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A Completion of Semi-simple MV-algebra

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Abstract. We first show that any complete MV-algebra whose Boolean subalgebra of idempotent elements is atomic, called a complete MV-algebra with atomic center, is isomorphic to a product of unit interval MV-algebra I's and finite linearly ordered MV-algebras of $A(m)$-type ($m \in \mathbb{Z}^+$). Secondly, for a semi-simple MV-algebra A, we introduce a completion $\delta(A)$ of A which is a complete, MV-algebra with atomic center. Under their intrinsic topologies (see §3) A is densely embedded into $\delta(A)$. Moreover, $\delta(A)$ has the extension universal property so that complete MV-algebras with atomic centers are epireflective in semi-simple MV-algebras

In his classical paper ([3]) C. C. Chang invented the notion of MV-algebra in order to provide an algebraic proof of the completeness theorem of Lukasiewicz axioms for infinite valued propositional logic. The Boolean algebra (ring) is the corresponding algebra for the classical two-valued logic. In Boolean algebras, (= commutative idempotent unitary ring) ring-operations $+, \cdot, 0$ and 1 define lattice-operations (= distributive complemented lattice) so that it forms a Boolean lattice. As in Boolean algebra, MV-algebraic-operations $+, \cdot, -, 0$ and 1 define lattice-operations so that it forms a bounded lattice (actually, distributive lattice). Using the fact that the set $C(A)$ of all idempotent elements of an MV-algebra A, called the center of A, forms a Boolean subalgebra, it is easy to prove that the category of Boolean algebras is a coreflective subcategory of the category of all MV-algebras and their homomorphisms.

Unlike Boolean algebra, not all MV-algebras are semi-simple. In [1], [2], Belluce and in [6] Hoo have developed MV-algebras in its algebraic properties and topological properties, in particular, they have characterized the semi-simple MV-

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algebra in terms of many different notions; in terms of Bold algebra of Fuzzy subset ([1]), Archimedeanness, quasi-locallyness, and the lattice-completeness and subdirect product of unit interval MV-algebras ([2]). Hoo has shown that A is semi-simple iff the space of maximal ideals of A is dense in the space of prime ideals ([6]). For each positive integer m, let $A(m) = \{0, \frac{1}{m}, \frac{2}{m}, \cdots, \frac{m-1}{m}, 1\}$. Then $(A(m), +, \cdot, -, 0, 1)$ is a MV-algebra under the operations $[p/m] + [q/m] = \min(1, [(p+q)/m])$, $[p/m]\cdot[q/m] = \max(0, [(p+q-m)/m])$ and $\overline{[p/m]} = [(m-p)/m]$ ([3]). In this paper, we first show that if A is a complete MV-algebra and its $C(A)$ is atomic then A is isomorphic to a product of a cube and $\Pi A(m)$, where cube means a product of unit interval MV-algebras and $\Pi A(m)$ is a product of finite MV-algebras $A(m)'s$. After the proof that any complete atomic one is an atomic center, it follows that if A is complete nonatomic and A has at least one atom, and if $A \cong B \times C$(Belluce's decomposition: Theorem 9 [2]) then the atomic part $B \cong \Pi A(m)$ and atomless part $C \cong I^\lambda$ (a cube). It follows immediately that if A is complete and atomic then A is the direct product $\Pi A(m)$ for some $m \in \Lambda \subset \mathbb{Z}$ (see Theorem 4.2 and the below remark [2]). Secondly we introduce an intrinsic topology on a semi-simple MV-algebra so that it is a topological MV-algebra, we show that every semi-simple MV-algebra A has a completion $\delta(A)$ which is a complete (or a compact) MV-algebra with atomic center and $\delta(A)$ contains A as a dense subalgebra. Furthermore A is a subdirect product of the type $\Pi J_\alpha \times \Pi A(m)$, of MV-algebras where J_α is a dense subalgebra of the unit interval MV-algebra I. Finally, we investigate further properties of the δ-completion, for example, $\delta(A)$ has an extension universal property.

Preliminaries.

We recall MV-algebra $A = (A, +, \cdot, -, 0, 1)$, which is $(2, 2, 1, 0, 0)$ type and the following equations are satisfied: for $x, y, z \in A$,

(i) $x + y = y + x$

(ii) $(x + y) + z = x + (y + z)$

(iii) $x \cdot y = \overline{(\bar{x} + \bar{y})}$

(iv) $x + 0 = x$ and $x + 1 = 1$

(v) $\bar{0} = 1$ and $\bar{1} = 0$

(vi) $\overline{(\bar{x} + y)} + y = \overline{(\bar{y} + x)} + x$.

If we define that for $x, y \in A$, $x \vee y = x + \bar{x}y$ and $x \wedge y = x(\bar{x} + y)$ then $(A, \vee, \wedge, 0, 1)$ is a bounded distribution lattice.

For all basic terminologies of MV-algebra, we refer to [1], [2] and [3]. For an MV-algebra A, $C(A)$ denotes the Boolean subalgebra of all idempotent elements of A. $C(A)$ is called *the center of* A, and its element is called a *center element* of A. For $a \in A$, $\downarrow a$ denotes the subset $\{x \in A \mid x \leq a\}$ and dually $\uparrow a$ denotes $\{x \in A \mid a \leq x\}$.

1. Atomic MV-algebras

Let A be an MV-algebra. For $x, y \in A$, *"y covers x"* means that $x < y$ and no elements between x and y. If a covers 0 then a is called *an atom* of A. In this section we show first that for any atomic complete MV-algebra the center $C(A)$ must be a power-set Boolean subalgebra, namely, $C(A)$ is an atomic complete Boolean algebra.

We first prove the following theorem

Theorem 1.1. *For an MV-algebra A, the following statements are equivalent, for $x, y \in A$,*

(i) *y covers x*

(ii) *$\bar{x}y$ covers 0 and $x\bar{y} = 0$*

(iii) *1 covers $x + \bar{y}$ and $\bar{x} + y = 1$(i.e., $x\bar{y} = 0$).*

Proof. (i) $\Rightarrow$ (ii). Assume that $0 < u < \bar{x}y$ for some $u \in A$. Then we have $x \le x + u \le x + \bar{x}y = y$. We claim that $x < x + u < y$ which is absurd. Indeed, if $x = x + u$ then $u = 0$ because $0 \le \bar{x}$ and $u < \bar{x}y \le \bar{x}$ imply $u = 0$ by Theorem 1.14 [3], which is a contradiction. Thus $x < x + u$. Now if $x + u = y$ then $\bar{x}(x + u) = \bar{x}y$, i.e., $\bar{x} \wedge u = \bar{x}y$. On the other hand, we have $u < \bar{x}y \le \bar{x}$, i.e., $\bar{x} \wedge u = u$. Thus $\bar{x}y = u$ which is also a contradiction. So we have $x + u < y$. Hence we have $x < x + u < y$ which is absurd to (i). Thus (ii) holds.

(ii) $\Rightarrow$ (i). Suppose that $x < z < y$ for some $z \in A$. Then we have $0 \le \bar{x}z \le \bar{x}y$. If $0 = \bar{x}z$ then $d(x, y) = \bar{x}y + \bar{y}x = \bar{x}z = 0$ since $x < z$. Thus $x = z$. Similarly if $\bar{x}z = \bar{x}y$ then $x + \bar{z} = x + \bar{y}$. Since $\bar{z}$ and $\bar{y}$ are both bounded by $\bar{x}$, we have $\bar{z} = \bar{y}$, i.e., $z = y$. Hence $0 < \bar{x}z < \bar{x}y$ which is a contradiction to (ii).

(ii) $\Leftrightarrow$ (iii). These (ii) and (iii) are dual each other. The proof is complete. $\square$

The following corollary is immediate.

Corollary ([2]). *If an MV-algebra has no atoms then it is densely ordered.*

The following lemma is immediate from Theorem 5 [2].

Lemma 1.2. *If A is a complete MV-algebra then so is $C(A)$, i.e., $C(A)$ is a complete Boolean subalgebra of A.*

Lemma 1.3. *Let A be a complete MV-algebra. For $S \subset A$ with $S \neq \emptyset$, if $c = \sup S^{\perp}$, then $c \in C(A)$, where $S^{\perp} = \{x \in A | x \wedge s = 0$ for all $s \in S\}$.*

Proof. For any $s \in S$, $s \wedge c = s \wedge \sup S^{\perp} = \sup(s \wedge \sup S^{\perp}) = 0$. Thus $c \in S^{\perp}$. Since $S^{\perp}$ is always an ideal of A, $2c \in S^{\perp}$, and hence $2c = c$. $\square$

Remark. *If A is a complete MV-algebra, and if $x \wedge y = 0$ and $c = \sup\{y\}^{\perp}$ for $0 \neq x$, $0 \neq y$, then $c \in C(A)$ and $0 \neq c \neq 1$.*

Proposition 1.4. *If A is a complete MV-algebra and a_0 is an atom of A, then there exists a unique atom c_0 of $C(A)$ such that $a_0 \leq c_0$.*

Proof. We have either

(i) $\{a_0\}^{\perp} = \{0\}$ or

(ii) $\{a_0\}^{\perp} \neq \{0\}$.

For case (i), if $y \in A$ with $y \neq 0$ then $a_0 \wedge y = a_0$, because either $a_0 \wedge y = a_0$ or $a_0 \wedge y = 0$ since a_0 is an atom of A, and if $y \wedge a_0 = 0$ then $y \in \{a_0\}^{\perp}$. Hence $y \in \uparrow a_0$. Thus $\uparrow a_0 = A - \{0\}$. Now let $c_0 = \inf\{c \in C(A) | a_0 \leq c\}$ then c_0 is an atom of $C(A)$.

For (ii), firstly we note that $\{a_0\} \subset \{a_0\}^{\perp\perp}$. Let $d_0 = \sup\{a_0\}^{\perp\perp}$. Then $d_0 \in C(A) \cap \uparrow a_0$. Clearly $d_0 \in \{a_0\}^{\perp\perp}$. So we have $\{a_0\}^{\perp\perp} = \downarrow d_0$. Now let $c_0 = \inf(C(A) \cap \uparrow a_0)$. Evidently $c_0 \in C(A) \cap \uparrow a_0$. We claim that c_0 is an atom of $C(A)$ and $a_0 \leq c_0$. Indeed, if there exists $e_0 \in C(A)$ such that $0 < e_0 < c_0 \leq d_0$, then $e_0 \in \{a_0\}^{\perp\perp} = \downarrow d_0$. Hence since a_0 is atom of A we have either $e_0 \wedge a_0 = a_0$ or $e_0 \wedge a_0 = e_0$ or $e_0 \wedge a_0 = 0$ which implies $a_0 \leq e_0$ or $e_0 \leq a_0$ or $e_0 \in \{a_0\}^{\perp}$ respectively. Thus we have $e_0 = c_0$ or $e_0 = a_0$ or $e_0 \in \{a_0\}^{\perp}$, respectively. But none of which is possible, because $e_0 < c_0$, $e_0 = a_0$ implies $e_0 = c_0$, and if $e_0 \in \{a_0\}^{\perp}$, then $e_0 = 0$, respectively.

For the uniqueness of such atoms, if c_0 and c_1 are two distinct atoms of $C(A)$ and $a_0 \leq c_0$, $a_0 \leq c_1$ then $a_0 \leq c_0 \wedge c_1 = 0$ and hence $a_0 = 0$, a contradiction. The proof is complete. □

The following corollary is obvious from Lemma 1.2 and the above proposition:

Corollary 1.5. *If A is a complete atomic MV-algebra, then $C(A)$ is a power set Boolean algebra.*

For a decomposition of an MV-algebra A, the center $C(A)$ plays a very important role, as in lattice theory.

If $C(A)$ is atomic for an MV-algebra A, A is said to have *atomic center.*

2. Decompositions of complete MV-algebras with atomic centers

It is well known that for an ideal P of an MV-algebra A, P is a prime ideal iff for $x, y \in A$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$. It is also known that the quotient MV-algebra A/P is linearly ordered for any prime ideal P of A([4]).

The following lemma is obvious:

Lemma 2.1. *If P is a prime ideal of A and $a \in C(A)$ then either $a \in P$ or $\bar{a} \in P$. Moreover, $a \in P$ iff $\bar{a} \notin P$.*

In the following lemma we prove firstly that the ideal generated by $\bar{a}$ is prime in a complete MV-algebra A for any atom a of $C(A)$, but it will turn out that it is actually a maximal ideal in the latter.

Lemma 2.2. *Let A be a complete MV-algebra. If a is an atom of $C(A)$, then $\downarrow \bar{a}$ is a prime ideal of A. Furthermore, the ideal $\downarrow a$ is linearly ordered.*

Proof. Clearly $\downarrow \bar{a}$ is a proper ideal of A. Assume that $\downarrow \bar{a}$ is not prime. Namely, there exists two elements x and y in A such that $x \wedge y \in \downarrow \bar{a}$ but $x \notin \downarrow \bar{a}$ and $y \notin \downarrow \bar{a}$. Then $ax \wedge ay = 0$ by Lemma 2 ([2]). Note that $ax \neq 0$ since $ax = 0$ implies $\bar{a} \vee x = \bar{a}$, and hence $x \leq \bar{a}$. Similarly $ay \neq 0$. Further $ax \neq 1$ since $\bar{a} + \bar{x} = 0$ implies $\bar{x} = 0$ and similarly $ay \neq 1$. By the Remark after Lemma 1.3, $c = \sup \{ay\}^{\perp} \in C(A)$ and $0 \neq c \neq 1$. Since $ax \in \{ay\}^{\perp}$, we have $0 < ax \leq c$; since $\bar{a} \in \{ay\}^{\perp}$ we have $\bar{a} \leq c$, i.e., $\bar{c} \leq a$. Note that $\bar{c} \neq a$ because if so, $ax \leq c = \bar{a}$ and hence $ax = a^2 x \leq a\bar{a} = 0$ which is impossible. If follows that $\bar{c} \in C(A)$ and $0 < \bar{c} < a$, which is a contradiction to the fact that a is an atom of $C(A)$. The second part of the Lemma follows from the first isomorphism Theorem; Let $f : A \to I_a = \downarrow a$ by $f(x) = ax$. By Proposition 3 ([2]), f is a MV-homomorphism of A onto I_a. Then the kernel of f is $I_a^{\perp} = \downarrow \bar{a}$. Thus the quotient of A modulo $\downarrow \bar{a}$ is isomorphic to $\downarrow a$. □

Remark. For a prime ideal P of A, there exists at most one atom a of $C(A)$ such that $\bar{a} \in P$. For, if there exist two such atoms a_1 and a_2 of $C(A)(a_1 \neq a_2)$, then $\bar{a_1} \vee \bar{a_2} = \bar{a_1} + \bar{a_2} = 1 \in P$.

Proposition 2.3. *If B is a complete subalgebra of the unit interval MV-algebra $I(= [0,\ 1])$, then B is either I itself or a finite MV-algebra $A(m)$ for some $m \in \mathbb{Z}^+$.*

Proof. If B has an atom b, say its order is m, then evidently B is isomorphic to $A(m)$. Now assume that B does not have any atom. Then B must be I. For, suppose $I - B \neq \emptyset$. Then for any $x \in I - B$ let $b_0 = \sup\{\downarrow x \cap B\}$ and $b_1 = \inf\{\uparrow x \cap B\}$ then b_1 covers b_0 in B since $b_0 < x < b_1$ and $x \notin B$. By Theorem 1.1, B has an atom. This is a contradiction. □

Lemma 2.4. *Let A be a complete MV-algebra with atomic center. Let a be an atom of $C(A)$. Then the MV-algebra $I_a = \langle \downarrow a,\ +,\ \cdot,\ -,\ 0,\ a \rangle$ is isomorphic to either $A(m)$ for some $m \in \mathbb{Z}^+$ or the unit interval MV-algebra.*

Proof. Since A is complete, so is I_a. Thus I_a is a complete semi-simple linearly ordered MV-algebra. It follows that I_a is Archimedean and hence I_a is locally finite (Theorem 31, 32 [1]). We have $A/I_{\bar{a}}$ is locally finite, since $A/I_{\bar{a}} \cong I_a$. It follows that I_a is embedded into I. (See the remark on page 2 [2]). By Proposition 2.3, I_a is isomorphic to either $A(m)$ for some $m \in \mathbb{Z}^+$ or I. □

Remark. In the above proof, since $A/I_{\bar{a}}$ is locally finite we have $I_{\bar{a}} = \downarrow \bar{a}$ is actually a maximal ideal of A by (Theorem 4.7 [3]).

Proposition 2.5. *Let A be a complete MV-algebra with atomic center and $\{a_\alpha | \alpha \in \Gamma\}$ be the set of all atoms of $C(A)$. Then A is isomorphic to $\Pi\{I_\alpha | \alpha \in \Gamma\}$, where $I_\alpha = I_{a_\alpha}$ for each $\alpha \in \Gamma$.*

Proof. Define $\varphi : A \to \Pi I_\alpha$ by $\varphi(x) = \langle a_\alpha x\rangle_{\alpha\in\Gamma}$ for each $x \in A$, and define $\psi : \Pi I_\alpha \to A$ by $\psi(\langle x_\alpha\rangle) = \sup\{x_\alpha | \alpha \in \Gamma\}$ for each element $\langle x_\alpha\rangle$ of ΠI_α. Then clearly φ and ψ are both MV-homomorphisms. By Theorem 5 [2], it is easy to see that $\psi \circ \varphi = \mathrm{id}_A$ and $\varphi \circ \psi = \mathrm{id}_{\Pi I_\alpha}$. □

In summary, the following theorem has completely characterized complete MV-algebras with atomic centers.

Theorem 2.6. *If A is a complete MV-algebra with atomic center, then A is isomorphic to a direct product of a cube and $\Pi\{A(m) \mid m \in \Lambda \subset \mathbb{Z}^+\}$, where $\Lambda \subseteq \mathbb{Z}^+$ and cube means a product of I's.*

In [2], it is shown that if A is a complete nonatomic MV-algebra and if A has at least one atom, then $A \cong B \times C$ where B is complete atomic, C is complete atomless MV-algebra. Furthermore if A is atomic then $A \cong B$ and C is disappeared as follows:

Corollary 2.7. *If A is a complete atomic MV-algebra then $A \cong \Pi A(m)$.*

3. The δ-completion of semi-simple MV-algebra

By a topological MV-algebra, we mean a pair (A, τ), where A is an MV-algebra and τ is a Hausdorff topology on A such that all operations $+, \cdot$ and $-$ are continuous.

Clearly every topological MV-algebra (A, τ) is also a topological distributive lattice and $C(A)$ is a closed subset of A.

The following lemma is well-known ([5]):

Lemma 3.1. *If (A, τ) is a compact topological MV-algebra, then*

(i) *A is a complete lattice.*

(ii) *$C(A)$ is a compact Boolean algebra, i.e., it is a power set Boolean algebra.*

By Theorem 2.6 we then have the following lemma:

Lemma 3.2. *If (A, τ) is a compact MV-algebra, then $A \cong I^\lambda \times \Pi\{A(m)|m \in \Lambda \subset \mathbb{Z}^+\}$ where the cube I^λ is the connected atomless part of A for some cardinal λ, and $\Pi\{A(m)\}$ is the totally disconnected atomic part of A for some subset Λ of $\mathbb{Z}^+$.*

Now we turn to characterize semi-simple algebras as subalgebras of a cube.

First of all, we note that the unit interval algebra $(I, \oplus, \odot, -, 0, 1)$ is a topological MV-algebra under the ordinal topology. For $x, y \in I$, $x \oplus y = \min\{1, x + y\} = \frac{1}{2}\{1 + x + y - |1 - x - y|\}$ and $\bar{x} = 1 - x$ are continuous and hence $x \odot y$ is continuous, where $+, -$ are the real operations of I.

Let A be a semi-simple MV-algebra and let $H = \hom(A, I)$ be the set of all homomorphisms of A to I.

Clearly, the cube $I^H = \Pi\{I_f | f \in H\}$ has the compact topology τ, its product topology, for which (I^H, τ) is a compact MV-algebra.

Let $e : A \to I^H$ be the evaluation map: $e(x) = \langle f(x) \rangle$ for each $x \in A$ and each $f \in H$. Since A is semi-simple, e is injective and hence A is embedded into I^H. Then $A \cong e(A) \subset I^H$. Hence $(e(A), \tau_{e(A)})$ is a topological MV-subalgebra of I^H under the relative topology $\tau_{e(A)}$ of τ.

It is known ([5]) that the closure of a subalgebra B of a topological universal algebra A is again a subalgebra of A.

Setting $\delta(A) = \Gamma(e(A))$ where Γ is the closure operation of I^H, we call $\delta(A)$ *the δ-completion of A*.

Evidently, $\delta(A)$ is a compact Hausdorff MV-algebra under its relative topology and hence $\delta(A)$ is a complete MV-algebra with atomic center.

Again by Theorem 2.6, we have that $\delta(A)$ has the following type: $\delta(A) \cong I^{H_0} \times \Pi\{A(m) | m \in \Lambda \subset \mathbb{Z}^+\}$, where $H_0 \subset H$.

Let A be a semi-simple MV-algebra. Then A is embedded into I^H. Since $A \cong (e(A), \tau_{e(A)})$, A has the topology τ_A so that $(A, \tau_A) \cong (e(A), \tau_{e(A)})$ is isomorphic algebraically and topologically. τ_A is called *the intrinsic topology of A*.

Then we have the following theorem:

Theorem 3.3. *Any semi-simple MV-algebra A is densely embedded into $I^{H_0} \times \Pi\{A(m) | m \in \Lambda \subset \mathbb{Z}^+\}$ under its intrinsic topology where $H_0 \subset H$, a subset Λ of $\mathbb{Z}^+$ and $|H_0| + |\Lambda| = |H|$, where $H = \hom(A, I)$. Furthermore, A is a subdirect product of $\prod_{f \in H_0} J_f \times \Pi A(m)$, where J_f is a dense subalgebra of I for each $f \in H_0$.*

Proof. The first part of the theorem already has been shown in the above. For the second part, let $\delta(A)$ be the δ-completion of A and $A \overset{e}{\hookrightarrow} e(A) \subset \delta(A) \subset I^H$. For each $f \in H$ and the f^{th} projection p_f of I^H onto I, setting $p_f(e(A)) = J_f$ for each $f \in H_0$, it is easy to see that J_f is a dense subalgebra of I. Note that for the atomic part $\Pi A(m)$, A has a exactly same copy of subalgebra as $\Pi A(m)$ because $p_m(e(A)) = A(m)$ for each $m \in \Lambda$, p_m is the m^{th} projection of I^H onto $A(m)$. □

Examples. We give several typical examples of dense subalgebras of I

1. I itself.
2. The subalgebra of all rationals in I.
3. The subalgebra of algebraic numbers in I.
4. The subalgebra of dyadic numbers in I.
5. The subalgebra of all numbers of type $r + s\sqrt{2}$ in I for all rationals r and s.

Here we study some important properties of the δ-completion; among those, a useful property in an extension property. From this property one can easily show that the category of a complete MV-algebras with atomic center is an epi-reflective subcategory of the category of all semi-simple MV-algebras.

Lemma 3.4. *Let A, B be two semi-simple MV-algebras and τ_A, τ_B are their intrinsic topologies respectively.*

If $\varphi : A \to B$ is an MV-homomorphism then $\varphi : (A, \tau_A) \to (B, \tau_B)$ is continuous.

Proof. Since (B, τ_B) is embedded into I^G, where $G = \hom(B, I)$, τ_B is the initial topology with respective to the source $G_B = \{g|_B \mid g \in I^G\}$. Since $g|_B \circ \varphi \in H_A$ for each $g \in I^G$, $g|_B \circ \varphi$ is continuous. Thus φ is continuous. □

Lemma 3.5. *If A is a complete MV-algebra with atomic center, then (A, τ_A) is a compact MV-algebra.*

Proof. Since $A \cong I^{H_0} \times \Pi A(m)$ by Theorem 3.3, A is closed-embedded into I^H. Therefore (A, τ_A) is compact. □

Now we prove the extension properties of the δ-completion.

Theorem 3.6. *Let A be a semi-simple MV-algebra with δ-completion $\delta(A)$ and let $\delta : A \to \delta(A)$ be a dense embedding. For each complete MV-algebra B with atomic center and a homomorphism $u : A \to B$, there exists a unique homomorphism $u^* : \delta(A) \to B$ with $u^* \circ \delta = u$.*

Proof. Let $H = \mathrm{Hom}(A, I)$, $G = \mathrm{Hom}(B, I)$ and let $e : A \to \delta(A)$ be the embedding.

Let $h = \coprod \{f \mid f \in \hom(A, I)\} = \coprod \hom(A, I) : A \to I^H$ and $k = \coprod \{g \mid g \in \hom(B, I)\} = \coprod \hom(B, I) : A \to I^G$. Then e is the image corestriction of h and $h = A \to \delta(A) \hookrightarrow I^H$.

By the assumption, $\hom(A, I)$ and $\hom(B, I)$ are point-separating, i.e., mono-sources; hence h and k are both injective homomorphisms. Furthermore, by the definition of intrinsic topology, they are indeed embeddings as topological algebras, so that δ is a dense embedding. Since B is a complete MV-algebra with atomic center, (B, τ_B) is compact, and hence k is a closed embedding.

Now take any homomorphism $u : A \to B$, let $\bar{u} = \coprod \{p_{gu} \mid g \in \hom(B, I)\} : I^H \to I^G$, where $p_{gu} : I^H \to I$ denotes the $g \circ u^{th}$ projection (note $g \circ u : A \to I \in \mathrm{Hom}(A, I) = H$), then for any $g \in \hom(B, I)$, $p_g \circ k \circ u = g \circ u = p_{gu} \circ h = p_g \circ \bar{u} \circ h = p_g \circ \bar{u} \circ j \circ \delta$, where j is the embedding of $\delta(A)$ into I^H and therefore $k \circ u = \bar{u} \circ j \circ \delta$. Since δ is dense and k is a closed embedding, by the Diagonalization Property ([7]), there is a unique continuous map $u^* : \delta(A) \to B$ with $u^* \circ \delta = u$, which is clearly a homomorphism because k is an embedding. □

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