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An-annular Complexes in 3-manifolds

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Abstract. Given a non-Haken, non Seifert fibred manifold we describe an algorithm that takes 2 (not necessarily distinct) Heegaard surfaces and produces a complex with certain useful properties (Properties 5.1). Our main tool is Rubinstein and Scharlemann's Cerf theoretic work ([5]).

1. Introduction

My goal in these notes is to explain a method for constructing an-annular complexes in 3-manifolds that behave somewhat like Heegaard surfaces. In attempt to keep this document short and as accessible as possible, applications of this machinery will not be discussed; for an application see "Invariant Heegaard surfaces for Manifolds with Involutions" (joint with Hyam Rubinstein [4]). This paper describes what properties such complexes must have (Properties 5.1), and proves that they can be obtained in closed non-Haken, non-Seifert fibred manifolds that contain more than one Heegaard surface (Theorem 6.6). We remark that the 2 surfaces need not be distinct, that is we may take 2 copies of the same surface, but distinct surfaces are usually more interesting.

Rubinstein and Scharlemann developed a Cerf-theoretic technique for getting the intersection of any two strongly irreducible Heegaard surfaces (for definition see Section 3) to a particularly nice configuration. After describing the properties of this intersection we show how to use this configuration to create a complex (at first simply the union of the two) and modify it, preserving all the important qualities achieved by Rubinstein and Scharlemann, and adding one more: the pieces from which this complex is built are of negative Euler characteristic. It is easy to explain why this is desirable: it makes Euler characteristic count possible.

2. Handlebodies, disks and spines

We work in the smooth and orientable category. A 3-manifold is a 2nd countable Hausdorff space, locally homeomorphic to $\mathbb{R}^3$ or $\mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3 | z \geq 0\}$ (in which case the points that are sent to $\{z = 0\}$ are the boundary of the manifold). All

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manifolds discussed are assumed to be compact, and if in addition the boundary is empty the manifold is called closed. We assume all manifolds to be smooth (which we may do since every 3-manifold carries a unique smooth structure). When discussing any object embedded in a manifold, it is always considered up to isotopy, which for surfaces must be proper (*i.e.* the boundary of the surface must remain on the boundary of the manifold at all times).

A **handlebody** is a neighborhood of a connected embedded graph in some 3-manifold M. It is an easy exercise to see that the diffeomorphism type of a handlebody is completely determined by the Euler characteristic of the graph, which equals the Euler characteristic of the handlebody and also half that of its boundary. However it is more common to consider the **genus** of a handlebody which is defined to be the genus of the boundary of the handlebody. Note that by definition the boundary of a handlebody is connected. Given a handlebody of genus other than one, an embedded graph without vertices of valence one or two whose neighborhood is the given handlebody is called a **spine.** (For genus zero the spine is a point and for genus one a circle with one vertex of valence two.) It is important to note that, except for genus zero and one handlebodies, the spine of a handlebody is not at all unique (see Figure 1). A choice of spine corresponds to a choice of compressing disks for the handlebody: given a spine, one can canonically pick disks, each corresponding to a point on an edge of the spine; this is demonstrated in Figure 2. We leave the other direction as an exercise: given non-parallel compressing disks for a handlebody that cut the handlebody up into balls, construct a spine for that handlebody.

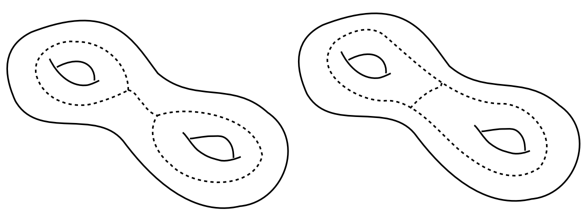

Figure 1: Distinct spines for a genus two handlebody.

A conclusion of the discussion in the previous paragraph is another well known fact: a connected 3-manifold is a handlebody if and only if it has disjointly embedded disks that cut it up into balls. These disks form very useful tools for studying handlebodies. A worthy exercise for which this tool is sufficient is proving that any essential surface (either closed or with boundary; for definition see, for example, [2]) in a handlebody is a disk. In their work Rubinstein and Scharlemann used a more refined tool, **sweepout**. Although we will not describe their proofs we mention what a sweepout is. Since a handlebody is a neighborhood of a graph in an orientable 3-manifold it is in fact the product of the given graph with a disk. The disk is foliated by concentric circles with a single singular leaf in the center

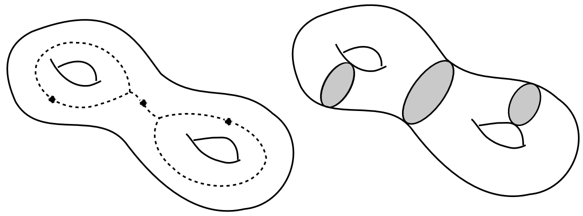

Figure 2: The correspondence between spines and disks in a handlebody.

(that leaf is a point). This foliation induces a foliation of the handlebody by *smooth* surfaces diffeomorphic to the boundary, with one singular leaf corresponding to the center of the disk. This foliation looks like the boundary collapsing onto the spine, and the trace of this collapse is the **sweepout** of the handlebody by surfaces.

3. Heegaard surfaces and strong irreducibility

A **Heegaard surface** for a closed 3-manifold M is a two sided, closed surface embedded in M whose exterior consists of two handlebodies. A **spine** of a 3-manifold is a union of two spines for the handlebodies. Since each handlebody is foliated by surfaces diffeomorphic to the Heegaard surface (and one exceptional leaf) the manifold is too foliated by surfaces diffeomorphic to the Heegaard surface with two exceptional leaves, the spine of the manifold. Formally, a spine is an embedded graph (necessarily disconnected) whose complement is diffeomorphic to the product of a surface with $\mathbb{R}$.

Given a Heegaard surface, one can obtain a Heegaard surface of higher genus by adding a **trivial** handle; a trivial handle addition is replacing a disk by a trivial once punctured torus as in Figure 3. (A once punctured torus in a ball is called trivial if it is unknotted, which can be seen in that figure.) A trivial handle addition is called a **stabilization**. After adding such handle, we see two disks, on opposite sides of the Heegaard surface, intersecting transversely in a single point. Such disks are called a **reducing pair**. By observing that a neighborhood of a reducing pair is a ball, and the Heegaard surface intersects that ball in a trivial once punctured torus, we conclude that any Heegaard surface that has a reducing pair is a stabilization of a lower genus Heegaard surface. The converse of a stabilization, removal of a trivial handle, is called a destabilization. If destabilization exists, the surface is said to be stabilized, else to be non-stabilized. It is not too hard to see that stabilization is unique. Destabilization is not unique, but that is quite hard to show.

A more subtle notion of reduction is due to Casson and Gordon ([1]). In their seminal work Casson and Gordon defined a **weak reduction** to be a pair of disks on opposite sides of a Heegaard surface whose boundaries are essential in the Heegaard surface and disjoint. It is an exercise to show that, except the genus one Heegaard surface in S^3, every other Heegaard surface that destabilizes also weakly

Figure 3: Trivial handle addition and destabilization.

reduces (hint: this only requires Figure 3). A Heegaard surface that supports no weak reductions is called **strongly irreducible**. The following result of Casson and Gordon ([1]) is very useful:

Theorem 3.1 (Casson–Gordon)**.** *Let M be an irreducible non-Haken closed manifold. Then any Heegaard surface for M that does not destabilize is strongly irreducible.*

From this point on, we will need all our Heegaard surfaces to be strongly irreducible.

4. Heegaard complexes

The main outcome of the Cerf theoretic work of [5] and [4] is placing any two strongly irreducible Heegaard surfaces in a very nice configuration which is described below. We first define:

Definition 4.1.

(1) Two surfaces embedded in a 3-manifold M and intersecting transversally are said to intersect **essentially** if every curve of intersection is essential in both surfaces.

(2) A graph embedded in a surface contains a spine of the surface if no piece of the surface cut open along the graph contains a curve that is non-trivial in the surface. (These pieces are not required to be simply connected.)

(3) The intersection of a Heegaard surface Σ with a surface F is called **spinal** if there exists a set of compressing disks Δ for both sides of Σ so that $F \cap (\Sigma \cup \Delta)$ contains a spine of F.

Theorem 4.2 (Rubinstein–Scharlemann, Rieck–Rubinstein)**.** *Let Σ_1 and Σ_2 be strongly irreducible Heegaard surfaces in a manifold M other than S^3. Then Σ_1 and Σ_2 can be isotoped to intersect essentially and spinally.*

Remark. The original result of Rubinstein and Scharlemann allows for at most one curve of $\Sigma_1 \cap \Sigma_2$ to be inessential. Removing this simple closed curve and obtaining essential intersection is the only contribution of [4] to this result.

Let M be a non-Haken, irreducible manifold, and let Σ_1 and Σ_2 be non-stabilized Heegaard surfaces for M. By Casson-Gordon Σ_1 and Σ_2 are strongly irreducible. We may therefore apply Theorem 4.2. We define C to be $\Sigma_1 \cup \Sigma_2$. For the rest of this section we study the properties of this complex, and in the next section we modify it. We start with a theorem that studies the components of M cut open along the $\Sigma_1 \cup \Sigma_2$, for the surfaces given to us by Rubinstein and Scharlemann. This theorem gives a convenient way of using spinal intersection.

Theorem 4.3. *Let Σ_1 and Σ_2 be a Heegaard surface intersecting spinally and essentially. Then the components of M cut open along $\Sigma_1 \cup \Sigma_2$ are handlebodies.*

Proof. Since the intersection is spinal there exists a complete set of compressing disks Δ for one of the surfaces (say Σ_2) so that $\Sigma_1 \cap (\Sigma_2 \cup \Delta)$ contains a spine of Σ_1. This implies that no compressing disk for Σ_1 is disjoint from $\Sigma_2 \cup \Delta$ (although components of Σ_1 cut open along $\Sigma_2 \cup \Delta$ may compress, these compressions are trivial in Σ_1).

We may assume that $\Sigma_1 \cap \Delta$ consists of arcs only: let γ be a simple closed curve in $\Sigma_1 \cap \Delta$. Since the intersection is spinal, γ bounds a disk in Σ_1. Passing to an innermost such, we see a disk whose interior intersects neither Δ nor Σ_2 (by essentiality). We now use this disk to isotope Δ and remove γ from $\Delta \cap \Sigma_1$.

Let B be some component of M cut open along $\Sigma_2 \cup \Delta$, say above Σ_2 (we picture Σ_2 as horizontal), and c some component of $\Sigma_1 \cap B$. We show that c is a disk: suppose c were not a disk. Since the intersection is spinal c is a punctured disk, and each puncture bounds a disk in Σ_1. Let γ be one of the punctures, and $D \subset \Sigma_1$ the disk it bounds. By assumption $\gamma \subset \partial B$, and near its boundary $D \cap B = \gamma$. Since the intersection of Σ_1 and Σ_2 is essential and $\Delta \cap \Sigma_1$ consists of arcs, γ must have parts on Σ_2 and parts on Δ above Σ_2. A non-empty subset of D is below Σ_2. But the boundary of this part of D is a non-empty collection of simple closed curves in $\Sigma_1 \cap \Sigma_2$, all trivial in Σ_1, contradicting essentiality.

Since the pieces of Σ_1 in each ball of M cut open along $\Sigma_2 \cup \Delta$ are disks, they further chop these balls up into balls, so M cup open along $\Sigma_1 \cup \Sigma_2 \cup \Delta$ consists of balls. Now the disks of Δ are chopped up into disks by Σ_1. Attaching the balls described above to each other via these disks we get handlebodies. $\square$

5. Producing an-annular complexes: the set up

For the remainder of this paper we fix a manifold M fulfilling the following assumptions:

(1) M is non-Haken.

(2) M is irreducible.

(3) M is not a Seifert Fibered Space.

(4) M is closed.

Thus in the previous section we saw that these assumptions allow us, using Casson and Gordon, to apply the work of Rubinstein and Scharlemann. We remark that for our work here assumption (1) can be weakened: M is a-toroidal suffices. However, in that case Casson and Gordon cannot be applied.

As remarked in the introduction, occasionally we need to use Euler characteristic. For that we want to bound the number of curves in $\Sigma_1 \cap \Sigma_2$, which at this point is impossible since there may be any number of annuli in $\Sigma_i \setminus (\Sigma_1 \cap \Sigma_2)$, $i = 1$ or 2. We now describe the procedure to remove such annuli. We consider a complex C, at first $\mathsf{C} = \Sigma_1 \cup \Sigma_2$. More precisely C is a finite collection of simple closed curves denoted $\mathrm{sing}(\mathsf{C})$ (the singular curves of C, currently $\Sigma_1 \cap \Sigma_2$) and a finite collection of surfaces with boundary whose boundaries are mapped to $\mathrm{sing}(\mathsf{C})$. Each of these surfaces is embedded and they do not intersect in their interiors. We call the closure of a component of $\mathsf{C} \setminus \mathrm{sing}(\mathsf{C})$ **sheet**.

Let g be the genus of a minimal genus Heegaard surface for M. Our goal is getting a complex C that fulfills the following conditions:

Proposition 5.1.

- **A.** $\chi(\mathsf{C}) \geq 4 - 4g$.
- **B.** *All components of M cut open along C are handlebodies.*
- **C.** *Every curve of $\mathrm{sing}(\mathsf{C})$ is the boundary of three sheets, one of negative Euler characteristic and two annuli, and these annuli close up, together with other such annuli, to form tori bounding solid tori. These solid tori do not intersect C in their interior.*

Note that conditions **A, B** have been achieved by Rubinstein and Scharlemann (recall Theorems 4.2 and 4.3). These conditions are to be our first **invariant**, that is to say they must be preserved as we modify C. We therefore mark them as an invariant:

Invariant 5.2. *Properties* **A, B** *are invariant.*

As we modify the complex there will be more and more invariant properties culminating to property **C**. We now explain this property.

Property **C**, at this point, does not hold. In fact, each curve of $\mathrm{sing}(\mathsf{C})$ bounds four sheets. If we were to replace C by the complex $[\mathsf{C} \setminus N(\mathrm{sing}(\mathsf{C}))] \cup (\partial N(\mathrm{sing}(\mathsf{C})))$ (see Figure 4, crossed with S^1) we would get a picture very similar to that required by property **C**, where annuli (of $\partial N(\mathrm{sing}(\mathsf{C}))$) are connected to each other, forming tori bounding solid tori (the solid tori are $N(\mathrm{sing}(\mathsf{C}))$) and C does not intersect these solid tori. One thing is missing: while sheets connected to such solid tori are of non-positive Euler characteristic by Rubinstein and Scharlemann, they may be annuli. Making sure they are not in the content of the algorithm below.

We remark that the complex C we are after is *not* the complex $[\mathsf{C} \setminus N(\mathrm{sing}(\mathsf{C}))] \cup (\partial N(\mathrm{sing}(\mathsf{C})))$ described above. The process described below is significantly different. It seems illustrative to consider the solid torus shown in Figure 5 (also crossed

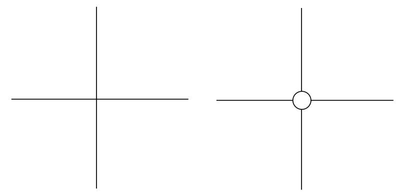

Figure 4: C and $[\mathsf{C} \setminus N(\mathrm{sing}(\mathsf{C}))] \cup (\partial N(\mathrm{sing}(\mathsf{C})))$.

with S^1). In that figure vertical arcs represent annuli from one of the Heegaard surfaces under consideration, and the horizontal from the other. We will refer to this figure again as we modify C.

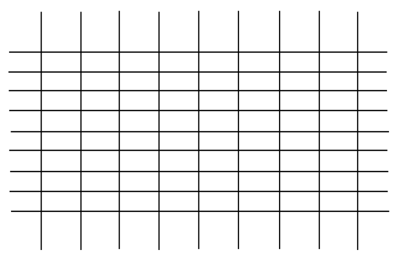

Figure 5: A solid torus with many annuli.

6. Producing an-annular complexes: the process

We finally describe the steps for modifying C. Note that this is an algorithm.

Step One: tori not bounding solid tori. Let $T \subset \mathsf{C}$ be a torus not bounding a solid torus. Then T bounds a piece (denoted X) that is a knot exterior in S^3. To see that, we first note that since M is a-toroidal T compresses to a sphere, and since M is irreducible this sphere bounds a ball. To retrieve T we tube the sphere. If this tubing is done outside the ball the result is a torus that bounds a solid torus, else the result is a ball with a tube drilled out, hence a knot exterior in S^3, which we denote X. Our assumption that T does not bound a solid torus implies that this knot exterior is non-trivial. The compressing disk for T is in $cl(M \setminus X)$, denoted by D_X. We get rid of such tori in the following way: first, we remove all pieces of C inside of X from C. By property **B** this strictly reduces the number of sheets in C. (Property **B** is temporarily lost.) Next, we modify the complex by cutting it along D_X, *unknotting the tube,* and gluing it back. (See Figure 6 for the cut-and-paste part of this step.) This does not change the homeomorphism type of $M \setminus int(X)$, so we may assume the complex had not been modified there. Property

B had been retrieved (without adding sheets). We continue this process (which obviously terminates) as long as we can. We have obtained a new invariant:

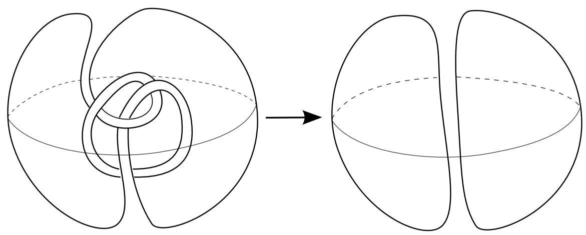

Figure 6: Cutting a knot exterior to produce a solid torus.

Invariant 6.1. *Every torus $T \subset \mathsf{C}$ bounds a solid torus.*

Remark.

(1) After completing this paper, Kobayashi and Rieck showed [3] that X must in fact be a solid torus to begin with, and this unknotting step is not required.

(2) Prior to step one, T gives a decomposition of M as follows: $M = X \cup_T (M \setminus X)$, where $M \setminus X$ is a solid torus connect sum M. The union is taken by attaching the meridian disk of the solid torus (*i.e.* D_X) to X to give the trivial filling of X, resulting in $S^3 \# M$. We can change this decomposition by modifying X, replacing it by any knot exterior we want, but to get property **B** we choose the exterior of the trivial knot.

Step Two: cleaning maximal solid tori. We start by defining:

Definition 6.2. A solid torus $V \subset M$ is called a **maximal solid torus** if $\partial V \subset \mathsf{C}$ and V is maximal with respect to inclusion among all such solid tori.

Denote the set of all maximal solid tori by $\{V_i\}_{i=1}^n$, which is finite since C is. We classify slopes on the boundary of a solid torus as meridional (bounding a disk in the solid torus) longitudinal (intersecting a meridian once) or cabled (all other slopes).

Lemma 6.3. *Any two maximal solid tori are either disjoint or intersect in a single simple closed curve.*

Proof. Given two maximal solid tori—say V_1 and V_2—that intersect in more than a single simple closed curve, either their intersection contains more than one component or that component is not a curve (*e.g.* an annulus), see Figure 7. In that case, the boundary of $W = V_1 \cup V_2$ consists of embedded tori. By Invariant 6.1 each of these tori bounds a solid torus, and by the maximality of V_1 and V_2 this solid cannot contain W. Thus M is the union of W with solid tori. It is now easy to see that either M is a Seifert Fibered Space, or it reduces. $\square$

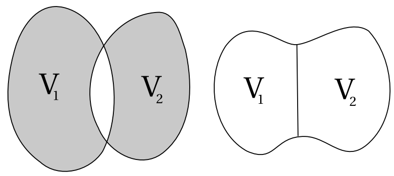

Figure 7: Maximal solid tori intersecting in more than a curve.

Let V_1 and V_2 be maximal solid tori so that $V_1 \cap V_2 \neq \emptyset$. If the slope of the intersection is longitudinal in one of the two, we amalgamate the solid tori into a single torus as in Figure 8. If, on both maximal solid tori, the slope is not

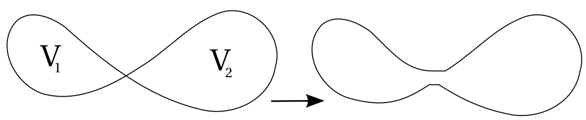

Figure 8: Amalgamating maximal solid tori.

longitudinal, it cannot be meridional on either solid torus: both slopes meridional implies M is reducible, one meridional and the other cabled implies that M contains a punctured lens space. We amalgamate the solid tori anyway, thus creating (in violation of property **B**) a complementary piece that is a Seifert Fibered Space over D^2 with two exceptional fibers. As we saw in the beginning of step one, either its boundary bounds a solid torus (impossible: M itself would be a Seifert Fibered Space, or reducible) or the new piece is compressible to the outside and contained in a ball, in which case we unknot it the way we did in step one.

We repeat this process (which reduces the number of singular curves and hence terminates) until we arrive at a complex where every two maximal solid tori are disjoint. Next we remove from C every sheet in the interior of maximal solid torus, and perturb C near the boundary of such solid torus so that each curve of $\mathrm{sing}(\mathsf{C})$ has valence three. We get the following invariant, which is stronger than Invariant 6.1 and so replaces it:

Invariant 6.4. *Any torus embedded in C bounds a solid torus, and any two tori embedded in C are disjoint and bound disjoint solid tori. The valence of singular curves on the boundary of a solid torus is three. The intersection of C with the interior of a maximal solid torus is empty.*

Remark. If we consider Figure 5, we see that at this point all the annuli in the interior of that solid torus have been removed. This illustrates the effectiveness of maximal solid tori and shows that after cleaning them (step two) the complex we are left with is quite substantially different than that we started with, and cannot be thought of as the union of two surfaces any longer.

Step Three: other singular curves. Any curve of sing(C) not on the boundary of a maximal solid torus is drilled out and the complex is modified in the following way: for any such curve $\gamma \in \mathrm{sing}(\mathsf{C})$ we replace C by $[\mathsf{C} \setminus N(\gamma)] \cup \partial(N(\gamma))$. After this all curves of sing(C) have valence three. Invariant 5.2 is easily seen to hold after this step, and we now verify Invariant 6.4.

Let $T \subset \mathsf{C}$ be any torus after step three. Consider $\hat{T}$ prior to step three. If $\hat{T}$ is the boundary of a maximal solid torus it was not modified in step three. So we may assume that $\hat{T}$ is not the boundary of a maximal solid torus. Thus some curve of sing(C) not on the boundary of any maximal solid torus, say γ, is on $\hat{T}$. Collapsing the solid tori we created in step three, we see that (unless T collapses to a curve) prior to that step $\hat{T}$ is almost embedded: it may have double curves on some curve of sing(C). But now an easy cut-and-paste argument gives us a torus embedded in C (still prior to step three) that is adjacent to γ. The ending is easy: this torus bounds a solid torus, which is included in a maximal solid torus, and so γ is on the boundary of a maximal solid torus, contradiction. To emphasize, we obtained that any torus in C (after step three) is either a torus bounding a maximal solid torus that existed prior to step three, or of the form $N(\gamma)$ from some $\gamma \in \mathsf{C}$ that was not on the boundary of a solid torus before step three. After renaming the collection of maximal solid tori we get:

Invariant 6.5. *Any curve of* sing*(C) is the boundary of exactly three sheets, two annuli on the boundary of maximal solid torus, and one other.*

Note that since any curve of sing(C) is on the boundary of a maximal solid torus, any annular sheet is either an annulus on the boundary of a maximal solid torus or connects two maximal solid tori, which (by Invariant 6.4) are distinct. Finally, we need to get sheets connecting maximal solid tori to have negative Euler characteristic, which is now easy:

Step Four: Annuli. If a sheet connecting maximal solid tori is an annulus, amalgamate the maximal solid tori together, as shown in Figure 9. If the resulting piece is not a solid torus, it is a Seifert Fibered Space over D^2 with two exceptional fibers and we unknot it the way we did at the end of steps one and two.

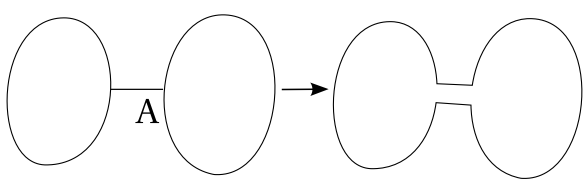

Figure 9: Amalgamating maximal solid tori along annuli.

Checking that all invariant are preserved is similar to the final paragraph of step three and will be omitted. The complex C now fulfills all the required properties.

In conclusion:

Theorem 6.6. *If M is a closed orientable non-Haken and not Seifert fibred manifold, and M contains 2 (not necessarily distinct) non-stabilized Heegaard splittings, then the algorithm described above produces an an-annular complex that fulfills Properties 5.1.*

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