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First Order Differential Subordinations and Starlikeness of Analytic Maps in the Unit Disc

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Abstract. Let α be a complex number with $\Re\,\alpha > 0$. Let the functions f and g be analytic in the unit disc $E = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = g(0) = 0$, $f'(0) = g'(0) = 1$. In the present article, we study the differential subordinations of the forms

$$\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \alpha \frac{z^2 g''(z)}{g(z)} + \frac{zg'(z)}{g(z)},\ z \in E,$$

and

$$\frac{z^2 f''(z)}{f(z)} \prec \frac{z^2 g''(z)}{g(z)},\ z \in E.$$

As consequences, we obtain a number of sufficient conditions for starlikeness of analytic maps in the unit disc. Here, the symbol ' $\prec$ ' stands for subordination.

1. Introduction

Let $\mathcal{H}$ denote the space of analytic functions in the unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$, with the topology of local uniform convergence. Denote by $\mathcal{A}$ and $\mathcal{A}'$, the subspaces of $\mathcal{H}$ consisting of functions f which are normalized by the conditions $f(0) = f'(0) - 1 = 0$ and $f(0) = 1$, respectively. Further, let

$$S^*(\alpha) = \left\{ f \in \mathcal{A} :\ \Re \frac{zf'(z)}{f(z)} > \alpha,\ z \in E,\ 0 \le \alpha < 1 \right\},$$

and

$$S(\alpha) = \left\{ f \in \mathcal{A} : \left| \arg \frac{zf'(z)}{f(z)} \right| < \alpha \frac{\pi}{2},\ z \in E,\ 0 < \alpha \le 1 \right\},$$

be the subspaces of $\mathcal{A}$ consisting of starlike functions of order α and strongly starlike functions of order α, respectively. Note that $S^*(0) = S(1) = S^*$ is the well-known space of normalized functions starlike (univalent) with respect to the origin. We denote by K, the family of all convex functions in E defined as under:

$$K = \left\{ f \in \mathcal{H} :\ f'(0) \neq 0,\ \Re \left[1 + \frac{zf''(z)}{f'(z)} \right] > 0,\ z \in E \right\}.$$

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A function $f \in \mathcal{H},\ f'(0) \neq 0$, is said to be close-to-convex in E if there exists a function $g \in K$, such that

$$\Re\, e^{i\lambda} \frac{f'(z)}{g'(z)} > 0,\ z \in E,$$

for some $\lambda,\ |\lambda| < \pi/2$. Let C denote the class of all such functions.

If f and g are analytic in E, we say that f is subordinate to g in E, written as $f(z) \prec g(z)$ in E, if g is univalent in E, $f(0) = g(0)$ and $f(E) \subset g(E)$.
Let h be a univalent function in E and let $\psi : \mathbb{C}^2 \to \mathbb{C}$, where $\mathbb{C}$ is the complex plane. An analytic function p is said to satisfy the first order differential subordination if

$$\psi(p(z),\ zp'(z)) \prec h(z), \quad h(0) = \psi(p(0),\ 0),\ z \in E. \tag{1}$$

A univalent function q is said to be the dominant of the differential subordination (1), if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ of (1) that satisfies $\tilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1).

In 1976, Lewandowski, Miller and Zlotkiewicz [1] proved the following criterion for starlikeness of a function $f \in \mathcal{A}$.

Theorem A. *If $f \in \mathcal{A}$ satisfies*

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0,$$

for all z in E, then $f \in S^*$.

Later, in 1995, Ramesha, Kumar and Padmanabhan [8] extended Theorem 1 by proving the following result:

Theorem B. *Let $f \in \mathcal{A}$. For $\alpha \geq 0$, let the condition*

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\frac{\alpha zf''(z)}{f'(z)} + 1\right)\right] > 0, \tag{2}$$

be satisfied for all z in E. Then $f \in S^*$.

Several similar results pertaining to starlikeness and strongly starlikeness of the analytic function f satisfying (2) have been obtained in [2], [5] and [6]. Recently, Ravichandran et.al. [10] proved the following theorem wherein a sufficient condition for starlikeness of order $\beta, 0 \leq \beta \leq 1$, has been obtained.

Theorem C. *For $0 \leq \alpha,\ \beta \leq 1$, if $f \in \mathcal{A}$ satisfies*

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\frac{\alpha zf''(z)}{f'(z)} + 1\right)\right] > \alpha\beta\left(\beta - \frac{1}{2}\right) + \beta - \frac{\alpha}{2},$$

for all $z \in E$, then $f \in S^*(\beta)$.

In the present article, we consider a first order differential subordination of the form

$$\frac{zf'(z)}{f(z)}\left(\frac{\alpha zf''(z)}{f'(z)}+1\right)\prec\frac{zg'(z)}{g(z)}\left(\frac{\alpha zg''(z)}{g'(z)}+1\right),\ z\in E, \tag{3}$$

where α is a complex number with $\Re\alpha>0$. Our purpose is to obtain the best dominant for the above differential subordination, which, in turn, will give a sharp order of starlikeness of f. We also study a special case of differential subordination (3), namely,

$$\frac{z^2f''(z)}{f(z)}\prec\frac{z^2g''(z)}{g(z)},\ z\in E,$$

to get some interesting results concerning starlikeness of f.

2. Preliminaries

We need the following definition and lemmas to prove our results.

Definition 2.1. A function $L(z,t), z\in E$ and $t\geq 0$ is said to be a subordination chain if $L(.,t)$ is analytic and univalent in E for all $t\geq 0$, $L(z,.)$ is continuously differentiable on $[0,\infty)$ for all $z\in E$ and $L(z,t_1)\prec L(z,t_2)$ for all $0\leq t_1\leq t_2$.

Lemma 2.1 [7, page 159]**.** *The function $L(z,t):E\times[0,\infty)\to\mathbb{C}$ of the form $L(z,t)=a_1(t)z+\cdots$ with $a_1(t)\neq 0$ for all $t\geq 0$, and $\lim\limits_{t\to\infty}|a_1(t)|=\infty$, is a subordination chain if, and only if, $\Re\left[\frac{z\partial L/\partial z}{\partial L/\partial t}\right]>0$ for all $z\in E$ and $t\geq 0$.*

Lemma 2.2 ([3]). *Let F be analytic in E and let G be analytic and univalent in $\overline{E}$ except for points ζ_0 such that $\lim\limits_{z\to\zeta_0}F(z)=\infty$, with $F(0)=G(0)$. If $F\not\prec G$ in E, then there is a point $z_0\in E$ and $\zeta_0\in\partial E$ (boundary of E) such that $F(|z|<|z_0|)\subset G(E)$, $F(z_0)=G(\zeta_0)$ and $z_0F'(z_0)=m\zeta_0G'(\zeta_0)$ for some $m\geq 1$.*

Lemma 2.3 ([4]). *Let Ω be a set in the complex plane $\mathbb{C}$ and suppose that the function $\psi:\mathbb{C}^2\times E\to\mathbb{C}$ satisfies the condition $\psi(iu,v;z)\notin\Omega$, for all $u,v,\ v\leq-(1+u^2)/2$ and all $z\in E$. If the function p, $p(z)=1+p_1z+p_2z^2+\cdots$, is analytic in E and if $\psi(p(z),zp'(z);z)\in\Omega$, then $\Re p(z)>0$ in E.*

3. Main results

We begin with the following theorem:

Theorem 3.1. *Let α be a complex number with $\Re\alpha>0$. If $f\in\mathcal{A}$ satisfies*

$$\Re\left[\frac{\alpha z^2f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right]>\frac{(\Im\,\alpha)^2(1-\beta)(1-2\beta)^2}{2(\Re\,\alpha)(3-2\beta)}+\frac{(\Re\,\alpha)(2\beta^2-\beta-1)}{2}+\beta, \tag{4}$$

for all $z\in E$, $0\leq\beta\leq 1$, then $f\in S^*(\beta)$.

Proof. Define a function $p(z)$ by

$$\frac{zf'(z)}{f(z)} = \beta + (1-\beta)p(z).$$

Then $p(z)$ is analytic in E and $p(0) = 1$. A simple calculation yields

$$\begin{aligned}
& \frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \\
= \;& \alpha[\beta + (1-\beta)p(z)]^2 + \alpha(1-\beta)zp'(z) + (1-\alpha)[\beta + (1-\beta)p(z)] \\
= \;& \alpha(1-\beta)zp'(z) + \alpha(1-\beta)^2p^2(z) + (1-\beta)(1+2\alpha\beta-\alpha)p(z) + \\
& \qquad \beta(\alpha\beta + 1 - \alpha) \\
= \;& \phi(p(z), zp'(z); z),
\end{aligned}$$

where,

$$\phi(u, v; z) = \alpha(1-\beta)v + \alpha(1-\beta)^2u^2 + (1-\beta)(1+2\alpha\beta-\alpha)u + \beta(\alpha\beta + 1 - \alpha).$$

For all real $u_2,\ v_1$ satisfying $v_1 \leq -(1+u_2^2)/2$ and $\alpha = a + ib$, say, we have

$$\begin{aligned}
(5) \qquad & \Re\phi(iu_2, v_1; z) \\
= \;& -a(1-\beta)^2u_2^2 + a\beta^2 - 2bu_2\beta(1-\beta) + a(1-\beta)v_1 + (1-a)\beta + bu_2(1-\beta) \\
\leq \;& -a(1-\beta)^2u_2^2 - \frac{a}{2}(1-\beta)(1+u_2^2) - 2b\beta(1-\beta)u_2 + b(1-\beta)u_2 \\
& + a\beta^2 + (1-a)\beta \\
= \;& \frac{-a(1-\beta)(3-2\beta)}{2}u_2^2 + b(1-\beta)(1-2\beta)u_2 + \frac{a(2\beta^2-\beta-1)}{2} + \beta \\
= \;& H(u_2), \text{ (say)} \\
\leq \;& \max H(u_2).
\end{aligned}$$

It can be easily verified that

$$\begin{aligned}
(6) \qquad \max H(u_2) \;&=\; H\left(\frac{b(1-2\beta)}{a(3-2\beta)}\right) \\
&=\; \frac{b^2(1-\beta)(1-2\beta)^2}{2a(3-2\beta)} + \frac{a(2\beta^2-\beta-1)}{2} + \beta.
\end{aligned}$$

Let

$$\Omega = \{w;\ \Re\, w > \frac{(\Im\, \alpha)^2(1-\beta)(1-2\beta)^2}{2\Re\, \alpha(3-2\beta)} + \frac{\Re\, \alpha(2\beta^2-\beta-1)}{2} + \beta\}.$$

Then from (4), we have $\phi(p(z),\ zp'(z);\ z) \in \Omega$ for all $z \in E$, but $\phi(iu_2,\ v_1;\ z) \notin \Omega$, in view of (5) and (6). Therefore, by virtue of Lemma 2.3, we conclude that $f \in$

$S^*(\beta)$. □

Example 3.1. Let $f \in \mathcal{A}$ satisfy

$$\Re\left[\left(\frac{1}{2}+\frac{i}{2}\right)\frac{z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right] > 1/4,\ z \in E,$$

then $\Re\,\frac{zf'(z)}{f(z)} > 1/2$.

Remark 3.1. Taking α as a positive real number, we obtain Theorem C. Particularly, if we take $\beta = 0$ and α, a positive real number in Theorem 3.1, we get the following result of Li and Owa [2]:

Corollary 3.1. *Let $f \in \mathcal{A}$ satisfy*

$$\Re\left[\alpha\frac{z^2 f''(z)}{f(z)}+\frac{zf'(z)}{f(z)}\right] > -\frac{\alpha}{2},\ z \in E,$$

for some $\alpha \geq 0$. Then $f \in S^*$.

Before stating next result, first we prove the following lemma.

Lemma 3.1. *Let α be a complex number with $\Re\,\alpha > 0$. Suppose that $q \in \mathcal{A}'$ is a convex univalent function which satisfies the following conditions*:

(a) $\Re\,q(z) > 0,\ z \in E,$ *when* $\Re\,\alpha \geq |\alpha|^2$;

(b) $\Re\,q(z) > \frac{|\alpha|^2-\Re\alpha}{2|\alpha|^2},\ z \in E,$ *when* $\Re\,\alpha < |\alpha|^2$.

If a function $p \in \mathcal{A}'$ satisfies the differential subordination

$$(1-\alpha)p(z)+\alpha(p(z))^2+\alpha zp'(z) \prec (1-\alpha)q(z)+\alpha(q(z))^2+\alpha zq'(z), \tag{7}$$

in E, then $p(z) \prec q(z)$ in E and q is the best dominant.

Proof. Let

$$h(z) = (1-\alpha)q(z)+\alpha(q(z))^2+\alpha zq'(z). \tag{8}$$

Clearly h is analytic in E and $h(0) = 1$. First of all, we will prove that h is univalent in E so that the subordination (7) is well-defined in E. From (8), we get

$$\frac{1}{\alpha}\frac{h'(z)}{q'(z)} = 2q(z)+\frac{\overline{\alpha}-|\alpha|^2}{|\alpha|^2}+1+\frac{zq''(z)}{q'(z)}.$$

In view of the conditions (a) and (b) above and the fact that q is convex in E, we obtain

$$\Re\,\frac{1}{\alpha}\frac{h'(z)}{q'(z)} > 0,\ z \in E.$$

Since $\Re\, \alpha > 0$, and q is convex univalent in E, we conclude that h is close-to-convex and hence, univalent in E. We now show that that $p \prec q$. Without any loss of generality, we can assume q to be analytic and univalent in $\overline{E}$.

If possible, suppose that $p \not\prec q$ in E. Then, by Lemma 2.2, there exist points $z_0 \in E$ and $\zeta_0 \in \partial E$ such that $p(z_0) = q(\zeta_0)$ and $z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$, $m \geq 1$. Thus,

$$(1-\alpha)p(z_0) + \alpha(p(z_0))^2 + \alpha z_0 p'(z_0) = (1-\alpha)q(\zeta_0) + \alpha(q(\zeta_0))^2 + m\alpha\zeta_0 q'(\zeta_0) \tag{9}$$

Consider a function

$$\begin{aligned} L(z,t) &= (1-\alpha)q(z) + \alpha(q(z))^2 + \alpha t z q'(z) \\ &= 1 + a_1(t)z + \cdots . \end{aligned} \tag{10}$$

The function $L(z,t)$ is analytic in E for all $t \geq 0$ and is continuously differentiable on $[0,\infty)$ for all $z \in E$. Now,

$$a_1(t) = \left[\frac{\partial L(z,t)}{\partial z}\right]_{(0,t)} = q'(0)(1+\alpha+\alpha t).$$

As the function q is univalent in E, therefore, $q'(0) \neq 0$. Also since $\Re\, \alpha > 0$, we get $|\arg(1+\alpha+\alpha t)| < \pi/2$. Therefore, it follows that $a_1(t) \neq 0$ and $\lim_{t\to\infty} |a_1(t)| = \infty$. A simple calculation yields

$$z\frac{\partial L/\partial z}{\partial L/\partial t} = 2q(z) + \frac{\overline{\alpha} - |\alpha|^2}{|\alpha|^2} + t\left(1 + \frac{zq''(z)}{q'(z)}\right).$$

Clearly,

$$\Re\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0,$$

for all $z \in E$ and $t \geq 0$, in view of given conditions (a) and (b) and the fact that q is convex in E. Hence, $L(z,t)$ is a subordination chain. Therefore, $L(z,t_1) \prec L(z,t_2)$ for $0 \leq t_1 \leq t_2$. From (10), we have $L(z,1) = h(z)$, thus we deduce that $L(\zeta_0, t) \notin h(E)$ for $|\zeta_0| = 1$ and $t \geq 1$. In view of (9) and (10), we can write

$$(1-\alpha)p(z_0) + \alpha(p(z_0))^2 + \alpha z_0 p'(z_0) = L(\zeta_0, m) \notin h(E)$$

where $z_0 \in E$, $|\zeta_0| = 1$ and $m \geq 1$ which is a contradiction to (7). Hence, $p \prec q$ in E. This completes the proof of our Lemma. □

Theorem 3.1 enables us to study only the real part of the functional $\frac{zf'(z)}{f(z)}$. We now prove our main result which is a subordination theorem providing us best dominant for a certain first order differential subordination, which, in turn, gives us exact region of variability of the functional $\frac{zf'(z)}{f(z)}$.

Theorem 3.2. *Let α be a complex number, with $\Re\, \alpha > 0$. For a function $g \in \mathcal{A}$, set $G(z) = \frac{zg'(z)}{g(z)}$. Assume that $G(z)$ is convex univalent function in E which satisfies the following conditions*:

(a) $\Re\, G(z) > 0,\ z \in E,$ *when* $\Re\, \alpha \geq |\alpha|^2$;

(b) $\Re\, G(z) > \frac{|\alpha|^2 - \Re\alpha}{2|\alpha|^2},\ z \in E,$ *when* $\Re\, \alpha < |\alpha|^2$.

If a function $f \in \mathcal{A}$ *satisfies the differential subordination*

$$\frac{\alpha z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec \frac{\alpha z^2 g''(z)}{g(z)} + \frac{zg'(z)}{g(z)},\ z \in E, \tag{11}$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)},\ z \in E.$$

Proof. The proof follows by setting $p(z) = \frac{zf'(z)}{f(z)}$ and $q(z) = \frac{zg'(z)}{g(z)}$ in Lemma 3.1.□

Taking α to be real such that $0 < \alpha \leq 1$, in Theorem 3.2, we obtain the following result (Also see Theorem 3 of Ravichandran [9]):

Corollary 3.2. *Let* $G(z)$ *be a convex univalent function which satisfies* $\Re\, G(z) > 0$, *for all* $z \in E$. *If* $f \in \mathcal{A}$ *satisfies the differential subordination*

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} \prec (1-\alpha)G(z) + \alpha G^2(z) + \alpha z G'(z),\ z \in E,$$

then $\frac{zf'(z)}{f(z)} \prec G(z)$ *for all* $z \in E$.

Again, assuming α to be real in Theorem 3.2, the limiting case when α approaches infinity, gives the following interesting result:

Theorem 3.3. *Let* $g \in \mathcal{A}$ *be a starlike function of order* $\delta,\ 1/2 \leq \delta \leq 1$ *and let* $\frac{zg'(z)}{g(z)} = G(z)$. *Assume that* $G(z)$ *is a convex univalent function in* E. *If* $f \in \mathcal{A}$ *satisfies the differential subordination*

$$\frac{z^2 f''(z)}{f(z)} \prec \frac{z^2 g''(z)}{g(z)},\ z \in E,$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{zg'(z)}{g(z)}$$

in E. *Note that the function* f, *too, belongs to* $S^*(\delta)$, *since* $\Re\frac{zg'(z)}{g(z)} > \delta,\ z \in E$.

4. Applications

I. Consider $G(z) = \frac{1+az}{1-z},\ |a| \leq 1$. Obviously, $G(z)$ is convex univalent in E. It can be easily verified that

(a) when $0 \leq \alpha \leq 1$, we have $\Re G(z) > \frac{1-a}{2} > 0,\ z \in E$, and

(b) when $\alpha > 1$, we get $\Re G(z) > \frac{1-a}{2} \geq \frac{\alpha-1}{2\alpha}$ provided $a \leq 1/\alpha$.

Thus, in view of Theorem 3.2, we obtain the following:

Corollary 4.1. *Let α, $\alpha \geq 0$, be a real number. Assume that a, $|a| \leq 1$, is a real number satisfying $a \leq 1/\alpha$ whenever $\alpha > 1$. Let $f \in \mathcal{A}$ satisfy*

$$\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec (1-\alpha)\left(\frac{1+az}{1-z}\right) + \alpha\left(\frac{1+az}{1-z}\right)^2 + \alpha\left(\frac{(a+1)z}{(1-z)^2}\right), \ z \in E.$$

Then,

$$\frac{zf'(z)}{f(z)} \prec \frac{1+az}{1-z}, \ z \in E.$$

Writing $a = 1 - 2\beta$, we obtain the following form of Theorem C:

Corollary 4.2. *Let α, $\alpha \geq 0$, be a real number. Assume that β, $0 \leq \beta \leq 1$, is a real number which satisfies $\beta \geq \frac{1}{2} - \frac{1}{2\alpha}$ for $\alpha > 1$. If an analytic function f in $\mathcal{A}$ satisfies*

$$\Re\left[\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right] > \alpha\beta^2 + \beta - \frac{\alpha}{2}(1+\beta), \ z \in E,$$

then $f \in S^*(\beta)$.

We observe that for $\alpha > 1$, $\beta \geq \frac{1}{2} - \frac{1}{2\alpha}$, therefore, $f \in S^*(\beta)$ implies that $f \in S^*(\frac{1}{2} - \frac{1}{2\alpha})$.

Taking $a = 1 - \alpha$ in Corollary 4.1, we get the following result of Li and Owa [2]:

Corollary 4.3. *If f in $\mathcal{A}$ satisfies*

$$\Re\left[\frac{zf'(z)}{f(z)}\left(\alpha \frac{zf''(z)}{f'(z)} + 1\right)\right] > -\frac{\alpha^2}{4}(1-\alpha), \ z \in E,$$

for some α $(0 < \alpha < 2)$, then $f \in S^*(\alpha/2)$.

Letting α tend to infinity in Corollary 4.2 (or an application of Theorem 3.3 with $G(z) = \frac{1+(1-2\beta)z}{1-z}$) gives us the following result:

Corollary 4.4. *Let an analytic function f in $\mathcal{A}$ satisfy*

$$\Re \frac{z^2 f''(z)}{f(z)} > \frac{2\beta^2 - \beta - 1}{2}, \ z \in E, 1/2 \leq \beta \leq 1.$$

Then $f \in S^*(\beta)$.

Writing $\beta = 1/2$ in Corollary 4.4 (or taking $G(z) = \frac{1}{1-z}$ in Theorem 3.3), we obtain the following:

Example 4.1. Let $f \in \mathcal{A}$ satisfy

$$\frac{z^2 f''(z)}{f(z)} \prec \frac{2z}{(1-z)^2}, \ z \in E.$$

Then $f \in S^*(1/2)$.

II. Setting $G(z) = \left(\frac{1+z}{1-z}\right)^\beta, \ 0 < \beta \leq 1$, in Theorem 3.2, we obtain:

Corollary 4.5. *Let α and β be real numbers such that $0 < \alpha, \beta \leq 1$. For all $z \in E$, let $f \in \mathcal{A}$ satisfy*

$$\alpha \frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)} \prec (1-\alpha)\left(\frac{1+z}{1-z}\right)^\beta + \alpha\left(\frac{1+z}{1-z}\right)^{2\beta} + \beta\alpha\left(\frac{1+z}{1-z}\right)^{\beta-1}\left(\frac{2z}{(1-z)^2}\right).$$

Then,

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^\beta$$

in E, i.e., $f \in S(\beta)$.

Taking $\beta = 1$ in Corollary 4.5, we obtain Corollary 3.1. Considering the case when $\beta = 1/2, \ \alpha = 2/3$, we get the following result of Obradović and Joshi [6]:

Corollary 4.6. *Let $f \in \mathcal{A}$ be such that*

$$\Re\left[\frac{2}{3}\frac{z^2 f''(z)}{f(z)} + \frac{zf'(z)}{f(z)}\right] > 0, \ z \in E.$$

Then $f \in S(1/2)$.

III. Taking $G(z) = 1+bz, \ 0 < b \leq 1/2$, we find that G satisfies all the conditions of Theorem 3.3. Thus, we get

Corollary 4.7. *Let $f \in \mathcal{A}$ be such that*

$$\left|\frac{z^2 f''(z)}{f(z)}\right| < b(b+2), \ z \in E, \ 0 < b \leq 1/2,$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < b$$

in E.

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