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Some Inequalities for Random Variables whose Probability Density Functions are Bounded Using an Improvement of Grüss Inequality

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Abstract. Some recent inequalities for expectation and cumulative distribution function are improved.

1. Introduction

In the recent paper [1] or [3], Barnett and Dragomir, using the pre-Grüss inequality, established some inequalities for expectation and the distribution function. In the paper [5], Cheng and Sun established the following variant of Grüss inequality.

Lemma. *Let $f,\ g : [a,b] \to R$ be two integrable functions such that*

$$m \leq f(x) \leq M, \quad \text{for all } x \in [a,b],$$

where $m,\ M \in R$ are constants. Then

$$\text{(1.1)} \quad \left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{(b-a)^2} \int_a^b f(x)dx \int_a^b g(x)dx \right| \leq \frac{(M-m)}{2(b-a)} \int_a^b \left| g(x) - \frac{1}{b-a} \int_a^b g(t)dt \right| dx.$$

Further, Cerone and Dragomir [4] have proved that $\frac{1}{2}$ in (1.1) is sharp constant. In this paper, using the above Lemma we shall improve the inequalities of the expection and the distribution function given by Barnett and Dragomir [1].

2. Some inequalities for expectation

Theorem 1. *Let X be a random variable having the probability density function*

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$f : [a,b] \to R$. Assume that there exist the constants M, m such that $0 \le m \le f(t) \le M \le 1$ a.e. t on $[a,b]$, then we have the inequality:

$$\left| E(x) - \frac{(a+b)}{2} \right| \le \frac{1}{8}(M-m)(b-a)^2, \tag{2.1}$$

where $E(X)$ is the expectation of the random variable X.

Proof. If we put $g(t) = t$ in (1.1), we obtain

$$\begin{aligned} &\left| \frac{1}{b-a}\int_a^b tf(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b tdt \right| \\ \le\ & \frac{(M-m)}{2(b-a)}\int_a^b \left| x - \frac{1}{b-a}\int_a^b tdt \right| dx. \end{aligned} \tag{2.2}$$

and as

$$\int_a^b tf(t)dt = E(X), \quad \int_a^b f(t)dt = 1, \quad \frac{1}{b-a}\int_a^b tdt = \frac{a+b}{2}$$

and

$$\begin{aligned} &\int_a^b \left| x - \frac{1}{b-a}\int_a^b tdt \right| dx \\ =\ & \int_a^{\frac{a+b}{2}} \frac{a+b}{2} - tdt + \int_{\frac{a+b}{2}}^b t - \frac{a+b}{2}dt \\ =\ & \frac{(b-a)^2}{4}, \end{aligned}$$

then by (2.2) we deduce (2.1). □

Remark 2. Theorem 1 is an improvement of Theorem 9 in [1].

To point out a result for the p-moments of the random variable X, $p \in R\backslash\{-1,0\}$, we need the following p-Logarithmic mean:

$$M_p(a,b) = \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{1/p},$$

where $0 < a < b$.

Theorem 3. *Let X and f be as in Theorem 1 and $E_p(X)$ be the p-moment of X, i.e.,*

$$E_p(X) = \int_a^b t^p f(t)dt,$$

which is assumed to be finite, then

$$\begin{aligned} (2.3) \qquad & \left|E_p(X) - M_p^p(a,b)\right| \\ \leq\ & \frac{(M-m)}{2}\left|\frac{2p}{p+1}M_p^{p+1}(a,b) - (a+b)M_p^p(a,b) + \frac{b^{p+1}+a^{p+1}}{p+1}\right|. \end{aligned}$$

Proof. Taking $g(t) = t^p$ in (1.1), we obtain

$$\begin{aligned} (2.4) \qquad & \left|\frac{1}{b-a}\int_a^b t^p f(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b t^p dt\right| \\ \leq\ & \frac{(M-m)}{2(b-a)}\int_a^b \left|t^p - M_p^p(a,b)\right| dt. \end{aligned}$$

Since

$$\begin{aligned} (2.5) \qquad & \int_a^b |t^p - M_p^p(a,b)|dt \\ =\ & \int_a^{M_p(a,b)} (M_p^p(a,b) - t^p)dt + \int_{M_p(a,b)}^b (t^p - M_p^p(a,b))dt \\ =\ & \frac{2p}{p+1}M_p^{p+1}(a,b) - (a+b)M_p^p(a,b) + \frac{b^{p+1}+a^{p+1}}{p+1}, \end{aligned}$$

if $p > 0$ and

$$\begin{aligned} (2.6) \qquad & \int_a^b |t^p - M_p^p(a,b)|dt \\ =\ & \int_a^{M_p(a,b)} (t^p - M_p^p(a,b))dt + \int_{M_p(a,b)}^b (M_p^p(a,b) - t^p)dt \\ =\ & -\frac{2p}{p+1}M_p^{p+1}(a,b) + (a+b)M_p^p(a,b) - \frac{b^{p+1}+a^{p+1}}{p+1}, \end{aligned}$$

if $p < 0$. By (2.4), (2.5) and (2.6), we obtain (2.3). □

Example 4. Let $p = 2$, $a = 1$ and $b = 2$ in (2.3) and (5.6) in [1], respectively. Then we have

$$(2.7) \qquad \left|E_2(X) - M_2^2(1,2)\right| \leq \frac{(14\sqrt{21}-54)}{27}(M-m)$$

and

$$(2.8) \qquad \left|E_2(X) - M_2^2(1,2)\right| \leq \frac{\sqrt{170}}{30}(M-m).$$

We note that the bound in (2.7) is better than the one in (2.8).

If we consider the logarithmic mean

$$M_{-1}(a,b) = L(a,b) = \frac{b-a}{\ln b - \ln a}, \qquad 0 < a < b,$$

and define the (-1)-moment of the random variable X by

$$E_{-1}(X) = \int_a^b \frac{f(t)}{t} dt,$$

then we have the following theorem.

Theorem 5. *Let X and f be as in Theorem 1, then*

$$(2.9)\quad \left|E_{-1}(X) - M_{-1}^{-1}(a,b)\right| \le \frac{(M-m)}{2}\Big[\ln\Big(\frac{M_{-1}^2(a,b)}{ab}\Big) + (a+b)M_{-1}^{-1}(a,b) - 2\Big].$$

The proof is similar to the proof of Theorem 2 and so we omit the details.

Example 6. Let $a = 1$ and $b = 2$ in (2.9) and (5.7) in [1], respectively. Then we have

$$(2.10)\qquad \left|E_{-1}(X) - M_{-1}^{-1}(1,2)\right| \le (\ln 2 - \ln(\ln 2) - 1)(M-m)$$

and

$$(2.11)\qquad \left|E_{-1}(X) - M_{-1}^{-1}(1,2)\right| \le \Big(\frac{1-2(\ln 2)^2}{8}\Big)^{\frac{1}{2}}(M-m).$$

We note that the bound in (2.10) is better than the one in (2.11).

The following theorem also holds.

Theorem 7. *Let X and f be as above. If*

$$\sigma_\mu(X) = \left[\int_a^b (t-\mu)^2 f(t)dt\right]^{1/2}, \qquad \mu \in [a,b],$$

then we have the inequality

$$(2.12)\quad |\sigma_\mu^2(X) - A(\mu)| \le (M-m)\Big[A(\mu)(\mu - \frac{a+b}{2}) + \frac{2}{3}A(\mu)^{\frac{3}{2}} + \frac{(b-\mu)^3 - (\mu-a)^3}{6}\Big]$$

where $A(\mu) = (\mu - \frac{a+b}{2})^2 + \frac{(b-a)^2}{12}$.

Proof. If we put $g(t) = (t-\mu)^2$ in (1.1), we get

$$(2.13)\qquad \left|\frac{1}{b-a}\int_a^b f(t)(t-\mu)^2 dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b (t-\mu)^2 dt\right|$$
$$\le \frac{(M-m)}{2(b-a)}\int_a^b \left|(t-\mu)^2 - \frac{1}{b-a}\int_a^b (t-\mu)^2 dt\right| dt,$$

and as $\int_a^b f(t)dt = 1$,

$$\begin{aligned}\frac{1}{b-a}\int_a^b (t-\mu)^2 dt &= \frac{(b-\mu)^3+(\mu-a)^3}{3(b-a)} = \frac{(b-\mu)^2-(b-\mu)(\mu-a)+(\mu-a)^2}{3} \\ &= (\mu-\frac{a+b}{2})^2+\frac{(b-a)^2}{12} = A(\mu) > 0,\end{aligned}$$

$$\begin{aligned}&\int_a^b \left|(t-\mu)^2 - \frac{1}{b-a}\int_a^b (t-\mu)^2 dt\right| dt = \int_a^b \left|(t-\mu)^2 - A(\mu)\right| dt \\ =& \int_a^{\mu+A(\mu)^{1/2}} (A(\mu)-(t-\mu)^2)dt + \int_{\mu+A(\mu)^{1/2}}^b ((t-\mu)^2-A(\mu))dt \\ =& \ 2A(\mu)(\mu-\frac{a+b}{2}) + \frac{4}{3}A(\mu)^{\frac{3}{2}} + \frac{(b-\mu)^3-(\mu-a)^3}{3},\end{aligned}$$

then by (2.13) we deduce (2.12). □

For $\mu = (a+b)/2$, we have the following corollary that improve the Corollary 13 in [1].

Corollary 8. *With the above assumptions and denoting* $\sigma_0(X) = \sigma_{(a+b)/2}(X)$, *we have the inequality*

$$\left|\sigma_0^2(X) - \frac{(b-a)^2}{12}\right| \le \frac{1}{36\sqrt{3}}(M-m)(b-a)^3.$$

The following theorem also holds.

Theorem 9. *Let* X *and* f *be as above. If*

$$A_\mu(X) = \int_a^b |t-\mu| f(t)dt, \quad \mu \in [a,b],$$

then we have the inequality

$$|A_\mu(X) - B(\mu)| \le (M-m)\left[\frac{(b-\mu)^2+(\mu-a)^2}{4} - \frac{(b-a)B(\mu)}{2} + B^2(\mu)\right]$$

where $B(\mu) = \frac{1}{b-a}[(\mu-\frac{a+b}{2})^2 + \frac{(b-a)^2}{4}]$.

Proof. If we put $g(t) = |t-\mu|$ in (1.1), we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b |t-\mu| f(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \cdot \frac{1}{b-a}\int_a^b |t-\mu| dt\right| \\ \le& \ \frac{(M-m)}{2(b-a)}\int_a^b \left| |t-\mu| - \frac{1}{b-a}\int_a^b |s-\mu| ds\right| dt,\end{aligned} \tag{2.14}$$

and as $\int_a^b f(t)dt = 1$,

$$\begin{aligned}\frac{1}{b-a}\int_a^b |t-\mu|dt &= \frac{1}{b-a}\left[\int_0^\mu \mu - t + \int_\mu^b t-\mu dt\right]\\ &= \frac{1}{b-a}\left[(\mu-\frac{a+b}{2})^2+\frac{(b-a)^2}{4}\right] = B(\mu),\end{aligned}$$

$$\begin{aligned}&\int_a^b \left||t-\mu| - \frac{1}{b-a}\int_a^b |s-\mu|ds\right| dt\\ =& \int_a^b ||t-\mu| - B(\mu)|\, dt\\ =& \int_a^\mu |\mu - t - B(\mu)|dt + \int_\mu^b |t-\mu-B(\mu)|dt\\ =& \int_a^{\mu-B(\mu)} (\mu - B(\mu) - t)dt + \int_{\mu-B(\mu)}^{\mu} (t-\mu+B(\mu))dt\\ &+ \int_\mu^{\mu+B(\mu)} (\mu + B(\mu) - t)dt + \int_{\mu+B(\mu)}^{b} (t-\mu-B(\mu))dt\\ =& \frac{(b-\mu)^2+(\mu-a)^2}{2} - (b-a)B(\mu) + 2B^2(\mu).\end{aligned}$$

Finally, using (2.14), we deduce the desired inequality. □

For $\mu = \mu_0 = \frac{a+b}{2}$ in Theorem 9, we have the following corollary that improve the Corollary 14 in [1].

Corollary 10. *With the above assumption, we have the inequality*

$$\left|A_{\mu_0}(X) - \frac{b-a}{4}\right| \le \frac{1}{16}(M-m)(b-a)^2.$$

3. Some inequalities for the cumulative distribution function

The following theorem contains an inequality which connects the expectation $E(X)$, the cumulative distribution function $Pr(X \le x) = F(x) = \int_a^x f(t)dt$, and the bounds M and m of the probability density function $f : [a,b] \to R$. In [2], Barnett and Dragomir have established the following equality:

$$(3.1)\qquad (b-a)F(x) + E(X) - b = \int_a^b p(x,t)dF(t) = \int_a^b p(x,t)f(t)dt,$$

where

$$p(x,t)=\begin{cases} t-a, & \text{if } a\le t\le x\le b,\\ t-b, & \text{if } a\le x<t\le b.\end{cases}$$

Theorem 11. *Let $X,\ f,\ E(X),\ F(\cdot),$ and $m,\ M$ be as above, then*

$$\left|E(X)+(b-a)F(x)-x-\frac{b-a}{2}\right|\le\frac{1}{8}(M-m)(b-a)^2 \tag{3.2}$$

for all $x\in[a,b]$.

Proof. Applying the equality (3.1) and putting $g(t)=p(x,t)$ in (1.1), we get

$$\begin{aligned}&\left|E(X)+(b-a)F(x)-b-\frac{1}{b-a}\int_a^b p(x,t)dt\cdot\int_a^b f(t)dt\right| \\ \le\ & \frac{(M-m)}{2}\int_a^b\left|p(x,t)-\frac{1}{b-a}\int_a^b p(x,s)ds\right|dt.\end{aligned} \tag{3.3}$$

Observe that

$$\frac{1}{b-a}\int_a^b p(x,t)dt=x-\frac{a+b}{2},\qquad \int_a^b f(t)dt=1,$$

$$\begin{aligned}&\int_a^b\left|p(x,t)-\frac{1}{b-a}\int_a^b p(x,s)ds\right|dt\\ =\ &\int_a^x\left|t-x+\frac{b-a}{2}\right|dt+\int_x^b\left|t-x-\frac{b-a}{2}\right|dt\end{aligned}$$

and

$$\begin{aligned}&\int_a^x\left|t-x+\frac{b-a}{2}\right|dt+\int_x^b\left|t-x-\frac{b-a}{2}\right|dt\\ =\ &\int_a^x(t-x+\frac{b-a}{2})dt+\int_x^{x+\frac{b-a}{2}}(x+\frac{b-a}{2}-t)dt+\int_{x+\frac{b-a}{2}}^b(t-x-\frac{b-a}{2})dt\\ =\ &\frac{(b-a)^2}{4},\end{aligned}$$

if $a\le x\le\frac{a+b}{2}$,

and

$$\begin{aligned}&\int_a^{x-\frac{b-a}{2}}(x-\frac{b-a}{2}-t)dt\int_{x-\frac{b-a}{2}}^x(t-x+\frac{b-a}{2})dt+\int_x^b(x+\frac{b-a}{2}-t)dt\\ =\ &\frac{(b-a)^2}{4},\end{aligned}$$

if $\frac{a+b}{2} < x \leq b$.

Using (3.3), we deduce (3.2). □

Remark 12. If in (3.2), we choose $x = (a+b)/2$, then we get the inequality

$$\left| E(X) + (b-a)Pr(X \leq \frac{a+b}{2}) - b \right| \leq \frac{1}{8}(M-m)(b-a)^2. \tag{3.4}$$

The inequality (3.4) is an improvement of inequality (5.21) in [1].

The following theorem also holds.

Theorem 13. *Let $X,\ f,\ F(\cdot)$, and $m,\ M$ be as above, then we have*

$$\left| E(X) + \frac{(b-a)}{2}F(x) - \frac{b+x}{2} \right| \leq \frac{1}{4}(M-m)\left[\frac{(b-a)^2}{4} + (x - \frac{a+b}{2})^2 \right], \tag{3.5}$$

for all $x \in [a,b]$.

Proof. Applying the equality (3.1), we get

$$(b-a)F(x) + E(x) - b = \int_a^x (t-a)f(t)dt + \int_x^b (t-b)f(t)dt, \tag{3.6}$$

for all $x \in [a,b]$.

Applying (1.1), we get, for $x \in [a,b]$,

$$\begin{aligned} &\left| \frac{1}{x-a}\int_a^x (t-a)f(t)dt - \frac{1}{x-a}\int_a^x (t-a)dt \cdot \frac{1}{x-a}\int_a^x f(t)dt \right| \\ \leq\ & \frac{(M-m)}{2(x-a)}\int_a^x \left| (t-a) - \frac{1}{x-a}\int_a^x (t-a)dt \right| dt, \\ =\ & \frac{1}{8}(M-m)(x-a) \end{aligned} \tag{3.7}$$

and, similarly,

$$\begin{aligned} &\left| \frac{1}{b-x}\int_x^b (t-b)f(t)dt - \frac{1}{b-x}\int_x^b (t-b)dt \cdot \frac{1}{b-x}\int_x^b f(t)dt \right| \\ =\ & \frac{1}{8}(M-m)(b-x). \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we can write

$$\left| \int_a^x (t-a)f(t)dt - \frac{x-a}{2}F(x) \right| \leq \frac{1}{8}(M-m)(x-a)^2 \tag{3.9}$$

and

$$\left| \int_x^b (t-b)f(t)dt + \frac{b-x}{2}(1-F(x)) \right| \leq \frac{1}{8}(M-m)(b-x)^2 \tag{3.10}$$

for all $x \in [a, b]$.

Summing (3.9) and (3.10) and using the triangle inequalities, we deduce that

$$\begin{aligned} & \left| \int_a^x (t-a)f(t)dt + \int_x^b (t-b)f(t)dt - \frac{b-a}{2}F(x) + \frac{b-x}{2} \right| \\ \leq & \frac{1}{8}(M-m)[(x-a)^2 + (b-x)^2] \\ = & \frac{1}{4}(M-m)[\frac{(b-a)^2}{4} + (x - \frac{a+b}{2})^2]. \end{aligned}$$

Using the identity (3.6), the desired inequality (3.5) is obtained. □

Remark 14. If we choose in (3.5) either $x = a$ or $x = b$, we get the inequality

$$\left| E(X) - \frac{a+b}{2} \right| \leq \frac{1}{8}(M-m)(b-a)^2, \tag{3.11}$$

and thus recapture (2.1). We note that the inequality (3.11) is an improvement of the inequality (5.29) in [1].

Remark 15. If in (3.5) we choose $x = (a+b)/2$, then we get

$$\left| E(X) + (\frac{b-a}{2})Pr(X \leq \frac{a+b}{2}) - \frac{a+3b}{4} \right| \leq \frac{1}{16}(M-m)(b-a)^2, \tag{3.12}$$

The inequality (3.12) is an improvement of the inequality (5.30) in [1].

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