \ y \in C.$$

In particular, T is said to be *asymptotically nonexpansive* [8] if $\lim_{n\to\infty} k_n = 1$, and it is said to be *nonexpansive* if $k_n = 1$ for all $n \in \mathbb{N}$. A set K satisfying $T(K) \subset K$ is said to be *invariant under* T or T*-invariant*. Let K be a nonempty subset of C. For each $x \in K$, we set

$$c_n(x; K) = \sup_{y\in K} (\|T^n x - T^n y\| - \|x - y\|) \vee 0.$$

We say that T is of *partly asymptotically nonexpansive type* if there exists a nonempty bounded closed convex and T-invariant subset K of C such that $c_n(x; K) \to 0$ for each $x \in K$. Recall that if $c_n(x) := c_n(x; C) \to 0$ for each $x \in C$, then T is said to be of asymptotically nonexpansive type (see [16]). A point $x \in C$ is a *fixed point* of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T; that is, $Fix(T) = \{x \in C : Tx = x\}$.

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In 1965, Kirk [15] proved that if C is a weakly compact convex subset of a Banach space with normal structure, then every nonexpansive self-mapping T of C has a fixed point, where a nonempty convex subset C of a normed linear space is said to have *normal structure* if each bounded convex subset K of C consisting of more than one point contains a nondiametral point; that is, a point $z \in K$ such that $\sup\{\|z-x\| : x \in K\} < \mathrm{diam(K)}$. Seven years later, in 1972, Goebel-Kirk [8] proved that if the space X is assumed to be uniformly convex, then every asymptotically nonexpansive self-mapping T of C has a fixed point. This was immediately extended to mappings of asymptotically nonexpansive type in a space with its characteristic of convexity, $\epsilon_o(X) < 1$, by Kirk [16] in 1974. More recently these results have been extended to wider classes of spaces, see for example [4], [6], [7], [14], [19], [18], [22]. In particular, Lim-Xu [19] and Kim-Xu [14] have demonstrated the existence of fixed points for asymptotically nonexpansive mappings in Banach spaces with uniform normal structure, see also [6] for some related results. Very recently, the result due to Kim-Xu [14] was extended to mappings of asymptotically nonexpansive type by Li-Sims [17] and Kim [10] independently.

On the other hand, fixed point theorems due to Lim-Xu [19] for asymptotically nonexpansive mappings defined on a weakly compact convex subset C in a Banach space X with either a weakly continuous duality mapping or for which $D(X) < 1$ having an additional condition, i.e., weak asymptotic regularity on C for T, where $D(X)$ is Maluta's constant (see [20]), were carried over continuous mappings of asymptotically nonexpansive type by Kim-Kim [13].

In this paper, we modify some results in [13] and carry over these to a wider class of continuous mappings of partly asymptotically nonexpansive type in a Banach space with weak uniform normal structure (see Theorem 3.2). Some applications and examples of non-Lipschitzian mappings of partly asymptotically nonexpansive type which are not of asymptotically nonexpansive type are also added.

2. Preliminaries

Let X be a real Banach space. First, let us introduce normal structure coefficient of X introduced by Bynum [5]. For $A \subset X$, diam(A) and $r_A(A)$ denote the *diameter* and the *self-Chebyshev radius* of A, respectively, i.e.,

$$\mathrm{diam(A)} = \sup_{\mathrm{x,y \in A}} \|\mathrm{x-y}\|,$$
$$r_A(A) = \inf_{x \in A} (\sup_{y \in A} \|x-y\|)$$

Recall that X has *uniform normal structure* (simply UNS) if $N(X) > 1$, where

$$N(X) = \inf\left\{\frac{\mathrm{diam(A)}}{\mathrm{r_A(A)}} : A \subset X \text{ bounded closed convex with } \mathrm{diam}(A) > 0\right\}.$$

Obviously, if $N(X) > 1$, then X has normal structure.

Recall that if X is a non-Schur Banach space, then the weakly convergent sequence coefficient of X, denoted by $WCS(X)$, is defined by

$$\begin{aligned} WCS(X) = \sup\{M > 0 :& \text{ for each weakly convergent sequence } \{x_n\}, \\ &\exists\, y \in \overline{co}(\{x_n\}) \text{ such that } M \cdot \limsup_{n\to\infty} \|x_n - y\| \le A(\{x_n\})\}, \end{aligned}$$

where $\overline{co}(K)$ denotes the closed convex hull of a set K and $A(\{x_n\})$ denotes the asymptotic diameter of $\{x_n\}$, i.e.,

$$A(\{x_n\}) = \lim_{n\to\infty} \sup\{\|x_i - x_j\| : i, j \ge n\}.$$

It is easy to give a sharp expression $WCS(X)$ as follows;

$$WCS(X) = \sup\{M : x_n \rightharpoonup u \;\Rightarrow\; M \cdot \limsup_{n\to\infty} \|x_n - u\| \le D(\{x_n\})\},$$

where $D(\{x_n\}) := \limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - x_m\|$ and "$\rightharpoonup$" means the weak convergence. For more details, see [5] and [12].

Note that if X is reflexive, then $1 \le N(X) \le BS(X) \le WCS(X) \le 2$ (cf., [5]), where $BS(X)$ means the *bounded sequence coefficient* of X, i.e.,

$$\begin{aligned} BS(X) = \sup\big\{M :& \text{ for any bounded sequence } \{x_n\} \text{ in } X, \\ &\exists\, y \in \overline{co}(\{x_n\}) \text{ such that } M \cdot \limsup_{n\to\infty} \|x_n - y\| \le A(\{x_n\})\big\}. \end{aligned}$$

While $N(X)$ and $BS(X)$ can be defined in every Banach space, $WCS(X)$ is well defined only in infinite dimensional reflexive spaces, where, by Eberlein-Šmulian theorem, we can assure the existence of weakly convergent sequences which do not converge.

The coefficient $WCS(X)$ plays important roles in fixed point theory. A space X such that $WCS(X) > 1$ is said to have *weak uniform normal structure*. It is well-known [5] that if $WCS(X) > 1$, then X has weak normal structure; that is, any weakly compact convex subset C of X with $\operatorname{diam}(C) > 0$ has a nondiametral point.

Let X be a Banach space. Recall that *Maluta's constant* $D(X)$ [20] of X is defined by

$$D(X) = \sup\left\{\frac{\limsup d(x_{n+1}, co(\{x_1, x_2, \cdots x_n\}))}{\operatorname{diam}(\{x_n\})}\right\},$$

where the supremum is taken over all bounded nonconstant sequences $\{x_n\}$ in X.

We remark the following properties for Maluta's constant given in [20].

Lemma 2.1. *Let X be a Banach space. Then*
(a) $D(X) \le \tilde{N}(X) := 1/N(X)$.
(b) $D(X) = \sup\{D(Y) : Y \subset X \text{ separable}\}$.
(c) $D(X) = 0$ *if and only if X is finite-dimensional.*

(d) *If X is reflexive, then $D(X) \leq 1/WCS(X)$.*
(e) *If $D(X) < 1$, then the Banach space X is reflexive and has normal structure.*

Remark 2.1. (i) The property (a) says that if X has uniform normal structure, then $D(X) < 1$. However, the converse does not hold (see Example 5.1 and Corollary 5.2 in [20]).

(ii) In view of (d), Maluta asked whether $D(X) = 1/WCS(X)$ holds true for every infinite dimensional reflexive space X. In 1985, Amir [2] gave a partial solution for this question. In other words, the converse inequality $D(X) \geq 1/WCS(X)$ holds if X satisfies *Opial's property*, i.e., for any sequence $\{x_n\}$ converging weakly to x, there holds the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad y\,(\neq x) \in X.$$

Five years later, this question was completely solved by Prus [21].

(iii) The converse of (e) also does not hold (see Example 4.1 in [20], $X = (\sum \bigoplus \ell_n)_2$ is reflexive and has normal structure although $D(X) = 1$).

Note that, by (e) of Lemma 2.1, if $D(X) < 1$, X has normal structure and hence the fixed point property for nonexpansive mappings; that is, for every weakly compact convex subset C of X, every nonexpansive map $T : C \to C$ has a fixed point. However, it is still open whether $D(X) < 1$ implies the fixed point property for asymptotically nonexpansive mappings. In 1994, Lim-Xu [19] gave a partial answer for this question as follows:

Theorem LX [19]. *Suppose that X is a Banach space such that $D(X) < 1$, that C is a closed bounded convex subset of X, and that $T : C \to C$ is an asymptotically nonexpansive mapping. Suppose, in addition, that T is weakly asymptotically regular on C, i.e., $T^{n+1}x - T^n x \rightharpoonup 0$ for all $x \in C$. Then T has a fixed point.*

Immediately, Theorem LX was extended to all mappings of asymptotically nonexpansive type by Kim-Kim (see Corollary 3.3 in [13]). In fact, under the assumption of weakly asymptotic regularity of T, the conditions for X and T can be weakened, in other words, Theorem LX can be extended to mappings of partly asymptotically nonexpansive type with $WCS(X) > 1$. Finally we need the following two well known properties for ultrafilters (for example, see [1]).

Lemma 2.2. *Let X be a Hausdorff topological linear space and let $\mathcal{U}$ be an ultrafilter on a set I. Then, the following properties hold.*

(i) *if $\{x_i\}_{i\in I}$ and $\{y_i\}_{i\in I}$ are two subsets of X and $\lim_{\mathcal{U}} x_i = x$ and $\lim_{\mathcal{U}} y_i = y$ both exists, then $\lim_{\mathcal{U}}(x_i + y_i) = x + y$ and $\lim_{\mathcal{U}}(\alpha x_i) = \alpha x$ for any scalar α.*

(ii) *K is a compact subset of X if and only if any set $\{x_i\}_{i\in I} \subset K$ is convergent over any ultrafilter $\mathcal{U}$ on I.*

3. Fixed point theorems

Let C be a nonempty subset of a Banach space X, and let $T : C \to C$ be a mapping. Suppose there exists a nonempty subset K of C and the weak limit $w\text{-}\lim_{\mathcal{U}} T^n x$ exists in K for each $x \in K$, where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$. We then can define a mapping $S : K \to K$ by

$$Sx = w\text{-}\lim_{\mathcal{U}} T^n x, \quad x \in K. \tag{1}$$

Note first that if K is weakly compact and T-invariant, by (ii) of Lemma 2.2, the weak limit $w\text{-}\lim_{\mathcal{U}} T^n x$ always exists in K for each $x \in K$. Furthermore, we can see that $Fix(T) \cap K \subset Fix(S)$. What are conditions on X and T for which the converse inclusion remains true? Our purpose is to find some conditions on X and T to answer the above question.

First, we exhibit the following easy lemma for our argument.

Lemma 3.1. *Let C be a nonempty subset of a reflexive Banach space X. If $T : C \to C$ is a continuous mapping of partly asymptotically nonexpansive type, then there exist a nonempty weakly compact convex and T-invariant subset K of C such that $c_n(x; K) \to 0$ for each $x \in K$, and a nonexpansive mapping $S : K \to K$.*

Proof. Since T is of partly asymptotically nonexpansive type and X is reflexive, there exists a nonempty weakly compact convex and T-invariant subset K of C such that $c_n(x; K) \to 0$ for each $x \in K$. Now defining $S : K \to K$ as in (1), S is nonexpansive. In fact, for $x, y \in K$, $Sx = w\text{-}\lim_{\mathcal{U}} T^n x$ and $Sy = w\text{-}\lim_{\mathcal{U}} T^n y$. By (i) of Lemma 2.2, we have $Sx - Sy = w\text{-}\lim_{\mathcal{U}}(T^n x - T^n y)$. Then there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $T^{n_k} x - T^{n_k} y \rightharpoonup Sx - Sy$ as $k \to \infty$. Since the norm $\|\cdot\|$ is weakly lower semicontinuous and $c_n(x; K) \to 0$ as $n \to \infty$ for each $x \in K$, we have

$$\begin{aligned} \|Sx - Sy\| &\leq \liminf_{k\to\infty} \|T^{n_k} x - T^{n_k} y\| \\ &\leq \limsup_{k\to\infty} [\|T^{n_k} x - T^{n_k} y\| - \|x - y\|] + \|x - y\| \\ &\leq \lim_{k\to\infty} c_{n_k}(x; K) + \|x - y\| = \|x - y\| \end{aligned}$$

for all $x, y \in K$. □

Now we will present a partial answer of the above question; that is, a sufficient condition for $Fix(S) \subset Fix(T) \cap K$, with a slight modification of the proof in Lemma 3.1 of [13]. Here we shall give the detailed proof for convenience sake.

Theorem 3.2. *Let C be a nonempty subset of a reflexive Banach space X with $WCS(X) > 1$. If $T : C \to C$ is a continuous mapping of partly asymptotically nonexpansive type and weakly asymptotically regular on C, then there exist a nonempty weakly compact convex and T-invariant subset K of C and a nonexpansive mapping $S : K \to K$ such that $Fix(T) \cap K = Fix(S) \neq \emptyset$.*

Proof. Let K and $S : K \to K$ be as in Lemma 3.1. Clearly, $Fix(S) \neq \emptyset$ by Kirk [15]. Now to complete the proof, it suffices to show that $Fix(S) \subset Fix(T) \cap K$.

To this end, let $x \in Fix(S)$; that is, $w\text{-}\lim_{\mathcal{U}} T^n x = x \in K$. Then there exists a subsequence $\{T^{n_k}x\}$ of the sequence $\{T^n x\}$ in K such that $T^{n_k}x \rightharpoonup x$ as $k \to \infty$. By the well known property of $WCS(X)$,

$$\limsup_{k\to\infty} \|T^{n_k}x - x\| \le \frac{1}{WCS(X)} D(\{T^{n_k}x\}). \tag{2}$$

By weakly asymptotic regularity of T, it follows that $T^{n_k+m}x \rightharpoonup x$ as $k \to \infty$ for any $m \ge 0$. On the other hand, for each $i, j \in \mathbb{N}$ with $i > j$, the weak lower semicontinuity of the norm $\|\cdot\|$ immediately yields that

$$\begin{aligned}
& \|T^{n_j}x - T^{n_i}x\| \\
\le\ & \big(\|T^{n_j}x - T^{n_j}(T^{n_i-n_j}x)\| - \|x - T^{n_i-n_j}x\|\big) + \|x - T^{n_i-n_j}x\| \\
\le\ & c_{n_j}(x;K) + \|x - T^{n_i-n_j}x\| \quad (T^{n_k+m}x \rightharpoonup x \text{ as } k \to \infty, \text{ with } m = n_i - n_j) \\
\le\ & c_{n_j}(x;K) + \liminf_{k\to\infty} \|T^{n_k+m}x - T^{n_i-n_j}x\| \\
\le\ & c_{n_j}(x;K) + c_{n_i-n_j}(x;K) + \limsup_{k\to\infty} \|x - T^{n_k}x\|.
\end{aligned}$$

Taking $\limsup_{i\to\infty}$ first and next $\limsup_{j\to\infty}$ on both sides, since $c_n(x;K) \to 0$ for each $x \in K$, this yields

$$D(\{T^{n_k}x\}) \le \limsup_{k\to\infty} \|x - T^{n_k}x\|,$$

and this together with (2) gives $(WCS(X)-1)\cdot \limsup_{k\to\infty} \|T^{n_k}x - x\| \le 0$, which in turn implies that $x = \lim_{k\to\infty} T^{n_k}x$. By the continuity and weak asymptotic regularity of T, we have $Tx = x$, i.e., $x \in Fix(T)$. □

Remark 3.1. (i) Note that if C is weakly compact convex, the reflexivity of X can be removed in Theorem 3.2.

(ii) Following (ii) of Remark 2.1, $D(X) = 1/WCS(X)$ for every infinite dimensional reflexive space X. Therefore, the assumption in Theorem 3.2 which X is a reflexive Banach space with $WCS(X) > 1$ can be replaced by $D(X) < 1$.

(iii) As a direct consequence of the proof of Theorem 3.2, we notice that, under the same assumptions of C, X and T, if $\{T^{n_k}x\}$ is a subsequence of $\{T^n x\}$ converging weakly to $x \in K$, then $\lim_{k\to\infty} T^{n_k}x = x$. However, if the whole sequence $\{T^n x\}$ converges weakly, the weakly asymptotic regularity on C for T is abundant.

Lemma 3.3. *Let C be a nonempty subset of a reflexive Banach space X with $WCS(X) > 1$. If $T : C \to C$ is a continuous mapping of partly asymptotically nonexpansive type, then $w\text{-}\lim_{n\to\infty} T^n x = x \in K \Rightarrow \lim_{n\to\infty} T^n x = x \in Fix(T)$.*

With the similar method of the proof as in Theorem 3.2, we observe the following

Theorem 3.4. *Let C be a nonempty bounded subset of a Banach space X with*

$D(X) < 1$. Let $T : C \to C$ be a continuous mapping of asymptotically nonexpansive type which is weakly asymptotically regular on C. Suppose there exists a nonempty closed convex subset K of C with the following property

$$x \in K \quad \Longrightarrow \quad \omega_w(x) \subset K, \tag{ω}$$

where $\omega_w(x)$ is the weak ω-limit set of T at x; namely, $\omega_w(x) = \{y \in X : y = w\text{-}\lim_{k\to\infty} T^{n_k}x$ for some $n_k \uparrow \infty\}$. Then there exists a nonexpansive mapping $S : K \to K$ such that $Fix(T) \cap K = Fix(S) \neq \emptyset$.

Proof. Since X is reflexive, K is weakly compact convex and $WSC(X) > 1$. Since the sequence $\{T^n x\}$ belongs to C, and $\overline{co}(C)$ is weakly compact, the weak limit $w\text{-}\lim_{\mathcal{U}} T^n x$ always exists in $\overline{co}(C)$ for each $x \in K$ by (ii) of Lemma 2.2. Define $Sx = w\text{-}\lim_{\mathcal{U}} T^n x$ for each $x \in K$. Then, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that $T^{n_k}x \rightharpoonup Sx$ as $k \to \infty$. By property of (ω), it follows that $Sx \in \omega_w(x) \subset K$. Therefore, $S : K \to K$ is well defined, and also nonexpansive. Thus, repeating the method of proof in Theorem 3.2, we can easily obtain the conclusion. □

It is clear that if C is a nonempty bounded subset of a Banach space X, and if $T : C \to C$ is an asymptotically nonexpansive mapping with its Lipschitz constant of T^n, $k_n \geq 1$, then T is a uniformly Lipschitzian mapping of asymptotically nonexpansive type. Therefore, we have the following easy result.

Corollary 3.5. *Let C be a nonempty bounded subset of a Banach space X with $D(X) < 1$. Let $T : C \to C$ be an asymptotically nonexpansive mapping which is weakly asymptotically regular on C. Suppose there exists a nonempty closed convex subset K of C with the property (ω). Then there exists a nonexpansive mapping $S : K \to K$ such that $Fix(T) \cap K = Fix(S) \neq \emptyset$.*

Let C be a weakly compact convex subset of a Banach space X. Consider a family $\mathcal{F}$ of subsets K of C which are nonempty, closed, convex, and satisfy the following property (ω). The weak compactness of C now allows one to use Zorn's lemma to obtain a minimal element (say) $K \in \mathcal{F}$. Therefore, as a direct consequence of Theorem 3.2 or 3.4, we have the following result due to Kim-Kim [13].

Corollary 3.6. *Let C be a nonempty weakly compact convex subset of a Banach space X with $WCS(X) > 1$. If $T : C \to C$ is a continuous mapping of asymptotically nonexpansive type and weakly asymptotically regular on C, then $Fix(T)$ is a nonempty nonexpansive retract of C.*

Proof. Note first that T is of partly asymptotically nonexpansive type with $K = C$. Since C is weakly compact and convex, in view of (i) of Remark 3.1, we can apply for Theorem 3.2 or 3.4, and hence $Fix(T) = Fix(S) \neq \emptyset$. Since S is nonexpansive, it follows from [3] that $Fix(S)$ is a nonempty nonexpansive retract of C. □

Recall that a Banach space X is said to be *uniformly convex in every direction* [9] if $\delta_z(\epsilon) > 0$ for all $\epsilon > 0$ and all $z \in X$ with $\|z\| = 1$, where $\delta_z(\cdot)$ means the *modulus of convexity of X in the direction z*, that is,

$$\delta_z(\epsilon) = \{1 - \|x+y\|/2 : \|x\| \leq 1,\ \|y\| \leq 1,\ x - y = \epsilon z\}.$$

There is clearly a space X which may be uniformly convex in every direction while failing to be uniformly convex. Obviously, such spaces are always strictly convex.

Theorem 3.7. *Suppose that X is a reflexive Banach space which is uniformly convex in every direction and for which $WCS(X) > 1$ and that C is a nonempty subset of X. Then, if $T : C \to C$ is a continuous mapping of partly asymptotically nonexpansive type, T has a fixed point.*

Proof. Use the same argument presented in the proof of Theorem 5 in [19] and Lemma 3.3. □

Finally, we shall give examples of non-Lipschitzian mappings of partly asymptotically nonexpansive type which are not of asymptotically nonexpansive type, inspired by the example 4.3 and 4.4 in [11]. These examples also satisfy all assumptions of Theorem 3.2.

Example A. Let $X = C = \mathbb{R}$, the set of real numbers, and let $|k| < 1$. For each $x \in C$, we define

$$Tx = \begin{cases} kx \sin \frac{1}{x}, & x \neq 0,\ |x| \leq 1/\pi; \\ 0, & x = 0; \\ \pi|x| - 1, & |x| > 1/\pi. \end{cases}$$

Then, clearly $c_n(1) = c_n(1; C) \geq T^n 1 - 1 \to \infty$, and so T is not of asymptotically nonexpansive type. Note further that $c_n(x) = c_n(x, C) \to \infty$ for all fixed $x \in C$. But if we take $K = [-1/\pi, 1/\pi]$, then K is T-invariant and also T is of partly asymptotically nonexpansive type. Indeed, it suffices to show that $c_n(x; K) \to 0$ for each $x \in K$. For fixed $x \in K$ and $n \in \mathbb{N}$, set

$$H_n(y) = |T^n x - T^n y| - |x - y|, \quad y \in K.$$

Then $H_n(\cdot)$ is continuous on K, and so it achieves its maximum in K, i.e., there exists a $y_n \in K$ such that $c_n(x; K) = H_n(y_n) \vee 0$. Since $T^n z \to 0$ uniformly on K, we have $c_n(x; K) \to 0$ for each $x \in K$.

Example B. Let $X = \mathbb{R}$ and $C = (-\infty, 1]$. First consider a continuous non-Lipschitzian mapping $f : [0, 1/2] \to [0, 1/4]$ defined by

$$f(x) = \begin{cases} \frac{n(2n+1)}{n+1}\left(x - \frac{1}{2n+1}\right), & \text{if } \frac{1}{2n+1} \leq x \leq \frac{1}{2n},\ n \geq 1; \\ -\frac{(n+1)(2n+1)}{n+2}\left(x - \frac{1}{2n+1}\right), & \text{if } \frac{1}{2(n+1)} \leq x \leq \frac{1}{2n+1},\ n \geq 1; \\ 0, & \text{if } x = 0. \end{cases}$$

Note first that for each $n \in \mathbb{N}$, the graph of f on each subinterval $[1/2(n+1), 1/2n]$ consists of two segments connecting three points $(1/2(n+1), 1/2(n+2))$, $(1/2n+1, 0)$ and $(1/2n, 1/2(n+1))$. For each $x \in C = (-\infty, 1]$, we now define

$$Tx = \begin{cases} \frac{x}{1-2x}, & \text{if } x \leq -\frac{1}{2}; \\ f(x), & \text{if } x \in [0, 1/2]; \\ -f(-x), & \text{if } x \in [-1/2, 0]; \\ x^2, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Obviously, $|T^n z| \leq \frac{1}{2(n+1)}$ for $|z| \leq \frac{1}{2}$, and so $T^n z \to 0$ uniformly on $[-1/2, 1/2]$. Also, since $|Tz| \leq 1/2$ for $z \leq -1/2$, we also have $T^n z \to 0$ uniformly on $(-\infty, -1/2]$. We thus obtain $T^n z \to 0$ uniformly on $(-\infty, 1/2]$. It is obvious that T is not of asymptotically nonexpansive type because $c_n(1) = 1$ for each n. However, if we take $K := [-1/2, 1/2]$, it is easy to see that K is T-invariant and T is of partly asymptotically nonexpansive type, i.e., $c_n(x; K) \to 0$ for each $x \in K$.

Remark 3.2. If we take $K := [-1/2, 0]$ in Example B, for this T-invariant closed interval K of C, we can further prove that $c_n(x) \to 0$ for each $x \in K$. Indeed, for $x \in K$, we set

$$\begin{aligned} &c_n(x) = \sup_{y \in C}(|T^n x - T^n y| - |x - y|) \vee 0 \\ &= \sup_{y \in (-\infty, 1/2]}(|T^n x - T^n y| - |x - y|) \vee \sup_{y \in [1/2, 1]}(|T^n x - T^n y| - |x - y|) \vee 0 \\ &:= A_n(x) \vee B_n(x) \vee 0. \end{aligned}$$

Since $T^n z \to 0$ uniformly on $(-\infty, 1/2]$, $A_n(x) \to 0$ as $n \to \infty$. Now it suffices to show that $\limsup_{n\to\infty} B_n(x) \leq 0$. For each $n \in \mathbb{N}$, there exists $y_n \in [1/2, 1]$ such that $B_n(x) = |T^n x - T^n y_n| - |x - y_n|$. If $y_n = 1$, since $-\frac{1}{2(n+1)} \leq T^n x \leq 0$, we have $|T^n x - 1| = 1 - T^n x \leq 1 - x = |x - 1|$ for sufficiently large n, and so $\limsup_{n\to\infty}(|T^n x - 1| - |x - 1|) \leq 0$. Also if $y_n \in [1/2, 1)$, we easily have

$$\limsup_{n\to\infty}(|T^n x - T^n y_n| - |x - y_n|) = -\liminf_{n\to\infty}|x - y_n| \leq 0.$$

Thus, $\limsup_{n\to\infty} B_n(x) \leq 0$ is obtained, and therefore $c_n(x) \to 0$ for each $x \in K$.

Finally, note that every sequence $\{T^n x\}$ converges uniformly to $0 \in Fix(T) \cap K = \{0\}$ for each $x \in K$.

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