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Examples of Quadratically Hyponormal Weighted Shifts

Wei He
Department of Mathematics, Fudan University, Shanghai 200-433, P. R. China
e-mail : hewei_neu@sohu.com

Chunji Li
Institute of System Science, College of Sciences, Northeastern University, Shenyang, Liaoning 110-004, P. R. China
e-mail : chunjili@hanmail.net

Abstract. In this paper, we have a further discussion about quadratically hyponormal weighted shifts with weight sequence $\alpha : 1, 1, \sqrt{a}, \left(\sqrt{b}, \sqrt{c}, \sqrt{d}\right)^{\wedge}$ on the basis of sufficient conditions for positively quadratically hyponormal weighted shifts. We set examples of quadratically hyponormal weighted shifts with weight sequence of the above form, and also establish a general method for setting examples.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is *hyponormal* if $T^*T \geq TT^*$, *weakly k-hyponormal* if $\lambda_1 T + \lambda_2 T^2 + \cdots + \lambda_k T^k$ is hyponormal for every $\lambda_i \in \mathbb{C}$, $i = 1, \cdots, k$. In particular, the weakly 2-hyponormal is often said to be *quadratically hyponormal.*

Let $\{e_n\}_{n=0}^{\infty}$ be the canonical orthonormal basis for $l^2(\mathbb{Z}_+)$, and let $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ be a bounded sequence of positive numbers. Let W_α be the *unilateral weighted shift* defined on $l^2(\mathbb{Z}_+)$ by $W_\alpha e_n := \alpha_n e_{n+1}$, for $n = 0, 1, 2, \cdots$. The numbers

$$\gamma_0 := 1, \;\; \gamma_1 := \alpha_0^2, \;\; \gamma_2 := \alpha_0^2 \alpha_1^2, \;\; \cdots, \; \gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2, \;\; \cdots$$

are called the *moments* of W_α. It is well known that W_α is hyponormal if and only if $\alpha_n \leq \alpha_{n+1}$, for $n = 0, 1, 2, \cdots$.

In the unilateral weighted shift operators, the recursively generated weighted shift operators provide a good role to detect the quadraticality of shift operators (cf. [3]). We recall here the unilateral weighted shift which is generated by only three weights $0 < \alpha_0 < \alpha_1 < \alpha_2$ (see [4, p. 370] for the general case). Given $\alpha : \alpha_0, \alpha_1, \alpha_2$ with $0 < \alpha_0 < \alpha_1 < \alpha_2$. Then $\gamma_0 := 1,\ \gamma_1 := \alpha_0^2,\ \gamma_2 := \alpha_0^2\alpha_1^2,\ \gamma_3 :=$

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$\alpha_0^2\alpha_1^2\alpha_2^2$. Let

$$v_0 := \begin{pmatrix} \gamma_0 \\ \gamma_1 \end{pmatrix}, \quad v_1 := \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad v_2 := \begin{pmatrix} \gamma_2 \\ \gamma_3 \end{pmatrix}.$$

Then the vectors v_0 and v_1 are linearly independent in $\mathbb{R}^2$, so there exists a unique real numbers Ψ_0, Ψ_1 such that

$$\Psi_0 v_0 + \Psi_1 v_1 = v_2. \tag{1.1}$$

In fact,

$$\Psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2 - \alpha_1^2)}{\alpha_1^2 - \alpha_0^2}, \qquad \Psi_1 = \frac{\alpha_1^2(\alpha_2^2 - \alpha_0^2)}{\alpha_1^2 - \alpha_0^2}. \tag{1.2}$$

Moreover, the equality (1.1) is $\Psi_0\gamma_0 + \Psi_1\gamma_1 = \gamma_2$ and $\Psi_0\gamma_1 + \Psi_1\gamma_2 = \gamma_3$. Let $\widehat{\gamma}_i := \gamma_i,\ i = 0, 1$ and let

$$\widehat{\gamma}_n := \Psi_0\gamma_{n-2} + \Psi_1\gamma_{n-1}, \quad (n \geq 2). \tag{1.3}$$

Since $\widehat{\gamma}_n > 0$ $(n \geq 0)$ (see [3, Proof of Theorem 3.5]), we define $\widehat{\alpha}_n := (\widehat{\gamma}_{n+1}/\widehat{\gamma}_n)^{1/2}$ $(n \geq 0)$ (so that $\widehat{\alpha}_n = \alpha_n$ for $0 \leq n \leq 2$). Hence we obtain a bounded sequence $\widehat{\alpha} := \{\widehat{\alpha}_i\}_{i=0}^{\infty}$ and we obtain the weighted shift operator $W_{\widehat{\alpha}}$ (or written as $W_{(\alpha_0,\alpha_1,\alpha_2)^\wedge}$).

In [6], Curto-Jung discussed weight sequence of the form, $\alpha : 1, 1, \alpha_2, \alpha_3, \cdots, (1 < \alpha_2 < \alpha_3 < \cdots)$ and presented a sufficient conditions for which W_α being positively quadratically hyponormal. Since the conditions of [6, Theorem 3.11] are very strict, it is not easy to give a concrete example. In this paper, we can give a concrete example for which the weight sequence of the form $\alpha : 1, 1, \sqrt{a}, \left(\sqrt{b}, \sqrt{c}, \sqrt{d}\right)^\wedge$ for [6, Theorem 3.11]. We can give many real examples of positively quadratically hyponormal weighted shift operators for Curto-Jung's theorem. Such examples are abundant!

2. Preliminaries and notations

We recall some notation which will be used frequently in this paper (cf. [6]). An operator $T \in \mathcal{L}(\mathcal{H})$ is *quadratically hyponormal* if $T + sT^2$ is hyponormal for every $s \in \mathbb{C}$. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for $\mathcal{H}$, let P_n denote the orthogonal projection onto the subspace generated by $e_0, \cdots, e_n$, and let W_α be a hyponormal weighted shift with a weight sequence $\alpha = \{\alpha_n\}_{n=0}^{\infty}$. For $s \in \mathbb{C}$, we let

$$D(s) := \left[\left(W_\alpha + sW_\alpha^2\right)^*, \left(W_\alpha + sW_\alpha^2\right)\right].$$

For $n \geq 0$, let

$$\begin{aligned} D_n(s) &= P_n \left[\left(W_\alpha + s W_\alpha^2 \right)^*, \left(W_\alpha + s W_\alpha^2 \right) \right] P_n \\ &= \begin{pmatrix} q_0 & \overline{r}_0 & 0 & \cdots & 0 & 0 \\ r_0 & q_1 & \overline{r}_1 & \cdots & 0 & 0 \\ 0 & r_1 & q_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & q_{n-1} & \overline{r}_{n-1} \\ 0 & 0 & 0 & \cdots & r_{n-1} & q_n \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} q_k &: = u_k + |s|^2 v_k, \\ r_k &: = s\sqrt{w_k}, \\ u_k &: = \alpha_k^2 - \alpha_{k-1}^2, \\ v_k &: = \alpha_k^2 \alpha_{k+1}^2 - \alpha_{k-1}^2 \alpha_{k-2}^2, \\ w_k &: = \alpha_k^2 \left(\alpha_{k+1}^2 - \alpha_{k-1}^2 \right)^2 \quad (k \geq 0), \end{aligned}$$

and $\alpha_{-1} = \alpha_{-2} := 0$. Hence, W_α is quadratically hyponormal if and only if $D_n(s) \geq 0$ for every $s \in \mathbb{C}$ and every $n \geq 0$. To do this, we consider $d_n(\cdot) := \det(D_n(\cdot))$. By direct computation we have

$$\begin{aligned} d_0 &= q_0, \\ d_1 &= q_0 q_1 - |r_0|^2, \end{aligned}$$

$$d_{n+2} = q_{n+2} d_{n+1} - |r_{n+1}|^2 d_n \quad (n \geq 0). \tag{2.1}$$

Clearly, d_n is a polynomial in $t := |s|^2$ of degree $n+1$, with Maclaurin expansion

$$d_n(t) := \sum_{i=0}^{n+1} c(n,i)\, t^i. \tag{2.2}$$

Theorem 2.1 ([4, Lemma 4.1, (ii)]). *If $d_n(t) > 0$ for all $t > 0$ and all $n \geq 0$, then W_α is quadratically hyponormal.*

Hence, it is sufficient to verify that $c(n,i) \geq 0$ and $c(n, n+1) > 0$ in (2.2). To do it, we introduce the following terminology.

Definition 2.2. Let $\alpha : \alpha_0, \alpha_1, \cdots$ be a weight sequence. We say that W_α is *positively quadratically hyponormal* if $c(n,i) \geq 0$ for all n, $i \geq 0$ with $0 \leq i \leq n+1$, and $c(n,\ n+1) > 0$ for all $n \geq 0$.

It is clear that positively quadratically hyponormal shifts are quadratically hyponormal. From (2.1), we have

$$\begin{aligned} c(0,0) &= u_0, \quad c(0,1) = v_0, \\ c(1,0) &= u_1 u_0, \; c(1,1) = u_1 v_0 + u_0 v_1 - w_0, \quad c(1,2) = v_1 v_0, \end{aligned}$$

$$c(n+2,i) = u_{n+2} c(n+1,i) + v_{n+2} c(n+1,i-1) - w_{n+1} c(n,i-1). \tag{2.3}$$

In particular, for our weight sequence $\alpha : 1, 1, \alpha_2, \alpha_3, \cdots (1 < \alpha_2 < \alpha_3 < \cdots)$, we can obtain directly

$$\begin{aligned} c(0,0) &= 1, \quad c(0,1) = 1, \\ c(1,0) &= 0, \quad c(1,1) = \alpha_2^2 - 1, \quad c(1,2) = \alpha_2^2, \\ c(2,0) &= 0, \quad c(2,1) = 0, \quad c(2,2) = (\alpha_2^2 - 1)\alpha_2^2\alpha_3^2, \\ c(2,3) &= \alpha_2^2(\alpha_2^2\alpha_3^2 - 1), \qquad c(3,0) = 0, \quad c(3,1) = 0, \\ c(3,2) &= \alpha_2^2(\alpha_2^2 - 1)(2\alpha_3^2 - \alpha_2^2\alpha_3^2 - 1), \\ c(3,3) &= \alpha_2^2\alpha_3^2(\alpha_2^2\alpha_3^2\alpha_4^2 - \alpha_3^2\alpha_4^2 - 2\alpha_2^4 + 3\alpha_2^2 - 1), \\ c(3,4) &= \alpha_2^2(\alpha_3^2\alpha_4^2 - \alpha_2^2)(\alpha_2^2\alpha_3^2 - 1), \end{aligned}$$

and

$$c(n,i) = u_n c(n-1,i) + v_n c(n-1,i-1) - w_{n-1} c(n-2,i-1). \tag{2.4}$$

for $n \geq 5, \ 0 \leq i \leq n+1$. From (2.4), we obtain

$$\begin{aligned} c(n,i) = \; & u_n c(n-1,i) + v_n v_{n-1} \cdots v_5 \left(v_4 c\left(3, i-n-3\right) - w_3 c(2, i-n-3)\right) \\ & + \{ \left(v_n u_{n-1} - w_{n-1}\right) c(n-2, i-1) \\ & + v_n (v_{n-1} u_{n-1} - w_{n-1}) c(n-3, i-2) + \cdots \\ & + v_n v_{n-1} \cdots v_6 \left(v_5 u_4 - w_4\right) c\left(3, i-n+4\right) \}. \end{aligned}$$

Hence, to satisfy $c(n,i) \geq 0$, we need only

$$\begin{aligned} v_4 c(3, i-n+3) - w_3 c(2, i-n+3) &\geq 0 \quad (i \geq 2) \\ v_{n+1} u_n - w_n &\geq 0 \quad (n \geq 4). \end{aligned}$$

It is exactly the following theorem..

Theorem 2.3 ([6, Theorem 3.11])**.** *Let $\alpha : 1, 1, \sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}, \alpha_6, \cdots$ be a weight sequence and assume that W_α is hyponormal. Suppose that*

(i) $(cad - ab)(2b - ab - 1) \geq b^2 (c-a)^2$,

(ii) $(cd - ab)\left(abc - bc - 2a^2 + 3a - 1\right) \geq (ab-1)(c-a)^2$, *and*

(iii) $\left(\alpha_{n+1}^2 \alpha_{n+2}^2 - \alpha_n^2 \alpha_{n-1}^2\right)\left(\alpha_n^2 - \alpha_{n-1}^2\right) \geq \alpha_n^2 \left(\alpha_{n+1}^2 - \alpha_{n-1}^2\right)^2 \quad (n \geq 4)$.

Then W_α is positively quadratically hyponormal.

In general, Theorem 2.3 (iii) is not easily checked. But, for the generated weighted sequence, we have the following result.

Proposition 2.4 ([6, Lemma 2.1]). *Let*

$$\alpha : \alpha_0, \alpha_1, \cdots, \alpha_{k-2}, (\alpha_{k-1}, \alpha_k, \alpha_{k+1})^\wedge$$

with $0 < \alpha_{k-1} < \alpha_k < \alpha_{k+1}$ $(k \geq 1)$ *and let* W_α *be the unilateral weighted shift associated with* α. *Then*

(i) $v_{n+1} = \Psi_1 (u_{n+1} + u_n)$ $\quad (n \geq k)$;

(ii) $w_n = u_n v_{n+1}$ $\quad (n \geq k)$ *and*

$$u_n = -\Psi_0 \frac{u_{n-1}}{\alpha_{n-2}^2 \alpha_{n-1}^2} \qquad (n \geq k+1),$$

where

$$\Psi_0 = -\frac{\alpha_k^2 \alpha_{k-1}^2 (\alpha_{k+1}^2 - \alpha_k^2)}{\alpha_k^2 - \alpha_{k-1}^2} \quad \text{and} \quad \Psi_1 = \frac{\alpha_k^2 (\alpha_{k+1}^2 - \alpha_{k-1}^2)}{\alpha_k^2 - \alpha_{k-1}^2}.$$

It is easy to see that if Proposition 2.4 (ii) holds, then Theorem 2.3 (iii) holds for $n \geq k$.

In particular, we consider the weight sequence of the first two equal weights, i.e., $\alpha : 1, 1, \alpha_2, \alpha_3, \cdots$.

3. Examples of Quadratically hyponormal weighted shift

In this section, we check the conditions of Theorem 2.3 for the following three weight sequences

$$\alpha : \begin{cases} 1, \left(1, \sqrt{a}, \sqrt{b}\right)^\wedge \\ 1, 1, \left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)^\wedge \\ 1, 1, \sqrt{a}, \left(\sqrt{b}, \sqrt{c}, \sqrt{d}\right)^\wedge \end{cases} \qquad (0 < a < b < c < d).$$

Example 3.1. First, we consider W_α with $\alpha : 1, \left(1, \sqrt{a}, \sqrt{b}\right)^\wedge$ ([6, Example 3.13]). Since

$$\gamma_0 = \gamma_1 = \gamma_2 = 1, \;\; \gamma_3 = a, \;\; \gamma_4 = ab.$$

From (1.1), we have

$$\Psi_0 = -\frac{a(b-a)}{a-1}, \quad \Psi_1 = \frac{a(b-1)}{a-1}.$$

Thus

$$\begin{aligned}\gamma_5 &= \Psi_0\gamma_3+\Psi_1\gamma_4=\frac{a^2(a-2b+b^2)}{a-1},\\ \gamma_6 &= \Psi_0\gamma_4+\Psi_1\gamma_5=\frac{a^2\left(2a^2b-4ab^2+ab^3+ab-a^2+b^2\right)}{(a-1)^2}.\end{aligned}$$

Taking the symbol of Theorem 2.3, we have

$$\begin{aligned}c &= \alpha_4^2=\frac{\gamma_5}{\gamma_4}=\frac{a\left(a-2b+b^2\right)}{(a-1)\,b},\\ d &= \alpha_5^2=\frac{\gamma_6}{\gamma_5}=\frac{2a^2b-4ab^2+ab^3+ab-a^2+b^2}{(a-1)\,(a-2b+b^2)}.\end{aligned}$$

If the condition (i) of Theorem 2.3 holds, then it must be

$$\frac{a^2\,(a-b)\,(b-1)^4}{(a-1)^2\,b}\geq 0,$$

This induces a contradiction for $a<b$. Hence by Theorem 2.3, there is no example of positively quadratically hyponormal weighted shift, in the case of the form α : $1,\left(1,\sqrt{a},\sqrt{b}\right)^\wedge$.

Example 3.2. We now consider W_α with $\alpha:1,1,\left(\sqrt{a},\sqrt{b},\sqrt{c}\right)^\wedge$. Since

$$\gamma_0=\gamma_1=\gamma_2=1,\quad \gamma_3=a,\quad \gamma_4=ab,\quad \gamma_5=abc.$$

From

$$\Psi_0\begin{pmatrix}1\\ a\end{pmatrix}+\Psi_1\begin{pmatrix}a\\ ab\end{pmatrix}=\begin{pmatrix}ab\\ abc\end{pmatrix},$$

we have

$$\Psi_0=\frac{ab(b-c)}{b-a},\quad \Psi_1=\frac{b(c-a)}{b-a}.$$

Thus

$$\gamma_6=\Psi_0\gamma_4+\Psi_1\gamma_5=\frac{ab^2\left(ab-2ac+c^2\right)}{b-a}.$$

Taking the symbol of Theorem 2.3,

$$d=\alpha_5^2=\frac{\gamma_6}{\gamma_5}=\frac{b\left(ab-2ac+c^2\right)}{c\,(b-a)}.$$

If the condition (i) of Theorem 2.3 holds, then it must be

$$-\frac{(-1+b)^2\,b\,(a-c)^2}{b-a}\geq 0.$$

So it is a contradiction for $a < b$ again. This means that there is no example of positively quadratically hyponormal weighted shift, in the case of the form $\alpha : 1, 1, \left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)^{\wedge}$ $(1 < a < b < c)$.

From the above two examples, we know that the condition of Theorem 2.3 is very strict, it is not easy to give an example of quadratically hyponormal weighted shift operators. But, it is lucky for us to give an example of quadratically hyponormal weighted shift with weight sequence of the form $\alpha : 1, 1, \sqrt{a}, \left(\sqrt{b}, \sqrt{c}, \sqrt{d}\right)^{\wedge}$ $(1 < a < b < c < d)$. This is the following result.

Theorem 3.3. *Let* $\alpha : 1, 1, \sqrt{\frac{4}{3}}, \left(\sqrt{x}, \sqrt{2}, \sqrt{d}\right)^{\wedge}$ $(\frac{4}{3} < x < 2 < d)$. *If* $d > 4$, *then* W_α *is quadratically hyponormal for all* x *in* $\left[\left(1 + d - \sqrt{1 - 4d + d^2}\right)/2,\ 2\right)$.

Proof. The condition (iii) of Theorem 2.3 is satisfied, due to the condition (ii) of Proposition 2.4. Let

$$H = \{x \,|4/3 < x < 2, \text{ and } W_\alpha \text{ is quadratically hyponormal}\}.$$

Our goal is finding the range of d such that H is non-empty. Taking $a = \frac{4}{3}$, $b = x$, $c = 2$ in the condition (i) of Theorem 2.3, we have

$$f(x) := 2x^2 - 2(d+1)x + 3d \leq 0. \tag{3.1}$$

Let x_1, x_2 $(x_1 \leq x_2)$ be the two roots of $f(x) = 0$.

Case 1: $x_1, x_2 \in \left(\frac{4}{3}, 2\right)$. In this case, $\Delta := 4d^2 - 16d + 4 \geq 0$ and $\frac{4}{3} < -\frac{-2(d+1)}{2\times 2} < 2$. That is, $d \geq 2 + \sqrt{3}$ and $\frac{5}{3} < d < 3$. It is impossible.

Case 2: only one of x_1 and x_2 is in $\left(\frac{4}{3}, 2\right)$. In this case, $f\left(\frac{4}{3}\right) \cdot f(2) < 0$. That is, $d > 4$. The range of x in (3.1) is

$$\frac{1}{2}\left(1 + d - \sqrt{1 - 4d + d^2}\right) \leq x < 2.$$

Case 3: $x_1 < \frac{4}{3}$ and $x_2 > 2$. In this case, $f\left(\frac{4}{3}\right) < 0$ and $f(2) < 0$. That is, $d < -\frac{8}{3}$ and $d > 4$. It is impossible.

Therefore, if we want $H \neq \emptyset$, then $d > 4$. Thus

$$H = \left\{x \,\middle|\, \left(1 + d - \sqrt{1 - 4d + d^2}\right)/2 \leq x < 2\right\}.$$

Similarly, we can discuss the condition (ii) of Theorem 2.3. From (ii), we obtain

$$g(x) := 12x^2 - 2(9d + 1)x + 3(5d - 2) \leq 0. \tag{3.2}$$

Let x_1' and x_2' $(x_1' \leq x_2')$ be the two roots of $g(x) = 0$.

Case 1: $x_1',\ x_2' \in \left(\frac{4}{3}, 2\right)$. In this case, $g(2) > 0$. That is, $d < \frac{38}{21}$. It is contradict to $d > 2$.

Case 2: only one of x_1', x_2' is in $\left(\frac{4}{3}, 2\right)$. In this case, $g\left(\frac{4}{3}\right) g(2) < 0$. That is, $\frac{38}{27} < d < \frac{38}{21}$. It is also contradict to $d > 2$.

Case 3: $x_1' < \frac{4}{3}$ and $x_2' > 2$. In this case, $d > 2$, $g\left(\frac{4}{3}\right) < 0$, $g(2) < 0$. That is, $d > 2$.

Therefore, if we want $H \neq \emptyset$, $d > 4$. □

Therefore, according to the above method, we can give many concrete examples of quadratically hyponormal weighted shifts satisfying Curto-Jung's theorem.

Example 3.4. In Theorem 3.3, we let $d = 10$. If

$$1.59488 \approx \frac{1}{2}\left(11 - \sqrt{61}\right) \leq x < 2,$$

then W_α is quadratically hyponormal.

We discussed one particular case for $a = \frac{4}{3}, c = 2$. In general, the problem is as following.

Problem 3.5. Let W_α be the weighted shift operator with

$$\alpha : 1, 1, \sqrt{a}, \left(\sqrt{x}, \sqrt{c}, \sqrt{d}\right)^\wedge \ (a < x < c < d).$$

Giving a, c such that $2a + c^2 - 2ac > 0$, find the range of d for which

$$H = \{x \,| a < x < c, \text{ and } W_\alpha \text{ is quadratically hyponormal.}\} \neq \emptyset.$$

For the problem, we give the following algorithm similar to the above method.

Algorithm 3.6. For the condition (i) of Theorem 2.3, we can obtain an quadratic inequality of x, $Ax^2 + Bx + C \leq 0$, where $A = 2a + c^2 - 2ac > 0$, $B = -(a + 2cd - acd)$, $C = cd$. Let $f(x) = Ax^2 + Bx + C$ and x_1, x_2 $(x_1 \leq x_2)$ be the two roots of $f(x) = 0$.

Case 1: $x_1, x_2 \in (a, c)$

$$d > c, \ \Delta \geq 0, \ f(a) > 0, \ f(c) > 0, \text{ and } a < -\frac{B}{2A} < c,$$

the set of solution E_1.

Case 2: only one of x_1, x_2 is in (a, c)

$$d > c \text{ and } f(a) f(c) < 0,$$

the set of solution E_2.

Case 3: $x_1 < a$ and $x_2 > c$

$$d > c, \ f(a) < 0 \text{ and } f(c) < 0,$$

the set of solution E_3.

Therefore, if we want $H \neq \emptyset$, then the range of d is

$$E = E_1 \cup E_2 \cup E_3.$$

Similarly, for the condition (ii) of Theorem 2.3, we can obtain the range F of d. Then if we choose d in the set $D = E \cap F$, then $H \neq \emptyset$.

Example 3.7. For $\alpha : 1,\ 1, \sqrt{a},\ \left(\sqrt{x}, \sqrt{c}, \sqrt{d}\right)^\wedge$, we let $a = \frac{3}{2},\ c = 2$. By taking Algorithm 3.6, we know that $D = \emptyset$. (Here, we omit the calculation). Hence, we can not give an example of quadratically hyponormal weighted shift operators with the weight sequence $\alpha : 1, 1, \sqrt{\frac{3}{2}}, \left(\sqrt{b}, \sqrt{2}, \sqrt{d}\right)^\wedge$, which means the condition of Theorem 2.3 is very strict.

Remark. From Example 3.1, Example 3.2 and Example 3.7, we know that the condition of Theorem 2.3 is very strict, especially, in most cases, the condition (i) is not satisfied. Hence, it is necessary to find another conditions. In [4], they presented a sufficient condition for $W_{\sqrt{x},\ (\sqrt{a},\sqrt{b},\sqrt{c})^\wedge}$. Using this result, [6, Example 3.13] discussed the range

$$R = \left\{(x, y) \,\middle|\, W_{1,(1,\sqrt{x},\sqrt{y})^\wedge} \text{ is quadratically hyponormal.} \right\}$$

It is clear that the condition of Theorem 4.3 in [4] is more weak than the condition of Theorem 2.3. Henceforth, we do need to weaken the condition of Theorem 2.3. Maybe, it can be done by changing the expression (2.4).

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