KYUNGPOOK Math. J. 45(2005), 561-570

Inverse of Frobenius Graphs and Flexibility

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Abstract. Weak Crossed Product Algebras correspond to certain graphs called lower subtractive graphs. The properties of such algebras can be obtained by studying this kind of graphs ([4], [5]). In [1], the author showed that a weak crossed product is Frobenius and its restricted subalgebra is symmetric if and only if its associated graph has a unique maximal vertex. A special construction of these graphs came naturally and was known as standard lower subtractive graph. It was a deep question that when such a special graph possesses unique maximal vertex?

This work is to answer the question partially and to give a particular characterization for such graphs at which the corresponding algebras are isomorphic. A graph that follows the mentioned characterization is called flexible. Flexibility is to some extend a generalization of the so-called Coxeter groups and its weak Bruhat ordering.

1. Introduction

Let $G = Gal(K/F)$ be a finite Galois group for an extension of fields K/F , and let $f : G \times G \to K$ be a weak 2-cocycle which satisfies: for $\sigma, \tau, \gamma \in G$,

(i) $f(\sigma, \tau)f(\sigma\tau, \gamma) = f^{\sigma}(\tau, \gamma)f(\sigma, \tau\gamma)$,

(ii) $f(1, \sigma) = f(\sigma, 1) = 1$,

(iii) $f(\sigma, \sigma^{-1}) = 0$ for $\sigma \neq 1$.

Consider a $K-$vector space on the basis $\{x_\sigma : \sigma \in G\}$, and define the following multiplications on it to make the structure into an algebra denoted by A_f

(1) $x_\sigma x_\tau = f(\sigma, \tau)x_{\sigma\tau}$,

(2) $k^\sigma x_\sigma = x_\sigma k$ for $k \in K$.

This algebra is called *weak crossed product*. It is associative and has an identity by (i), (ii) above [3]. If $e : G \times G \to \{0, 1\}$ is defined by $e(\sigma, \tau) = 0$ if and only if $f(\sigma, \tau) = 0$ for $\sigma, \tau \in G$. So we get an *idempotent weak 2-cocycle* associated to f whose algebra A_e is called restricted algebra over the base field F.

Received May 7, 2004.
2000 Mathematics Subject Classification: 47L65.
Key words and phrases: weak crossed products, lower subtractive graphs, frobenius algebras, restricted algebras, symmetric algebras, weak 2–cocycles, inverse graphs, flexible graphs, coxeter groups, weak bruhat ordering.

Define a relation "$\leq$" on G by $\sigma \leq \tau \iff f(\sigma^{-1}, \sigma\tau) \neq 0$ (equivalently $\iff e(\sigma, \tau) = 1$). It is straightforward that this relation is partial order and can be represented in a rooted graph. It is also satisfying

$$\sigma \leq \gamma \leq \tau \iff \sigma^{-1}\gamma \leq \sigma^{-1}\tau.$$

The later property is known as *lower subtractivity.* Intuitively, it means in the graphical language every subgraph upstairs has a copy subgraph downstairs and this property is powerful in proving some statements in this work, in more specific words if things do not happen in the lower part of the graph then they can not happen in the upper part of the graph ([3], [6]). For instance, there are nine lower subtractive graphs defined on S_3 with two generators (elements of level 1). Only one of them is Frobenius [2].

However, one can start with a pair (G, S), a finite group G with generating set $S = \{s_1, s_2, \cdots, s_t\}$ and construct a lower subtractive graph in the following manner:

Put 1 as a root of the graph (level 0), and then put all the generators $s_1,\ s_2,\ \cdots,\ s_t$ right above the identity (level 1). For level 2, every element of $s_1s_1,\ s_1s_2,\ \cdots,\ s_1s_t$ should be put right above s_1 unless it appeared in level 0 or 1. The elements among $s_1s_1,\ s_1s_2,\ \cdots,\ s_1s_t$ which appeared in a lower level should be ignored. The elements $s_2s_1,\ s_2s_2,\ \cdots,\ s_2s_t$ are put right above s_2 except any element appeared in a lower level. Continue in this process until the elements of G are all exhausted. The resulting graph is lower subtractive and has the *Catenary property* which means each element has a length (level) and the length is unique. We keep the notation $l(g)$ to indicate the length function that can be defined mathematically as $l(1) = 0$ and for $g \neq 1$, $l(g) = \min\{d \in N : g = s_{i_1}s_{i_2}\cdots s_{i_d},\ s_{i_j} \in S\}$. Unfortunately, not all lower subtractive graphs arise this way. These special graphs are called *standard lower subtractive graphs* or briefly *standard graphs* and denoted by $\Gamma(G, S)$ [1]. In this work we focus on standard graphs and try to answer questions such as

- When has $\Gamma(G, S)$ a unique maximal vertex?
- When is $\Gamma(G, S)$ a lattice?
- What kind of graphs one can get where the group is cyclic?
- When is the "inverse" graph isomorphic to the original?
- What are the relations between the generators and the top element?

It turned out that attacking such questions is not an easy task, so we have come up with partial results which may open the road to more sophisticated studies.

2. Graph inverse

Definition 2.1. Let $\Gamma(G, S)$ be a standard graph having a unique maximal element γ. The *inverse graph* is denoted by $\gamma^{-1}\Gamma(G, S)$ and defined on $G = \{\gamma^{-1}g : g \in G\}$ by

$$\gamma^{-1}g' \prec \gamma^{-1}g \iff g \leq g' \quad \text{in} \quad \Gamma(G, S).$$

Note that the graph $\gamma^{-1}\Gamma(G,S)$ needs to be flipped over in order to be read regularly. We also use "$\prec$" and "$\leq$" to indicate the relations on $\gamma^{-1}\Gamma(G,S)$ and $\Gamma(G,S)$ respectively.

If the graph is standard lower subtractive with a unique maximal element, we abbreviate that by SFG which stands for standard Frobenius graph and if it is only lower subtractive with unique maximal element, we denote it by FG. Before going on, we assert that the inverse graph of an FG or SFG is lower subtractive.

Proposition 2.2.

(i) *The graph is an FG with maximal element γ if and only if its inverse is an FG with unique maximal element γ^{-1}.*

(ii) *The graph $\Gamma(G,S)$ is an SFG with maximal element γ if and only if $\gamma^{-1}\Gamma(G,S)$ is an SFG with unique maximal element γ^{-1}. Moreover, $l_\gamma(\gamma^{-1}g)$ in the inverse graph is equal to $l(\gamma)-l(g)$ in $\Gamma(G,S)$.*

Proof. (i) In $\gamma^{-1}\Gamma(G,S)$, suppose that $\gamma^{-1}\alpha \prec \gamma^{-1}\beta$. We need to show that $\gamma^{-1}\alpha \prec \gamma^{-1}\delta \prec \gamma^{-1}\beta$ if and only if $\alpha^{-1}\delta \prec \alpha^{-1}\beta$. Since $\gamma^{-1}\alpha \prec \gamma^{-1}\beta$, so $\beta \leq \alpha$ in $\Gamma(G,S)$. Now

$$\begin{aligned}
& \alpha^{-1}\delta \prec \alpha^{-1}\beta \\
\iff\ & \gamma^{-1}\gamma\alpha^{-1}\delta \prec \gamma^{-1}\gamma\alpha^{-1}\beta \\
\iff\ & \gamma\alpha^{-1}\beta \leq \gamma\alpha^{-1}\delta \\
\iff\ & \gamma\alpha^{-1}\beta \leq \gamma\alpha^{-1}\delta \leq \gamma \quad (\text{since } \gamma \text{ is the maximal element in } \Gamma(G,S)) \\
\iff\ & \beta^{-1}\alpha\gamma^{-1}\gamma\alpha^{-1}\delta \leq \beta^{-1}\alpha\gamma^{-1}\gamma \quad (\text{by lower subtractivity in } \Gamma(G,S)) \\
\iff\ & \beta^{-1}\delta \leq \beta^{-1}\alpha \\
\iff\ & \beta \leq \delta \leq \alpha \quad (\text{by lower subtractivity of } \Gamma(G,S) \text{ and } \beta \leq \alpha) \\
\iff\ & \gamma^{-1}\alpha \prec \gamma^{-1}\delta \prec \gamma^{-1}\beta \quad (\text{by definition}).
\end{aligned}$$

So $\gamma^{-1}\Gamma(G,S)$ is lower subtractive.

(ii) Part(i) is taking care of proving lower subtractivity. We first show that $\gamma^{-1}\Gamma(G,S)=\Gamma(G,S^{-1})$. If $l(\gamma)=m$, and $g=s_{i_1}s_{i_2}\cdots s_{i_{m-1}}$ lies right below γ with length $m-1$ then there is a generator s such that $\gamma = s_{i_1}s_{i_2}\cdots s_{i_{m-1}}s$ and hence $\gamma^{-1}g=s^{-1}$, so the set of generators of the inverse graph is precisely S^{-1}. If $l(h)=m-2$, and $h=s'_{i_1}s'_{i_2}\cdots s'_{i_{m-2}}$, then there exist two generators s', s'' such that $\gamma = s'_{i_1}s'_{i_2}\cdots s'_{i_{m-2}}\, s's''$, but then $\gamma^{-1}h = s''^{-1}s'^{-1}$ and we claim that this element in the second level in the graph $\gamma^{-1}\Gamma(G,S)$. To see this, notice that if $s''^{-1}s'^{-1}$ appears in a lower level then either $s''^{-1}s'^{-1}=1$ or $s''^{-1}s'^{-1}=s_0^{-1}$ for some generator $s_0\in S$. Hence $\gamma = s'_{i_1}s'_{i_2}\cdots s'_{i_{m-2}}$ or $\gamma = s'_{i_1}s'_{i_2}\cdots s'_{i_{m-2}}s_0$, so $l(\gamma)<m$ (contradiction). Same idea can be applied to show that if $l(\sigma)=m-i$ in $\Gamma(G,S)$ then $l_\gamma(\gamma^{-1}\sigma)=i$ in $\gamma^{-1}\Gamma(G,S)$. $\square$

Let $e : G \times G \to \{0,1\}$ be an idempotent weak 2-cocycle associated to f and defined by $e(\sigma,\tau) = 0$ if and only if $f(\sigma,\tau) = 0$ for $\sigma,\ \tau \in G$. So we get an F-algebra A_e called *restricted algebra* over the base field F. The multiplication in A_e is given by

$$x_\sigma x_\tau = \begin{cases} x_{\sigma\tau} & \sigma \leq \sigma\tau \\ 0 & \textit{otherwise.} \end{cases}$$

Now, if $\Gamma(G,S)$ is an SFG with unique maximal element γ and associated idempotent weak 2-cocycle e, then how is the idempotent weak 2-cocycle e_γ of $\gamma^{-1}\Gamma(G,S)$ related to e? The next result answers the question.

Lemma 2.3. *If $\Gamma(G,S)$ is an SFG with unique maximal element γ and associated idempotent weak 2-cocycle e, then the idempotent weak 2-cocycle e_γ of $\gamma^{-1}\Gamma(G,S)$ is given by $e_\gamma(\sigma,\tau) = e(\gamma\sigma\tau,\tau^{-1})$, $\sigma,\ \tau \in G$.*

Proof. We first show that e_γ is indeed 2-cocycle. This means that $e_\gamma(\sigma,\tau)e_\gamma(\sigma\tau,\delta) = e^\sigma_\gamma(\tau,\delta)e_\gamma(\sigma,\tau\delta)$ for $\sigma,\ \tau,\ \delta \in G$. Equivalently, $e^\sigma(\gamma\tau\delta,\delta^{-1})e(\gamma\sigma\tau\delta,\delta^{-1}\tau^{-1}) = e(\gamma\sigma\tau,\tau^{-1})e(\gamma\sigma\tau\delta,\delta^{-1})$. But this is true if and only if $\gamma\sigma\delta \leq \gamma\tau$ and $\gamma\sigma\tau\delta \leq \gamma\sigma \iff \gamma\sigma\tau \leq \gamma\sigma$ and $\gamma\sigma\tau\delta \leq \gamma\sigma\tau$. Since γ is the maximal element, the later statement can be replaced by $\gamma\sigma\delta \leq \gamma\tau \leq \gamma$ and $\gamma\sigma\tau\delta \leq \gamma\sigma \leq \gamma \iff \gamma\sigma\tau \leq \gamma\sigma \leq \gamma$ and $\gamma\sigma\tau\delta \leq \gamma\sigma\tau \leq \gamma$. Using the lower subtractivity yields $\delta^{-1} \leq \delta^{-1}\tau^{-1}\sigma^{-1}$ and $\tau^{-1} \leq \tau^{-1}\sigma^{-1} \iff \delta^{-1} \leq \delta^{-1}\tau^{-1}$ and $\delta^{-1}\tau^{-1} \leq \delta^{-1}\tau^{-1}\sigma^{-1}$. Thus, it is sufficient to show that $\delta^{-1} \leq \delta^{-1}\tau^{-1}\sigma^{-1}$ and $\tau^{-1} \leq \tau^{-1}\sigma^{-1} \iff \delta^{-1} \leq \delta^{-1}\tau^{-1} \leq \delta^{-1}\tau^{-1}\sigma^{-1}$:

($\Longleftarrow$) Clearly $\delta^{-1} \leq \delta^{-1}\tau^{-1} \leq \delta^{-1}\tau^{-1}\sigma^{-1}$ implies that $\delta^{-1} \leq \delta^{-1}\tau^{-1}\sigma^{-1}$ and multiplying from the left by δ and using lower subtractivity give $\tau^{-1} \leq \tau^{-1}\sigma^{-1}$.

($\Longrightarrow$) We point out that the relation $\sigma \leq \tau$ is equivalent -in algebra level- to the existence of an element $x_g \in \{x_\sigma : \sigma \in G\}$ such that $x_\sigma x_g = x_{\sigma\tau} (\neq 0)$, where $g = \tau$ in this case. We translate the given relations: $\tau^{-1} \leq \tau^{-1}\sigma^{-1} \Longrightarrow x_{\tau^{-1}}x_{\sigma^{-1}} = x_{\tau^{-1}\sigma^{-1}} (\neq 0)$ and $\delta^{-1} \leq \delta^{-1}\tau^{-1}\sigma^{-1} \Longrightarrow x_{\delta^{-1}}x_{\tau^{-1}\sigma^{-1}} = x_{\delta^{-1}\tau^{-1}\sigma^{-1}}$. From these two equations we get $x_{\delta^{-1}}x_{\tau^{-1}}\,x_{\sigma^{-1}} = x_{\delta^{-1}\tau^{-1}\sigma^{-1}} (\neq 0)$. Associativity of the algebra A_e implies that $x_{\delta^{-1}}x_{\tau^{-1}} = x_{\delta^{-1}\tau^{-1}} \neq 0$ i.e., $\delta^{-1} \leq \delta^{-1}\tau^{-1}$ and $x_{\delta^{-1}\tau^{-1}}x_{\sigma^{-1}} \neq 0$ which means that $\delta^{-1}\tau^{-1} \leq \delta^{-1}\tau^{-1}\sigma^{-1}$. To complete the proof, we see that for $\sigma,\ \tau \in G : e_\gamma(\sigma,1) = e(\gamma\sigma,1) = 1$ and $e_\gamma(1,\tau) = e(\gamma\tau,\tau^{-1}) = 1$ because it is always true that $\gamma\tau \leq \gamma$. Finally, for $\sigma \neq 1, e_\gamma(\sigma,\sigma^{-1}) = e(\gamma,\sigma) = 0$, since γ never be less than any other element. $\square$

We make use of this Lemma and show a strong relation between the algebras associated to an SFG and its inverse.

Lemma 2.4. *If G is abelian, then $\sigma \leq \sigma\tau$ if and only if $\tau \leq \sigma\tau$.*

Proof. Observe that both statements mean that $l(\sigma\tau) = l(\tau\sigma) = l(\sigma) + l(\tau)$. $\square$

Theorem 2.5. *Let $A_e,\ A_{e_\gamma}$ be the algebras associated to an SFG $\Gamma(G,S)$ and its inverse graph $\gamma^{-1}\Gamma(G,S)$, and suppose that G is abelian. Then $A_e \cong A_{e_\gamma}$ as F-algebras.*

Proof. On the basis $\{x_\sigma : \sigma \in G\}$, define the map $\psi : A_e \to A_{e_\gamma}$ by $\psi(x_\sigma) = x_{\sigma^{-1}}, \psi(a) = a$ for $a \in F$. We show that ψ is homomorphism. Notice that $\psi(x_\sigma x_\tau) = \psi(e(\sigma,\tau)x_{\sigma\tau}) = e^\psi(\sigma,\tau)x_{\tau^{-1}\sigma^{-1}}$. On the other hand $\psi(x_\sigma)\psi(x_\tau) = x_{\sigma^{-1}}x_{\tau^{-1}} = e_\gamma(\sigma^{-1},\tau^{-1})x_{\sigma^{-1}\tau^{-1}} = e(\gamma\sigma^{-1}\tau^{-1},\tau)x_{\tau^{-1}\sigma^{-1}}$ (by Lemma 2.3 and commutativity of G). Now, we must complete the proof by demonstrating that the definition $e^\psi(\sigma,\tau) = e(\gamma\sigma^{-1}\tau^{-1},\tau)$ is consistent, i.e., taking 0 to 0 and 1 to 1. Suppose $e(\gamma\sigma^{-1}\tau^{-1},\tau) = 1$. So, $\gamma\sigma^{-1}\tau^{-1} \leq \gamma\sigma^{-1}$ or by maximality of $\gamma, \gamma\sigma^{-1}\tau^{-1} \leq \gamma\sigma^{-1} \leq \gamma$. Lower subtractivity gives $\tau \leq \tau\sigma = \sigma\tau$. But this happens if and only if $\sigma \leq \sigma\tau$ as Lemma 2.4 asserts. The other direction can be followed backward. $\square$

Remark 2.6. The words "Standard" and "Frobenius" are necessary in the previous theorem. Take the following examples to figure out this necessity: On Z_5, let $\Gamma_1 = \Gamma(Z_5,\{1,2\})$ and Γ_2 be defined by $0 \leq 1 \leq 2 \leq 3$, $0 \leq 4 \leq 3$. Both are lower subtractive, but none of them gives the above isomorphism.

3. Flexible graphs

Definition 3.1. Let $\Gamma(G,S)$ be an SFG with unique maximal element γ. Then Γ is called *flexible* if for all $a,\ b \in G : a \leq b \Longleftrightarrow a \prec b$.

Proposition 3.2.

(i) *For any FG $\Gamma(G,S)$ with unique maximal element γ, if $T =$ the set of cogenerators $= \{g \in G : g$ lies right below $\gamma\}$. Then $|T| = |S|$.*

(ii) *The unique maximal element in a flexible graph Γ must be of order 2. That is, $\gamma^{-1} = \gamma$.*

Proof. (i) For each $s \in S$, we have $\gamma s^{-1} \in T$ and for each $g \in T$, $l(g) = l(\gamma) - 1$, so there exists $s_0 \in S$ such that $gs_0 = \gamma$ or $g = \gamma s_0^{-1}$. This shows that $T = \{\gamma s^{-1} : s \in S\}$. But the mapping $s \mapsto \gamma s^{-1}$ is bijective.

(ii) By the construction of the inverse graph, the unique maximal element in $\gamma^{-1}\Gamma$ is γ^{-1}. At the same time, since $a \leq \gamma$ in Γ for all $a \in G$, Definition 3.1. implies that $a \prec \gamma$ for all $a \in G$. Therefore γ is a unique maximal element in $\gamma^{-1}\Gamma$ as well. Hence $\gamma^{-1} = \gamma$ or $\gamma^2 = 1$. $\square$

The following statement characterizes flexibility.

Theorem 3.3. *Let $\Gamma(G,S)$ be an SFG with unique maximal element γ. The following statements are equivalent*:

(i) *Γ is flexible.*

(ii) *$g \leq gs \Longleftrightarrow g \prec gs$ for all $s \in S, g \in G$.*

(iii) *$S = S^{-1}$.*

(iv) *$l(g) = l(g^{-1})$ for all $g \in G$.*

Proof. (i) $\Longrightarrow$ (ii): Clear from the definition of flexibility.

(ii) $\Longrightarrow$ (iii): Assume $s_i^{-1} \notin S$ for some $s_i \in S \Longrightarrow l(s_i^{-1}) > 1$. Now, $\gamma s_i \leq \gamma s_i s_i^{-1} = \gamma$. Thus, $l(\gamma s_i)+l(s_i^{-1}) = l(\gamma)$, but since $l(s_i^{-1}) > 1$ we get $l(\gamma s_i) < l(\gamma)-1$ or there exists $s \in S$ such that $\gamma s_i \leq \gamma s_i s \Longrightarrow \gamma^{-1}\gamma s_i s \prec \gamma^{-1}\gamma s_i$ or $s_i s \prec s_i$. Using (ii) yields $s_i s \leq s_i$. Therefore, $s_i^{-1} = s \in S$ (contradiction). A similar method is applicable if we assume $s \notin S^{-1}$.

(iii) $\Longrightarrow$ (iv): If $S = S^{-1}$ and $l(g) = m \Longrightarrow g = s_1 s_2 \cdots s_m$ (reduced expression) $\Longrightarrow g^{-1} = s_m{-}1 s_2^{-1} \cdots s_1^{-1}$, and since $s_i^{-1} \in S, l(g^{-1}) \leq m$. If $l(g^{-1}) < m$ this would imply that $l(g) < m$ so $l(g^{-1}) = m$.

(vi) $\Longrightarrow$ (iii): For each $s \in S$, we have $l(s) = l(s^{-1}) = 1 \Longrightarrow S^{-1} \subseteq S$. But $|S^{-1}| = |S|$. So $S^{-1} = S$.

(iii) $\Longrightarrow$ (ii): Since $S^{-1} = S$, the generators of $\gamma^{-1}\Gamma(G,S)$ is S. Thus $\gamma^{-1}\Gamma(G,S) = \Gamma(G,S)$. In particular $g \leq gs \Longleftrightarrow g \prec gs$ for all $s \in S,\ \ g \in G$.

(ii) $\Longrightarrow$ (i): Let $a \leq b$ and $l(b)-l(a) = n$. So there exist generators $s_1, s_2, \cdots, s_n$ with $a \leq as_1 \leq as_1s_2 \leq \cdots \leq as_1s_2\cdots s_{n-1} \leq b$. Apply the hypothesis to $a \leq as_1$ to get $a \prec as_1$, apply it again to $as_1 \leq as_1s_2$ in order to have $as_1 \prec as_1s_2$. Continue to obtain $a \prec as_1 \prec as_1s_2 \prec \cdots \prec as_1s_2\cdots s_{n-1} \prec b \Longrightarrow a \prec b$. The converse is similar. $\square$

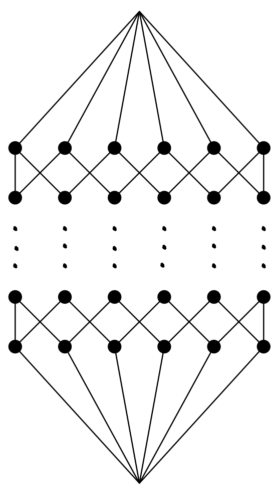

Fig.1.

Corollary 3.4. *If G is a Coxeter group with generating set S, then the graph of the weak Bruhat ordering built on the elements of G is flexible.*

Proof. In any Coxeter group we have $s = s^{-1}$ for every generator $s \in S$. Hence $S = S^{-1}$. $\square$

Flexibility can be fully understood from part (iii) in Theorem 3.3, and this shows that if Γ is flexible then Γ and $\gamma^{-1}\Gamma$ are not only isomorphic but they are exactly the same if we flip the graph $\gamma^{-1}\Gamma$ up side down. As a result their algebras are isomorphic (in fact, the same) by the identity map.

Example 3.4. Consider the dihedral group $D_4 = \{s, r : s^2 = r^4 = 1, rs = sr^3\}$ and construct $\Gamma(D_4, \{s, r, r^3\})$. The graph you will get is an SFG with unique maximal element sr^2, and since $\{s, r, r^3\}^{-1} = \{s, r, r^3\}$ the graph $\Gamma(D_4, \{s, r, r^3\})$ is flexible, so it is the same as $(sr^2)^{-1}\ \Gamma(D_4, \{s, r, r^3\})$. Furthermore, their algebras are the same although G is non-abelian.

Example 3.5. In $Z_8 = \langle r \rangle$, let $S = \{r^2,\ r^3,\ r^5\}$. Notice that $\Gamma(Z_8,\ S)$ has a unique maximal element which is r. the inverse graph $r^{-1}\Gamma(Z_8, S)$ has also a unique maximal element $r^7 = r^{-1}$. Their algebras are isomorphic since Z_8 is abelian.

4. Graphs on cyclic groups

One may ask when standard construction gives a unique maximal vertex? We introduce this section which deals with such a question by some definitions.

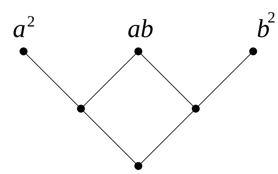

Fig.2.

Definition 4.1. A rectangular shape graph is called a *lattice*, if the rectangle is incomplete, we call it a *broken lattice*. In both cases, we call the graph a *subnet*. A graph taking shape as in the Figure 1 is called a *tower* of *thickness* d and denoted by T_d. And, if the graph is merely a cycle then we call it a *necklace*.

It should be pointed out that if the graph has one generator, it must be a chain and the group must be cyclic. This case is trivial, so we consider the case when the generating set contains more than one element. Also, occasionally we may write $\Gamma(G, \{s_1, s_2, \cdots, s_k\}) \to \gamma_1, \gamma_2, \cdots, \gamma_m$ to indicate that the standard graph Γ which is generated by $\{s_1, s_2, \cdots, s_k\}$ has maximal elements $\gamma_1, \gamma_2, \cdots, \gamma_m$. We write the elements of cyclic groups Z_n multiplicatively and denote r^a by a, r^{2a} by a^2, $r^a r^b$ by ab and so on.

Lemma 4.2. *Let* $(Z_n,\ S)$ *be a cyclic group of order* $n \geq 5$ *with generating set* $S = \{a, b\}$. *Then,*

(i) *If* $a \neq b^{-1}$ *and* $a^k \neq b^k$ *for all* $k \in N$, *then* $\Gamma(Z_n, S)$ *is a subnet.*

(ii) *If* $a \neq b^{-1}$ *and* $a^2 = b^2$, *then* $\Gamma(Z_n, S)$ *is* T_2.

(iii) $a = b^{-1}$, *then* $\Gamma(Z_n, S)$ *is a necklace.*

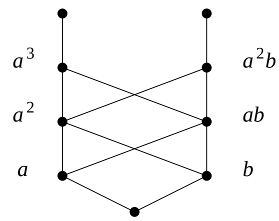

Fig.3.

Proof. (i) We construct the graph and get the part in Figure 2 at the first step. According to the hypotheses, no two of ab, a^2, b^2 are equal. To construct the next level, we get a^2b right above a^2 and ab. Similarly, ab^2 is above ab and b^2. We claim that a^2b and $ab^2$are not equal and do not equal any element in a lower level. If so then either $a = b, a^2 = b$ or $a = b^2$. The first equality contradicts the choice of the generators. If the other two equalities hold simultaneously then $a^2b^2 = ab \Longrightarrow (ab)^2 = ab \Longrightarrow ab = 1 \Longrightarrow a = b^{-1}$ contradicting the assumption. If only one of the last two equalities holds, then we still get a subnet of width 2 unless $a^3 = b^3$ but this can not happen by the hypothesis.

(ii) We first notice that if $a^2 = b^2$ then n must be even because otherwise we would have $2a \equiv 2b \bmod n$ or $\frac{n+1}{2}(2a) \equiv \frac{n+1}{2}(2b) \bmod n$ which implies that $a \equiv b \bmod n$ (contradiction). So, we construct the graph based on the given equality $a^2 = b^2$ as in Figure 3.

In the next step, we get a^3 above both a^2 and ab since if $a^3 = a^2$ then $a = 1$, and $a^3 = ab \Longrightarrow a^2 = b$ or $b^2 = b \Longrightarrow b = 1$. Also, $a^3 \neq a$ because this would imply $a^2 = b^2 = 1$ and then we only have four elements in this group. A similar argument shows that a^2b lies above a^2 and ab. Moreover, $a^3 \neq a^2b$ clearly. We continue in this process to get finally a tower of thickness 2 with unique maximal element, namely $a^{\frac{n}{2}}$.

(iii) If $a = b^{-1}$, then $Z_n = \langle a \rangle$, and we do not have a net since we only have two paths $1, a, a^2, a^3, \cdots$ and $1, a^{-1}, a^{-2}, a^{-3}, \cdots$. Catenary property forces the top of the graph to be unique which is $a^k = a^{-k}$ where $k = \frac{n}{2}$. So, the order must be even in this case, too. $\square$

Corollary 4.3. *If $a \neq b^{-1}$ and $a^2 = b^2$ or $a = b^{-1}$, then $\Gamma(Z_{2n+1}, \{a, b\})$ has more than one maximal element.*

Proceeding the same way as in the proof of Lemma 4.2, one can generalize the statement to the following

Lemma 4.4. *Let (Z_n, S) be a cyclic group of order $n \geq 5$ with generating set $S = \{a, b\}$. Define the width ϵ and the thickness δ of the graph respectively by $\epsilon = \min\{k \in N : a^k = b$ or $b^k = a\}$ and $\delta = \min\{k \in N : a^k = b^k\}$. Then,*

(i) *If $a \neq b^{-1}$ and $\epsilon < \delta$, then $\Gamma(Z_n, S)$ is a subgraph of a net (subnet).*

(ii) *If* $a \neq b^{-1}$ *and* $\delta < \epsilon$, *then* $\Gamma(Z_n, S)$ *is* T_δ.

(iii) *If* $a = b^{-1}$, *then* $\Gamma(Z_n, S)$ *is a necklace.*

Theorem 4.5. *Let* $(Z_n,\ S)$ *be a cyclic group of order* $n \geq 5$ *with generating set* $S = \{a, b\}$. *Suppose that* $\epsilon < \delta$. *Then,* $\Gamma(Z_n, S)$ *has a unique maximal element* γ *if and only if* $\epsilon | n$. *Moreover, in this situation,* $\Gamma(Z_n, S)$ *is a lattice and* $\gamma = a^{-1}$.

Proof. With Lemma 4.4 in hand, it is sufficient to show that the mentioned subnet is in fact complete (rectangle). If $\epsilon | n$ then $a^\epsilon = b$ and a^ϵ does not appear in the graph while we see b instead. each element $a^i b^j$, $i < \epsilon$, $j < \delta$ appears exactly once in a standard lexicographical order because otherwise we would have $a^\lambda = b$ with $\lambda < \epsilon$. Each of such elements above the generators and below the highest two levels has exactly two edges coming in and two edges going out. Also we have $b^{\frac{n}{\epsilon}} = (a^\epsilon)^{\frac{n}{\epsilon}} = a^n = 1$, so we get a chain of powers of b, namely $1, b, b^2, \cdots, b^{\frac{n}{\epsilon}-1}$ which is of length $\frac{n}{\epsilon}$. Thus we get a planer graph taking a rectangular shape with dimensions ϵ and $\frac{n}{\epsilon}$. Since the elements of Z_n are exhausted by this graph, we obtain a unique maximal element lying on the corner of the rectangle and can be computed by multiplying $b^{\frac{n}{\epsilon}-1}$ by $a^{\epsilon-1}$. This gives $\gamma = a^{\epsilon-1} b^{\frac{n}{\epsilon}-1} = a^{-1} b^{\frac{n}{\epsilon}} = a^{-1}$.

Conversely, the hypotheses of the theorem insures that no cancellation in constructing the graph except for the cases $a^\epsilon = b, 1 = b^{\frac{n}{\epsilon}}$. Thus we obtain two paths $1, a, a^2, \cdots, a^{\epsilon-1}$ and $1, b, b^2, \cdots, b^{\frac{n}{\epsilon}-1}$ which are of length ϵ and $\frac{n}{\epsilon}$ respectively. Hence $\epsilon | n$. $\square$

Corollary 4.6. *If* $p > 4$ *is prime, then* $\Gamma(Z_p, \{a, b\})$ *has at least two maximal vertices.*

Proof. If $a^k = b^k$ with $1 < k < p$, then $p | k(a - b)$ which implies that $p | k$ or $p | a - b$. But both are impossible. Likewise, if $a^k = b^{-k}$, then $a^{2k} = 1 \Longrightarrow p | 2ka$ which is also impossible since $2,\ k,\ a < p$. Therefore, the condition $\epsilon < \delta$ is automatically satisfied and the statement of Theorem 4.5 completes the proof. $\square$

Examples 4.7. (i) $\Gamma(Z_{24}, \{1, 7\}) \to 20$, and this graph is T_4. Notice that $1^4 = 7^4$ and $7^7 = 1, 1^7 = 7 \Longrightarrow 4 = \delta < \epsilon = 7$. Furthermore, $4 | 24$.

(ii) $\Gamma(Z_{27}, \{1, 7\}) \to 20, 26$, and this graph has two maximal vertices because $1^7 = 7$ and $1^9 = 7^9 \Longrightarrow 7 = \epsilon < \delta = 9$. But $7 \nmid 27$ although $9 | 27$.

(iii) In $(Z_{21}, \{1, 3\})$, we have $1^3 = 3$ and $3^7 = 1$, so $\epsilon = 3 | 21$. Note that δ does not exist. Hence, $\Gamma(Z_{21}, \{1, 3\})$ is Frobenius and $\gamma = 1^{-1} = 20$.

Corollary 4.7. *If the graph* $\Gamma(Z_n, \{a, b\})$ *is flexible then* n *is even.*

Proof. Flexibility of $\Gamma(Z_n, \{a, b\})$ implies that $\{a, b\} = \{a^{-1}, b^{-1}\} \Longrightarrow a = a^{-1}$ and $b = b^{-1}$ or $a = b^{-1}$. The first case implies that $a^2 = b^2 = 1$ and hence $n = 4$ and G is not cyclic. The group in the second case is shown in Lemma 4.2(iii) to have an even order. $\square$

References

[1] A. Aljouiee, *On Weak Crossed Products, Frobenius Algebras and Weak Bruhat Ordering*, J. Algebra, **287(1)**(2005), 88-102.

[2] A. Aljouiee, *On the Brauer Monoid of* $S3$, Lobachevskii Journal of Mathematics, **14**(2004), 3-16.

[3] D. Haile, *On Crossed Product Algebras Arising from Weak Cocycles*, J. Algebra, **l74**(1982), 270-280.

[4] D. Haile, *The Brauer Monoid of a Field*, J. Algebra, **91**(1983), 521-539.

[5] D. Haile, R. Larson and M. Sweedler, *Almost Invertible Cohomology Theory and the Classification of Idempotent Cohomology Classes and Algebras by Partially Ordered Sets with a Galois Group Action*, Amer. J. Math., **105**(1983), 689-814.

[6] R. Stimets, *Weak Galois Cohomology and Group Extension*, Communications in Algebra, **19(1)**(2000), 44-61.