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Variants of Compactness in Pointfree Topology

Dedicated to Professor Han Taidong in honor of his 80th birthday

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Abstract. This paper introduces compactness notions for frames which are expressed in terms of the convergence of suitably specified general filters. It establishes several preservation properties for them as well as their coreflectiveness in the setting of regular frames. Further, it shows that supercompact, compact, and Lindelöf frames can be described by compactness conditions of the present form so that various familiar facts become consequences of these general results. In addition, the Prime Ideal Theorem and the Axiom of Countable Choice are proved to be equivalent to certain conditions connected with the kind of compactness considered here.

0. Introduction

We recall that pointfree topology deals with frames and their homomorphisms where a *frame* is a complete lattice L in which

$$a \wedge \bigvee S = \bigvee \{a \wedge t \mid t \in S\}$$

for all $a \in L$ and $S \subseteq L$ and a *frame homomorphism* is a map $h : L \to M$ between frames which preserves finitary meets, including the unit (= top) e, and arbitrary joins, including the zero (= bottom) 0. The category thus determined will be denoted **Frm**.

For general notions and results concerning frames we refer to Johnstone [15] or Vickers [16].

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Given the central rôle the convergence of filters has played in classical topology since Bourbaki's Topologie générale, it is natural to consider the corresponding notion in pointfree topology as well, despite the absence of points in that setting. Modelled in an obvious way on the classical case, a filter F in a frame L is called *convergent* if it meets every cover of L ([12]); further, it is called *strongly convergent* if it contains a completely prime filter ([9]). We note that, in the case of the frame $\mathfrak{O}X$ of open subsets of a space X, these two concepts coincide and amount to convergence in the classical sense. In general, strong convergence evidently implies convergence though not conversely, as is easily seen, but the two notions are equivalent for regular frames ([5], [6], [13]).

On the other hand, somewhat similar to the classical fact that the convergence of sequences in a topological space is not sufficient to describe its structure, it turns out that the usual notion of filter is inadequate in various situations in pointfree topology. The remedy in that case is to replace it by the more general notion of T-valued filter on a frame, for an arbitrary frame T ("of truth values") ([1]), the ordinary filters then appearing as the **2**-valued case, and it is well established that this provides the proper tool for the study of certain aspects of frames ([6], [7]), [8]). For convenience, we recall that this notion may be described in terms of *the propositional theory of* (*proper*) *filters on a frame* L which is given by

a *basic proposition* $a \in F$ for each element a of L and

the *axioms*

$$0 \in F \vdash \bot, \ \top \vdash e \in F$$
$$a \in F \wedge b \in F \vdash a \wedge b \in F$$
$$a \in F \vdash b \in F \text{ whenever } a \leq b.$$

where the last one may be replaced by

$$a \wedge b \in F \vdash a \in F \wedge b \in F.$$

For the details concerning such theories in general we refer to [2].

Now, in accordance with the usual terminology, a T-valued filter on L for some frame T is a T-valued model of this theory which amounts to a $0 \wedge e$ homomorphism $\varphi : L \to T$. In particular, for $T = \mathbf{2}$, these φ are just the characteristic functions of ordinary filters.

In the following, T-valued filters on a frame L are simply called the *filters* on L, while the ordinary filters are to referred to as *classical filters.*

Regarding the concept of compactness and its various relatives it is clear that any classical notion which is expressed as a condition on covers in the lattice of open sets has its immediate counterpart for frames, giving rise to the pointfree notions of compactness, Lindelöfness, paracompactness, and the like. On the other hand, there is the familiar fact that the compactness of spaces can also be expressed as the requirement that certain filters be convergent, and even though properties of this type seem a priori to be intimately tied to points it turns out that they also have meaning in the pointfree setting. It is the purpose of this paper to study notions of compactness of this particular kind in pointfree topology.

In detail, this will proceed as follows. Section 1 introduces the notions of convergence and strong convergence as the obvious counterparts of the corresponding concepts for classical filters described above and discusses a number of questions which naturally arise from the definitions. Next, in Section 2, we present a general setting for compactness notions defined in terms of convergent, or strongly convergent, filters and establish a number of preservation results for them, involving closed quotients, certain kinds of weak retracts, and coproducts, as well as their coreflectiveness in the case of regular and similar types of frames. Further, in Section 3, the case of some familiar types of filters is considered. We show that they fall under the general scheme dealt with in Section 2 and give concrete characterizations for them which then demonstrates that various familiar facts, such as the coreflectiveness of compact regular, completely regular, and zero-dimensional frames, are all consequences of the general results of Section 2. Finally, in Section 4, we obtain new equivalents of the Prime Ideal Theorem and the Axiom of Countable Choice, respectively, in terms of the present compactness notions.

Regarding foundations, we mainly work in Zermelo-Fraenkel set theory; the few cases in which some choice principle or other is used in a proof will be marked, as is customary, by an asterisk.

1. Convergence

This section deals with the notion of convergence for filters on a frame which will be central to the compactness conditions to be considered later.

In view of the definition of the convergence of classical filters on a frame mentioned earlier, it is natural to take the present notion as given by adding the following axiom to the theory of filters on a frame L:

$$\top \vdash \bigvee\{a \in F \mid a \in C\}, \text{ for each cover } C \text{ of a frame } L,$$

and consequently a *convergent* filter is a filter $\varphi : L \to T$ such that $\bigvee\{\varphi(a) | a \in C\} = e$ for each cover C of L, that is, φ takes covers to covers.

Further, in order to formulate the notion of strong convergence in the present context, we first need that of a *completely prime filter.* This is obviously expressed by the axiom

$$\bigvee S \in F \vdash \bigvee\{a \in F \mid a \in S\}, \text{ for each } S \subseteq L,$$

saying that a filter $\varphi : L \to T$ is completely prime iff

$$\varphi(\bigvee S) \leq \bigvee\{\varphi(a) \mid a \in S\}$$

for each $S \subseteq L$ which evidently holds iff φ preserves all joins and hence is a *frame homomorphism.*

Finally, in accordance with the corresponding notion for classical filters, this leads to the following condition where $\leq$ is understood in the usual argumentwise sense:

A filter $\varphi : L \to T$ is called *strongly convergent* if $h \leq \varphi$ for some frame homomorphism $h : L \to T$.

Evidently, any strongly convergent filter is convergent but not conversely, as the following example shows.

For the frame

a

$L \;=$ c e , $\{x \in L \,|\, x < c\} \cong \omega,$

b

the familiar $0 \wedge e$-homomorphism $\downarrow : L \to \mathfrak{J}L$ into its ideal lattice $\mathfrak{J}L$, taking each element x to the corresponding principal ideal $\downarrow x$, is clearly a convergent filter since any cover C of L contains a and b. Suppose then it is strongly convergent so that there exists a frame homomorphism $h : L \to \mathfrak{J}L$ such that $h \leq \downarrow$. Now $h(a) \neq \downarrow a$ implies $h(a) \subseteq \downarrow c$ and hence $h(a), h(b) \subseteq \downarrow b$, a contradiction since $h(a) \vee h(b) = \downarrow e$. Consequently $h(a) = \downarrow a$ and analogously $h(b) = \downarrow b$, showing that

$$h(c) = h(a) \wedge h(b) = \downarrow a \wedge \downarrow b = \downarrow c.$$

At the same time,

$$h(c) = \bigvee\{h(x) \mid x < c\} \subseteq \bigvee\{\downarrow x \mid x < c\} = \{x \in L \mid x < c\}$$

and hence $c < c$, again a contradiction.

By way of contrast, there are quite naturally arising frames for which every convergent filter is indeed strongly convergent, we show this holds for

(i) the regular frames,

(ii) the supercompact frames, and

(iii) any frame in which the elements c such that $\downarrow c$ is dually well-ordered form a cover (noting that any product of dually well-ordered frames is of this type).

Concerning (i), if $\varphi : L \to T$ is any convergent filter on the regular frame L let $\varphi^\circ : L \to T$ be defined such that

$$\varphi^\circ(a) = \bigvee\{\varphi(x) \mid x \prec a\}$$

where $x \prec a$ means that $a \vee x^* = e$ for the pseudocomplement x^* of x. Then this is a filter by the familiar properties of $\prec$ and convergent by the regularity of L: for any cover C of L, $\{x \in L \mid x \prec s$ for some $s \in C\}$ is a cover of L and hence

$$e = \bigvee\{\varphi(x) \mid x \prec s, s \in C\} = \bigvee\{\varphi^\circ(s) \mid s \in C\}.$$

Further, φ° preserves all joins: if $a = \bigvee S$ in L then, for each $x \prec a$, $S \cup \{x^*\}$ is a cover, hence $\bigvee\{\varphi^\circ(t) \mid t \in S\} \vee \varphi^\circ(x^*) = e$ which implies $\varphi(x) \leq \bigvee\{\varphi^\circ(t) \mid t \in S\}$ since $\varphi^\circ(x^*) \leq \varphi(x^*)$, and therefore in all $\varphi^\circ(a) \leq \bigvee\{\varphi^\circ(t) \mid t \in S\}$, the non-trivial part of the desired identity. Hence φ° is a frame homomorphism below φ, showing that φ is strongly convergent.

In the second case, if u is the largest element less than e of the supercompact frame L, then for any convergent filter $\varphi : L \to T$ (which actually means *any* filter here since every cover of L contains e) lies above the homomorphism

$$h : L \to \uparrow u \cong \mathbf{2} \to T$$

taking any $x \leq u$ to 0.

Finally, concerning (iii), if $a = \bigvee S$ then $a \wedge c = \bigvee\{x \wedge c \mid x \in S\} = x_0 \wedge c$ for some $x_0 \in S$ since $\downarrow c$ is dually well-ordered; hence it follows for any convergent filter $\varphi : L \to T$ that

$$\varphi(a) \wedge \varphi(c) = \varphi(a \wedge c) = \varphi(x_0 \wedge c) \leq \varphi(x_0) \leq \bigvee \varphi[S],$$

and taking the join over all these c shows that $\varphi(a) \leq \bigvee \varphi[S]$. Thus φ itself is a frame homomorphism and therefore trivially strongly convergent.

We mention in passing two further facts concerning the frames of the present type: any coproduct of such frames and any lax retract of such a frame is again such a frame, where a frame L is called a *lax retract* of a frame M if there exist frame homomorphisms $h : L \to M$ and $k : M \to L$ such that $kh \leq id_L$. We omit the details, the proofs being straightforward applications of the definitions.

Note that in the earlier example of a convergent filter which is not strongly convergent, the frame involved is actually spatial but the filter is not classical. The latter is not by accident: in fact, in the frame of open sets of a space X, any convergent classical filter is strongly convergent since it converges in X in the usual sense. Actually, this is a special case of a considerably more general result, as follows.

Call a frame L *spatial at the top* if, for any $a < e$ in L, there exists a homomorphism $\xi : L \to \mathbf{2}$ such that $\xi(a) = 0$. Note that for certain types of frames (such as the regular ones) spatiality at the top implies spatiality, but in general this is certainly not the case. Indeed, any frame obtained by adding a new top to a given frame will trivially be of that kind but will not be spatial unless the original frame is spatial. Consider now any classical convergent filter F in a frame L which is spatial at the top. Then $s = \bigvee\{a \in L \mid a \notin F\} < e$ since the set involved here cannot be a cover by the definition of convergent filters, and by hypothesis there exists $\xi : L \to \mathbf{2}$ such that $\xi(s) = 0$. It follows that $a \notin F$ implies $\xi(a) = 0$ and hence F contains the completely prime filter $P = \{a \in L \mid \xi(a) = 1\}$, showing it is strongly convergent.

Regarding the existence of convergent filters, there is the obvious fact that *any* frame has such filters, namely the identity map $L \to L$. On the other hand, though, there need not be any convergent classical filter on a given frame L. Thus, for Boolean L, if $\varphi : L \to \mathbf{2}$ is a convergent filter then, by the regularity of Boolean frames, the $\varphi^\circ : L \to \mathbf{2}$ considered earlier is a frame homomorphism and as such determined by an *atom* of L. Hence an atomless Boolean frame does not have any convergent classical filter.

We close this section with a discussion of the uniqueness question naturally arising from the notion of strong convergence in obvious analogy with the classical case of filter convergence in topological spaces.

The key property of frames involved here is that of being *separated* (= strongly Hausdorff [14], or T_2-frames [10], [11]), saying for a frame L that the codiagonal $L \bigoplus L \to L$ represents it as a closed quotient of the coproduct $L \bigoplus L$. Alternatively, this is equivalent to the following condition (see [11]): If $f, g : L \to M$ are frame homomorphisms such that $f(x) \wedge g(y) = 0$ whenever $x \wedge y = 0$ in L then $f = g$. For present purposes, it seems convenient to use this characterization.

On the other hand, we say that a frame *has unique strong convergence* if, for any strongly convergent filter $\varphi : L \to T$ on L, the frame homomorphism $h \leq \varphi$ is unique. The result is now the following analogue of Bourbaki's characterization of Hausdorff spaces in terms of convergent filters.

A frame has unique strong convergence iff it is separated.

To see ($\Rightarrow$) let $i, j : L \to L \bigoplus L$ be the coproduct maps, $\nabla : L \bigoplus L \to L$ the codiagonal map such that $\nabla i = id_L = \nabla j$,

$$s = \bigvee\{i(x) \wedge j(y) \mid x \wedge y = 0 \text{ in } L\},$$

$\nu : L \bigoplus L \to \uparrow s$ the homomorphism $(\cdot) \vee s$, and $h : \uparrow s \to L$ the homomorphism such that $\nabla = h\nu$ induced by ∇ since $\nabla(s) = 0$. Then h is dense so that its right adjoint $h_* : L \to \uparrow s$ is a filter. Further, $\nu i \leq h_*$ since $h\nu i = id_L$, and similarly $\nu j \leq h_*$. Now, for any homomorphisms $f,\ g : L \to M$ such that $f(x) \wedge g(y) = 0$ whenever $x \wedge y = 0$, let $k : L \bigoplus L \to M$ be the homomorphism such that $ki = f$ and $kj = g$. Then $k(s) = 0$ since $k(i(x) \wedge j(y)) = f(x) \wedge g(y) = 0$ if $x \wedge y = 0$ and hence k induces a homomorphism $l : \uparrow s \to M$ such that $l\nu = k$. Consequently, by the earlier observation concerning νi and νj, $f = ki = l\nu i \leq lh_*$ and similarly $g \leq lh_*$, and the present hypothesis then implies $f = g$, showing L is separated.

Conversely, if $f, g : L \to T$ are frame homomorphisms such that $f, g \leq \varphi$ for some filter $\varphi : L \to T$ then

$$f(x) \wedge g(y) \leq \varphi(x) \wedge \varphi(y) = \varphi(x \wedge y) = 0$$

whenever $x \wedge y = 0$. Hence if L is separated then $f = g$, showing L has unique strong convergence.

2. Compactness

As indicated earlier, the compactness notions to be considered here will be defined in terms of the convergence of filters, specifically as the requirement that all filters of a certain type be convergent. The following describes a convenient general format for this.

An object function $\mathbb{F}$ on the category **Frm** will be called an *admissible selection of filters* (for short : a *filter selection*) if $\mathbb{F}(L)$ is a class of filters $\varphi : L \to T$ for each frame L such that

(FS1) every frame homomorphism $L \to M$ belongs to $\mathbb{F}(L)$, and

(FS2) $\mathbb{F}(L)$ is closed under composition, that is, for any $\varphi : L \to M$ in $\mathbb{F}(L)$ and $\psi : M \to N$ in $\mathbb{F}(M)$, $\psi\varphi$ belongs to $\mathbb{F}(L)$.

Note that there is the following alternative description of filter selections. If $\mathbf{S}$ is the category of all frames and all bounded meet-semilattice homomorphisms between them then any filter selection $\mathbb{F}$ determines a subcategory $\mathbf{S}(\mathbb{F})$ of $\mathbf{S}$ in which the maps from L to M are exactly the $L \to M$ belonging to $\mathbb{F}(L)$, and by (FS1) this is then a category extension of **Frm** in **S**. Conversely, any subcategory $\mathbf{K}$ of $\mathbf{S}$ of that kind determines a filter selection $\mathbb{F}(\mathbf{K})$ which assigns to each frame L the class of filters

$$\bigcup\{\mathbf{K}(L, M) \mid M \in \mathbf{Frm}\}.$$

Furthermore, the correspondence $\mathbb{F} \mapsto \mathbf{S}(\mathbb{F})$ and $\mathbf{K} \mapsto \mathbb{F}(\mathbf{K})$ are clearly inverse to each other, and in all, then,

the filter selections are essentially the same as the subcategories of **S** *containing* **Frm**.

We note in passing that a somewhat similar setting plays a rôle in a recent study of general notions of projectivity for frames ([4]).

Now, for any filter selection $\mathbb{F}$, a frame L is called
$\mathbb{F}$-*compact* if every $\varphi \in \mathbb{F}(L)$ is convergent, and
strongly $\mathbb{F}$-*compact* if every $\varphi \in \mathbb{F}(L)$ is strongly convergent.

Trivially, strong $\mathbb{F}$-compactness implies $\mathbb{F}$-compactness for any $\mathbb{F}$, but not conversely: we may take $\mathbb{F}(L)$ as the class of all convergent filters for each L so that any L is trivially $\mathbb{F}$-compact but any frame with a convergent filter which is not strongly convergent (such as exhibited earlier) then fails to be strongly $\mathbb{F}$-compact. On the other hand, for any frame such that every convergent filter is strongly convergent the two compactness notions trivially coincide.

For any filter selection $\mathbb{F}$ of filters, a frame L is called an $\mathbb{F}$-*lax retract* of a frame M if there exists a frame homomorphism $h : L \to M$ and a filter $\varphi : M \to L$ in

$\mathbb{F}(M)$ such that $\varphi h \leq id_L$. Note that, by (FS1), any lax retract is an $\mathbb{F}$ -lax retract, but the converse does not hold. To see this let L be any non-trivial frame. Then the unique homomorphism $\mathbf{2} \to L$ shows that $\mathbf{2}$ is an $\mathbb{F}$-lax retract of L for any filter selection $\mathbb{F}$ such that $\mathbb{F}(L)$ contains the characteristic function $\varphi : L \to \mathbf{2}$ of the classical filter $\{e\} \subseteq L$. On the other hand, $\mathbf{2}$ is trivially not a lax retract of any frame which has no frame homomorphism into $\mathbf{2}$ such as any atomless Boolean frame.

Proposition 1. *For any filter selection $\mathbb{F}$, closed quotients and $\mathbb{F}$-lax retracts of $\mathbb{F}$-compact frames are $\mathbb{F}$-compact, and the same holds for strong $\mathbb{F}$-compactness.*

Proof. We first deal with $\mathbb{F}$-compactness. Let L be any $\mathbb{F}$-compact frame, $a \in L$, and $\nu : L \to \uparrow a$ the usual homomorphism $(\cdot) \vee a$. Then, for any $\varphi : \uparrow a \to T$ in $\mathbb{F}(\uparrow a)$, $\varphi\nu : L \to T$ belongs to $\mathbb{F}(L)$ and is therefore convergent. Further, any cover C of $\uparrow a$ is also a cover of L, and since it is its own image by ν, $\varphi[C]$ is a cover, saying φ is convergent.

Concerning the second case, let M be an $\mathbb{F}$-compact frame, $h : L \to M$ any frame homomorphism such that $\varrho h \leq id_L$ for some $\varrho : M \to L$ in $\mathbb{F}(M)$ and $\varphi : L \to T$ in $\mathbb{F}(L)$. Then $\varphi\varrho \in \mathbb{F}(M)$, making it and hence also $\varphi\varrho h$ convergent, and since $\varphi\varrho h \leq \varphi$ it follows that φ is convergent.

Next, let L now be strongly $\mathbb{F}$-compact and, with the same notation as above, consider any $\varphi : \uparrow a \to T$ in $\mathbb{F}(\uparrow a)$. Then $\varphi\nu : L \to T$ belongs to $\mathbb{F}(L)$ as before, implying now that there exists a frame homomorphism $h : L \to T$ such that $h \leq \varphi\nu$. It follows that $h(a) \leq \varphi\nu(a) = 0$ and hence h factors through ν, that is, $h = k\nu$ for the induced $k : \uparrow a \to T$. Consequently $k \leq \varphi$, showing $\uparrow a$ is strongly $\mathbb{F}$ -compact.

Finally, if M is strongly $\mathbb{F}$-compact and $h : L \to M, \varrho : M \to L$ exhibit L as an $\mathbb{F}$-lax retract of M as before then, for any $\varphi : L \to T$ in $\mathbb{F}(L)$, again $\varphi\varrho \in \mathbb{F}(M)$ and hence $k \leq \varphi\varrho$ for some frame homomorphism $k : M \to T$ by the present hypothesis. It follows that $kh \leq \varphi\varrho h \leq \varphi$, the latter since $\varrho h \leq id_L$, showing φ is strongly convergent which proves L is strongly $\mathbb{F}$-compact. □

Next we derive the analogous result for coproducts, but in this case only for strong compactness. As a first step towards this we need the following

Lemma 1. *For any frame homomorphism $h : L \to T$ and any filter $\varphi : L \to T$, if $h|X \leq \varphi|X$ for some generating set X of L then $h \leq \varphi$.*

Proof. Let $M = \{x \in L \mid h(x) \leq \varphi(x)\}$. Then $0,\ e \in M$ and $x \wedge y \in M$ for any $x,\ y \in M$. Further, for any subset S of M,

$$h(\bigvee S) = \bigvee\{h(t) \mid t \in S\} \leq \bigvee\{\varphi(t) \mid t \in S\} \leq \varphi(\bigvee S)$$

and hence $\bigvee S \in M$. Thus M is a subframe of L, and since it contains the generating set X of L it is equal to L. □

***Proposition 2.** *For any filter selection $\mathbb{F}$, coproducts of strongly $\mathbb{F}$-compact frames are strongly $\mathbb{F}$-compact.*

Proof. For any strongly $\mathbb{F}$-compact L_α, let $L = \bigoplus L_\alpha$ with coproduct maps $i_\alpha : L_\alpha \to L$ and consider any $\varphi : L \to T$ in $\mathbb{F}(L)$. Then $\varphi i_\alpha : L_\alpha \to T$ belongs to $\mathbb{F}(L_\alpha)$ so that there exist frame homomorphisms $h_\alpha : L_\alpha \to T$ below φi_α, and choosing such $h_\alpha : L_\alpha \to T$ for each α we obtain a frame homomorphism $h : L \to T$ such that $hi_\alpha = h_\alpha$. It then follows that $hi_\alpha \leq \varphi i_\alpha$ for each α, and since L is generated by the union of the $Im(i_\alpha)$ Lemma 1 shows $h \leq \varphi$. □

Remark 1. (1) We do not know whether the same result holds for mere $\mathbb{F}$-compactness although this is the case for certain particular $\mathbb{F}$ (see Proposition 6).

(2) If all L_α in the above proof are separated then the $h_\alpha \leq \varphi i_\alpha$ are unique and hence h is obtained without the Axiom of Choice.

Proposition 3. *For any filter selection $\mathbb{F}$, the $\mathbb{F}$-compact regular frames are coreflective in* **Frm**.

Proof. Note that $\mathbb{F}$-compactness = strong $\mathbb{F}$-compactness and strong convergence is unique in this setting. Also, since the regular frames are coreflective in **Frm** it is enough to argue within the category **RFrm** of these frames. Now, since coequalizers in **RFrm** are closed quotients Propositions 1 and 2 show that the only thing left to check here is the existence of a Solution Set. We claim this is provided, for any frame L, by the set of all $\mathbb{F}$-compact regular quotients of the downset frame $\mathfrak{D}L$ of L. To see this, let $h : M \to L$ be any homomorphism from an $\mathbb{F}$-compact regular frame M into L and consider its dense-onto factorization

$$h : M \xrightarrow{\nu} \uparrow s \xrightarrow{k} L$$

where $s = h_*(0)$, $\nu = (\cdot) \vee s$, and k such that $k\nu = h$. Then $\uparrow s$ is $\mathbb{F}$-compact regular, as closed quotient of M, and k is dense. It follows that the right adjoint $k_* : L \to \uparrow s$ of k is a filter and hence induces a frame homomorphism $l : \mathfrak{D}L \to \uparrow s$ such that $l(\downarrow a) = k_*(a)$. Furthermore, $\uparrow s$ is generated by $Im(k_*)$ since it is regular and consequently l is onto. In all this shows $\uparrow s$ is isomorphic to a quotient of $\mathfrak{D}L$ which proves the claim. □

Remark 2. Obviously, the same proof leads to the corresponding result for completely regular and for zero-dimensional frames.

We close with a criterion for strong $\mathbb{F}$-compactness in terms of a single specific construct which applies to certain filter selections $\mathbb{F}$. We begin by describing this.

For any frame L, $\mathbb{F}(L)$ determines a nucleus $n_{\mathbb{F}L}$ on $\mathfrak{D}L$ given by

$$n_{\mathbb{F}L}(U) = \bigcap\{\overline{\varphi}_*\overline{\varphi}(U) \mid \varphi \in \mathbb{F}(L)\}$$

where $\overline{\varphi} : \mathfrak{D}L \to T$ is the frame homomorphism associated with the filter $\varphi : L \to T$ and $\overline{\varphi}_*$ is its right adjoint so that $\overline{\varphi}_*\overline{\varphi}$ is the nucleus determined by $\overline{\varphi}$. Note that $n_{\mathbb{F}L}(U) \subseteq \downarrow(\bigvee U)$ ($\bigvee$ taken in L) because $id_L \in \mathbb{F}(L)$ and hence $\bigvee n_{\mathbb{F}L}(U) = \bigvee U$, showing that the homomorphism $\bigvee : \mathfrak{D}L \to L$ given by taking joins in L induces a frame homomorphism $k : Fix(n_{\mathbb{F}L}) \to L$. Further, $n_{\mathbb{F}L}(\downarrow a) = \downarrow a$ for each $a \in L$, and consequently we have $\downarrow : L \to Fix(n_{\mathbb{F}L})$.

In the following we put $\mathfrak{F}L = Fix(n_{\mathbb{F}L})$.

Next, a filter selection $\mathbb{F}$ is called *natural* if the filter $\downarrow : L \to \mathfrak{F}L$ belongs to $\mathbb{F}(L)$. Note that, for any filter selection $\mathbb{F}$, the frame homomorphisms $\overline{\varphi} : \mathfrak{D}L \to T$ determined by the $\varphi : L \to T$ in $\mathbb{F}(L)$ all factor through $\downarrow : L \to \mathfrak{F}L$, and hence the naturality of $\mathbb{F}$ means that $\mathbb{F}(L)$ consists exactly of the filters $L \to T$ for which this is the case.

The criterion referred to above is now given by

Proposition 4. *For any natural filter selection* $\mathbb{F}$*, the following are equivalent.*

(1) *L is strongly $\mathbb{F}$-compact.*

(2) *$\downarrow : L \to \mathfrak{F}L$ is strongly convergent.*

(3) *L is a lax retract of $\mathfrak{F}L$.*

Proof. (1) ⇒ (2). Immediate by the definition of natural $\mathbb{F}$.

(2) ⇒ (3). By hypothesis, there exists a frame homomorphism $h : L \to \mathfrak{F}L$ such that $h \leq \downarrow$, and for the homomorphism $k : \mathfrak{F}L \to L$ introduced above we then have $kh \leq id_L$.

(3) ⇒ (1). By Proposition 1, it is enough to show that $\mathfrak{F}L$ is strongly $\mathbb{F}$-compact. Consider then any $\varphi : \mathfrak{F}L \to T$ in $\mathbb{F}(\mathfrak{F}L)$. Then $\psi = \varphi \downarrow : L \to T$ belongs to $\mathbb{F}(L)$ by the naturality of $\mathbb{F}$ and hence the corresponding homomorphism $\overline{\psi} : \mathfrak{D}L \to T$ determines a homomorphism $h : \mathfrak{F}L \to T$ such that $hn_{\mathbb{F}L} = \overline{\psi}$. Now

$$h(\downarrow a) = hn_{\mathbb{F}L}(\downarrow a) = \overline{\psi}(\downarrow a) = \psi(a) = \varphi(\downarrow a),$$

and since the $\downarrow a$ generate $\mathfrak{F}L$ Lemma 1 implies that $h \leq \varphi$, showing $\mathfrak{F}L$ is strongly $\mathbb{F}$-compact. □

3. Special cases

Here we consider a number of particular situations which are quite familiar from other contexts.

Generally, a natural way to define filter selections is to impose certain *primeness conditions*. In terms of the propositional theory of filters on a frame L, these are expressed by axioms of the form

$$\bigvee S \in F \vdash \bigvee\{a \in F \mid a \in S\}$$

where S is taken from a specified collection of subsets of L, and the corresponding models are then the $0 \wedge e$-homomorphisms $L \to T$ which preserve the joins of the given S. We have already observed that taking *any* subset of L for S expresses the notion of completely prime filter in L which has as its models the frame homomorphisms on L.

Other familiar cases are as follows

choice of S	type of filter	model $\varphi : L \to T$
$S = \emptyset$	filter	filter
$S = \{a, b\}$	prime filter	bounded lattice homomorphism
S at most countable	σ-prime filter	σ-frame homomorphism
S updirected	Scott open filter	preframe homomorphism

It is obvious from the description of the models involved here that these specifications indeed define filter selections. In the following, these will be denoted

$$\mathbb{A},\ \mathbb{P},\ \mathbb{S},\ \mathbb{D}$$

respectively.

Proposition 5. *All these filter selections are natural.*

Proof. We begin with a more general consideration. Let $\mathfrak{A}$ be any collection of subsets of a frame L such that

$$\{a \wedge t \mid t \in S\} \in \mathfrak{A}$$

for each $S \in \mathfrak{A}$ and $a \in L$, and call a filter $\varphi : L \to T$ $\mathfrak{A}$-*prime* if $\varphi(\bigvee S) = \bigvee \varphi[S]$ for all $S \in \mathfrak{A}$. Further, let $\mathfrak{C} \subseteq \mathfrak{D}L$ be the closure system of all $U \in \mathfrak{D}L$ such that $S \subseteq U$ implies $\bigvee S \in U$ for all $S \in \mathfrak{A}$ and l the corresponding closure operator on $\mathfrak{D}L$. Then l is a nucleus and $\mathfrak{C}$ a frame, as is readily seen by the fact that the operator l_0 on $\mathfrak{D}L$ such that

$$l_0(U) = U \cup \bigcup\{\downarrow(\bigvee S) \mid S \subseteq U \text{ in } \mathfrak{A}\}$$

is a prenucleus with $Fix(l_0) = Fix(l) = \mathfrak{C}$. For this, note that trivially $U \subseteq l_0(U)$ and $l_0(U) \subseteq l_0(W)$ whenever $U \subseteq W$, while $l_0(U) \cap W \subseteq l_0(U \cap W)$ because $a \leq \bigvee S$ for $S \subseteq U$ in $\mathfrak{A}$ and $a \in W$ implies that $\{a \wedge t \mid t \in S\} \subseteq U \cap W$ which belongs to $\mathfrak{A}$ and has join a.

Further, $\downarrow a \in \mathfrak{C}$ for each $a \in L$ so that we have the filter $\downarrow : L \to \mathfrak{C}$, and since

$$\bigvee\{\downarrow t \mid t \in S\} = l(\bigcup\{\downarrow t \mid t \in S\}) = \downarrow(\bigvee S)$$

for any $S \in \mathfrak{A}$ (where the first join is in $\mathfrak{C}$) this is $\mathfrak{A}$-prime. Finally, for any $\mathfrak{A}$-prime filter $\varphi : L \to T$, the induced frame homomorphism $\overline{\varphi} : \mathfrak{D}L \to T$ has the property that $\overline{\varphi}(l_0(U)) = \overline{\varphi}(U)$, as seen by straightforward calculation, and consequently also $\overline{\varphi}(l(U)) = \overline{\varphi}(U)$. It therefore follows that $l(U) \subseteq \overline{\varphi}_*\overline{\varphi}(U)$ which shows that

$$l(U) = \bigcap\{\overline{\varphi}_*\overline{\varphi}(U) \mid \varphi : L \to T \ \mathfrak{A}\text{-prime filter}\},$$

equality since $\downarrow : L \to \mathfrak{C}$ is one of the φ and the term corresponding to it is actually $l(U)$.

Now, if $\mathbb{F}$ is any of the above filter selections then, for any frame L, the condition assumed above for $\mathfrak{A}$ clearly holds for the $S \subseteq L$ specified in each of these cases. It follows that $\downarrow : L \to \mathfrak{F}L$ corresponds to the above $\downarrow : L \to \mathfrak{C}$ and hence belongs to $\mathbb{F}(L)$, showing $\mathbb{F}$ is natural, as claimed. □

Remark 3. The above proof also identifies the corresponding $\mathfrak{F}L$ for the different $\mathbb{F}$ involved here as follows:

$\mathbb{A}$ - $\mathfrak{D}L$
$\mathbb{P}$ - $\mathfrak{J}L$
$\mathbb{S}$ - the frame $\mathfrak{H}L$ of σ-ideals
$\mathbb{D}$ - the frame $\mathfrak{S}L$ of Scott closed downsets.

Next we characterize the $\mathbb{F}$-compact frames for $\mathbb{F} = \mathbb{A}, \mathbb{P}$, and $\mathbb{S}$ by internal conditions.

Proposition 6. *A frame L is*

(1) $\mathbb{A}$-*compact iff it is supercompact,*

(2) $\mathbb{P}$-*compact iff it is compact, and*

*(3) $\mathbb{S}$-*compact iff it is Lindelöf.*

Proof. (1) ($\Rightarrow$) In particular, $\downarrow : L \to \mathfrak{D}L$ is convergent so that $\{\downarrow s \mid s \in C\}$ is a cover of $\mathfrak{D}L$ for any cover C of L but $\bigcup\{\downarrow s \mid s \in C\} = \downarrow e$ implies $e \in C$, showing L is supercompact.

($\Leftarrow$) Since any cover C of L contains e the same holds for $\varphi[C]$ where $\varphi : L \to T$ is any filter so that φ is trivially convergent.

(2) ($\Rightarrow$) Since $\downarrow : L \to \mathfrak{J}L$ is convergent, $\{\downarrow s \mid s \in C\}$ is a cover of $\mathfrak{J}L$ for any cover C of L and hence the ideal generated by it is $\downarrow e$. Consequently, there exist $s_1, s_2, \cdots, s_n \in C$ such that $s_1 \vee s_2 \vee \cdots \vee s_n = e$, showing L is compact.

($\Leftarrow$) Any bounded lattice homomorphism $\varphi : L \to T$ takes each finite cover to a cover, and for compact L this says it takes every cover to a cover, that is, it is convergent.

(3) ($\Rightarrow$) Again, since $\downarrow : L \to \mathfrak{H}L$ is convergent, $\{\downarrow s \mid s \in C\}$ is a cover of $\mathfrak{H}L$ for any cover C of L and hence the σ-ideal generated by it is $\downarrow e$. Further, if the Axiom of Countable Choice is assumed this σ-ideal consists of all $a \leq \bigvee X$ for the countable $X \subseteq C$ and hence C has a countable subcover.

($\Leftarrow$) Use the same kind of argument as for (2). □

Remark 4. This proposition together with Proposition 3 and Remark 2 immediately implies that the regular, completely regular, and zero-dimensional compact frames are coreflective in **Frm** and the same for the corresponding Lindelöf frames provided the Axiom of Countable Choice is assumed. Note these are familiar results, originally obtained in various ad hoc ways which appear here as consequences of a single general principle. Incidentally, nothing much along these lines results for supercompactness: a regular supercompact frame evidently has at most two elements, and the coreflection provided by the general result is just given by the initial homomorphism $\mathbf{2} \to L$.

It might be of interest to consider what happens when the compactness hypotheses in Proposition 6 are weakened to requiring the convergence only of the relevant *classical* filters. The results are as follows.

(1) A frame L is supercompact if every classical filter on L converges because L is trivially supercompact if the classical filter $\{e\}$ meets every cover.

(2) A frame L is compact if every classical prime filter on L converges provided the Prime Ideal Theorem holds. Given any cover C of a frame L, let J be the ideal in L generated by C and suppose this is proper. Then the Prime Ideal Theorem implies there exists a (classical) prime filter P disjoint from J. Hence if every classical prime filter of L converges then $J = \downarrow e$ which says C has a finite subcover.

Regarding the rôle of the Prime Ideal Theorem in this argument, it will be shown in Section 4 that this is actually equivalent to the assertion that the convergence of every classical prime filter in a frame L implies the compactness of L.

(3) For any set X of nonmeasurable cardinal, any classical σ-prime filter in $\mathfrak{P}X$ is fixed and hence convergent, but for uncountable X $\mathfrak{P}X$ is obviously not Lindelöf. Hence the convergence of the *classical* σ-prime filters does not characterize the Lindelöf frames.

In view of Proposition 5, the cases of strong compactness for $\mathbb{A}$, $\mathbb{P}$, and $\mathbb{S}$ are governed by Proposition 4; here we add some additional characterizations.

Proposition 7. *A frame is*

(1) *strongly $\mathbb{A}$-compact iff it is supercompact,*

(2) *strongly $\mathbb{P}$-compact iff it is a lax retract of a coherent frame, and*

*(3) *strongly $\mathbb{S}$-compact iff it is a lax retract of a σ-coherent frame.*

Proof. (1) It is immediate from Proposition 6 together with the fact that $\mathbb{A}$-compactness = strong $\mathbb{A}$-compactness for supercompact frames.

(2) ($\Rightarrow$) By Propositions 4 and 5 L is a lax retract of its ideal lattice $\mathfrak{J}L$ which is coherent.

($\Leftarrow$) by Proposition 1, it is enough to prove that any coherent frame M is strongly $\mathbb{P}$-compact. Let $\varphi : M \to T$ then be any bounded lattice homomorphism. Then its restriction to the sublattice $K \subseteq M$ of all compact elements induces a frame homomorphism $f : \mathfrak{J}K \to T$ such that $f(\downarrow c) = \varphi(c)$ for each $c \in K$. Further, if $g : M \to \mathfrak{J}K$ is the inverse of the familiar isomorphism $\bigvee : \mathfrak{J}K \to M$ and $h = fg : M \to T$ then

$$h(c) = fg(c) = f(\downarrow c) = \varphi(c)$$

for all $c \in K$, proving $h \leq \varphi$ by Lemma 1 since K generates M.

(3) The proof is the exact analogue of that of (2), with ideal lattices replaced by σ-ideal lattices and compact elements by Lindelöf elements, using the Axiom of Countable Choice in the appropriate places. □

It immediately follows from this proposition that certain familiar frames are strongly $\mathbb{P}$-compact. Recall for this that a frame L is called *continuous* if $a = \bigvee\{x \in L \mid x \ll a\}$ for each $a \in L$ where $x \ll a$ (x is way below a) means that $a \leq \bigvee S$ implies $x \leq \bigvee T$ for some finite $T \subseteq S$, and furthermore *stably continuous* if, in addition, it is compact and $x \ll a \wedge b$ whenever $x \ll a$ and $x \ll b$. It is a familiar fact that the stably continuous frames are exactly the retracts of coherent frames. Further, any compact regular frame is stably continuous.

Similarly, the retracts of σ-coherent frames are characterized by the analogous conditions with countable joins replacing the finite ones, and the regular Lindelöf frames are of this type.

With these notions, we now have the following immediate consequences of Proposition 7.

Corollary 1. (1) *Every stably continuous frame is strongly* $\mathbb{P}$*-compact.*
*(2) *Every stably* σ*-continuous frame is strongly* $\mathbb{S}$*-compact.*

Remark 5. Whereas Propositions 6 and 7 show that $\mathbb{A}$-compactness is the same as strong $\mathbb{A}$-compactness, the situation for $\mathbb{P}$ and $\mathbb{S}$ is different. Thus the frame

a

$L \;=\qquad c \qquad e \quad , \quad \{x \in L \,|\, x < c\} \cong \omega,$

b

already considered earlier is obviously compact and hence $\mathbb{P}$-compact but since the prime filter $\downarrow : L \to \mathfrak{J}L$ is not strongly convergent, as shown, L is not strongly $\mathbb{P}$-compact. Similarly, for the case of $\mathbb{S}$, the frame M obtained from L by replacing the part isomorphic to ω with the first uncountable ordinal Ω is also compact and hence trivially Lindelöf, and here the σ-prime filter $\downarrow : M \to \mathfrak{H}M$ is not strongly convergent: if $h : M \to \mathfrak{H}M$ is a homomorphism such that $h \leq \downarrow$ then the same argument as in the previous case shows that $h(c) = \downarrow c$ so that

$$\downarrow c = \bigvee\{h(x) \mid x < c\} = \bigvee\{\downarrow x \mid x < c\} = \{x \in M \mid x < c\},$$

the last step by the properties of Ω, and hence again the contradiction $c < c$. In all, then, M is $\mathbb{S}$-compact but not strongly so.

4. Choice Principles

We conclude by showing that the two choice principles employed earlier to prove certain results are in fact themselves consequences of the results in question so that we obtain a couple of new equivalents in this area.

Proposition 8. *The Prime Ideal Theorem holds iff every frame in which all classical prime filters are convergent is compact.*

Proof. We only need to show ($\Leftarrow$) and do this by proving that the hypothesis implies the Tychonoff Product Theorem for compact Hausdorff spaces, which is known to be equivalent to the Prime Ideal Theorem. For this, let X_α ($\alpha \in I$) be any family of such spaces, $X = \prod X_\alpha$, and $h : \bigoplus \mathfrak{O}X_\alpha \to \mathfrak{O}X$ the homomorphism induced by the maps $\mathfrak{O}X_\alpha \to \mathfrak{O}X$ resulting from the product projections $X \to X_\alpha$. Now, for any classical prime filter $\varphi : \mathfrak{O}X \to \mathbf{2}$, φh strongly converges since $\bigoplus \mathfrak{O}X_\alpha$ is compact regular, saying that $\xi \leq \varphi h$ for some homomorphism $\xi : \bigoplus \mathfrak{O}X_\alpha \to \mathbf{2}$. On the other hand, h is known to be the reflection map to spatial frames; hence there exists a homomorphism $\zeta : \mathfrak{O}X \to \mathbf{2}$ such that $\xi = \zeta h$, and since h is onto it follows that $\zeta \leq \varphi$. Now, the given hypothesis shows that $\mathfrak{O}X$, and hence X, is compact, as desired. □

Proposition 8. *The Axiom of Countable Choice holds iff every $\mathbb{S}$-compact frame is Lindelöf.*

Proof. Again, only ($\Leftarrow$) has to be shown. Now, by the proof for (3) $\Rightarrow$ (1) of Proposition 4, together with Proposition 5 and Remark 3, any $\mathfrak{H}L$ is (strongly) $\mathbb{S}$-compact; this makes it Lindelöf in the present context, and the desired result then follows by the familiar fact that the Axiom of Countable Choice holds whenever $\mathfrak{H}L$ is Lindelöf for any Boolean frame L ([3]). □

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