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OPERATIONS OF INTUITIONISTIC FUZZY IDEALS/FILTERS IN LATTICES

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Abstract. The notion of intuitionistic fuzzy convex sublattices is introduced, and its characterization is given. Natural equivalence relations on the set of all intuitionistic fuzzy ideals/filters of a lattice are investigated. Operations on intuitionistic fuzzy sets of a lattice is introduced. Some results of intuitionistic fuzzy ideals/filters under these operations are provided. Using these operations, characterizations of intuitionistic fuzzy ideals/filters are given.

1. Introduction

Yon and Kim [6] introduced the notion of intuitionistic fuzzy sublattices and intuitionistic fuzzy ideals/filters in a lattice, and then investigated their properties. Hur et al. [5] discussed the relationship between intuitionistic fuzzy ideals and intuitionistic fuzzy congruences on a distributive lattice, and they proved that the lattice of intuitionistic fuzzy ideals is isomorphic to the lattice of intuitionistic fuzzy congruences on a generalized Boolean algebra. They also obtained a necessary and sufficient condition for an intuitionistic fuzzy ideal on the direct sum of lattices to be representable as a direct sum of intuitionistic fuzzy ideals on each lattice. In this paper, we introduce the notion of intuitionistic

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fuzzy convex sublattices, and give its characterizations. We investigate natural equivalence relations on the set of all intuitionistic fuzzy ideals/filters of a lattice. We also introduce operations on intuitionistic fuzzy sets of a lattice, and we provide some results of intuitionistic fuzzy ideals/filters under these operations. Using these operations, we give characterizations of intuitionistic fuzzy ideals/filters.

2. Preliminaries

A mapping $\mu : L \to [0,1]$, where L is an arbitrary non-empty set, is called a *fuzzy set* in L. The *complement* of μ, denoted by $\bar{\mu}$, is the fuzzy set in L given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in L$. Let $\mathbf{0}$ and $\mathbf{1}$ be fuzzy sets in L defined by $\mathbf{0}(x) = 0$ and $\mathbf{1}(x) = 1$ for all $x \in L$. For any fuzzy set μ in L and any $t \in [0,1]$ we define two sets

$$U(\mu;t) = \{x \in L \mid \mu(x) \geq t\} \quad \text{and} \quad L(\mu;t) = \{x \in L \mid \mu(x) \leq t\},$$

which are called an *upper* and *lower t-level cut* of μ and can be used to the characterization of μ. Let μ_A and γ_A be two functions from L to $[0,1]$ such that

$$(\forall x \in L)\,(0 \leq \mu_A(x) + \gamma_A(x) \leq 1).$$

By the original definition of Atanassov in [4], an *intuitionistic fuzzy set* is an object of the form: $A = \{(x, \mu_A(x), \gamma_A(x) \mid x \in X\}$. We consider it in a form of an ordered triple: $A = \langle L, \mu_A, \gamma_A \rangle$ where L, μ_A and γ_A are as above. Let $\mathbf{0}_\sim = \langle L, \mathbf{0}, \mathbf{1} \rangle$ and $\mathbf{1}_\sim = \langle L, \mathbf{1}, \mathbf{0} \rangle$ be intuitionistic fuzzy sets in L.

Definition 2.1. [5, 6] An IFS $A = \langle L, \mu_A, \gamma_A \rangle$ in a lattice $L = (L, +, \cdot)$ is called an *intuitionistic fuzzy sublattice* of L if it satisfies:

$$(\forall x, y \in L)\,(\mu_A(x+y) \wedge \mu_A(x \cdot y) \geq \mu_A(x) \wedge \mu_A(y)),$$

$$(\forall x, y \in L)\,(\gamma_A(x+y) \vee \gamma_A(x \cdot y) \leq \gamma_A(x) \vee \gamma_A(y)).$$

Definition 2.2. [5, 6] An IFS $A = \langle L, \mu_A, \gamma_A \rangle$ in a lattice $L = (L, +, \cdot)$ is called an *intuitionistic fuzzy filter* of L if it satisfies:

- $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy sublattice of L,
- $A = \langle L, \mu_A, \gamma_A \rangle$ is intuitionistic monotonic, i.e., $\mu_A(x) \leq \mu_A(y)$ and $\gamma_A(x) \geq \gamma_A(y)$ whenever $x \leq y$.

Definition 2.3. [5, 6] An IFS $A = \langle L, \mu_A, \gamma_A \rangle$ in a lattice $L = (L, +, \cdot)$ is called an *intuitionistic fuzzy ideal* of L if it satisfies:

- $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy sublattice of L,
- $A = \langle L, \mu_A, \gamma_A \rangle$ is intuitionistic antimonotonic, i.e., $\mu_A(x) \geq \mu_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$ whenever $x \leq y$.

3. Intuitionistic fuzzy sublattices/ideals/filters

In what follows, let L denote a lattice unless otherwise specified. We first give an example of intuitionistic fuzzy sublattice.

Example 3.1. Let $\mathbb{N}$ be the set of natural numbers and let L be the set consisting of the empty set $\emptyset$, $\mathbb{N}$ and the set of all the singletons of $\mathbb{N}$, that is,

$$L = \{\emptyset, \mathbb{N}\} \cup \{\{n\} \mid n \in \mathbb{N}\}.$$

Then L is a lattice under the ordering of set inclusion with $\emptyset$ as its least element and $\mathbb{N}$ the greatest element (see [1]). Consider all the finite sublattices of L of the form

- $L_1 := \{\emptyset, \mathbb{N}\}$,
- $L_n := \{\emptyset, \mathbb{N}\} \cup \{\{i\} \mid i \leq n-1\}$, for each $n \in \mathbb{N}$ and $n \geq 2$.

Define an IFS $A = \langle L, \mu_A, \gamma_A \rangle$ in L as follows:

$$\mu_A(x) := \begin{cases} 1 & \text{if } x \in L_1, \\ \frac{1}{n} & \text{if } x \in L_n \sim L_{n-1}, \text{ for } n \geq 2 \end{cases}$$

$$\gamma_A(x) := \begin{cases} 0 & \text{if } x \in L_1, \\ \alpha_n & \text{if } x \in L_n \sim L_{n-1}, \text{ for } n \geq 2 \end{cases}$$

where $\alpha_n \in [0,1]$ with $\alpha_n + \frac{1}{n} \leq 1$. Then $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy sublattice of L.

Proposition 3.2. *Let $A = \langle L, \mu_A, \gamma_A \rangle$ be an IFS in L. Then the following are equivalent:*

(i) $(\forall x, y \in L)$ $(x \leq y \Rightarrow \mu_A(x) \geq \mu_A(y),\ \gamma_A(x) \leq \gamma_A(y))$.

(ii) $(\forall x, y \in L)$ $(\mu_A(x \cdot y) \geq \mu_A(x) \vee \mu_A(y),\ \gamma_A(x \cdot y) \leq \gamma_A(x) \wedge \gamma_A(y))$.

(iii) $(\forall x, y \in L)$ $(\mu_A(x + y) \leq \mu_A(x) \wedge \mu_A(y),\ \gamma_A(x + y) \geq \gamma_A(x) \vee \gamma_A(y))$.

PROOF. Assume that (i) is valid. For any $x, y \in L$, we have $x \cdot y \leq x$ and $x \cdot y \leq y$. It follows from (i) that $\mu_A(x \cdot y) \geq \mu_A(x)$, $\gamma_A(x \cdot y) \leq \gamma_A(x)$, $\mu_A(x \cdot y) \geq \mu_A(y)$, $\gamma_A(x \cdot y) \leq \gamma_A(y)$ so that $\mu_A(x \cdot y) \geq \mu_A(x) \vee \mu_A(y)$ and $\gamma_A(x \cdot y) \leq \gamma_A(x) \wedge \gamma_A(y)$. Now for any $x, y \in L$, we get $x \leq x + y$ and $y \leq x + y$. Using (i), we have $\mu_A(x) \geq \mu_A(x+y)$, $\gamma_A(x) \leq \gamma_A(x+y)$, $\mu_A(y) \geq \mu_A(x+y)$, $\gamma_A(y) \leq \gamma_A(x+y)$. Hence $\mu_A(x+y) \leq \mu_A(x) \wedge \mu_A(y)$ and $\gamma_A(x+y) \geq \gamma_A(x) \vee \gamma_A(y)$. Therefore (ii) and (iii) are valid. Suppose that (ii) is true and let $x, y \in L$ be such that $x \leq y$. Then $x \cdot y = x$, and so $\mu_A(x) = \mu_A(x \cdot y) \geq \mu_A(x) \vee \mu_A(y)$, $\gamma_A(x) = \gamma_A(x \cdot y) \leq \gamma_A(x) \wedge \gamma_A(y)$. Therefore $\mu_A(x) \geq \mu_A(y)$ and $\gamma_A(x) \leq \gamma_A(y)$, and thus (i) is true. Finally assume that (iii) holds and let $x, y \in L$ be such that $x \leq y$. Then $x + y = y$, and so $\mu_A(y) = \mu_A(x + y) \leq \mu_A(x) \wedge \mu_A(y)$, $\gamma_A(y) = \gamma_A(x + y) \geq \gamma_A(x) \vee \gamma_A(y)$. It follows that $\mu_A(y) \leq \mu_A(x)$ and $\gamma_A(y) \geq \gamma_A(x)$. This completes the proof. □

Dually, we have

Proposition 3.3. *Let $A = \langle L, \mu_A, \gamma_A \rangle$ be an IFS in L. Then the following assertions are equivalent:*

(i) $(\forall x, y \in L)$ $(x \leq y \Rightarrow \mu_A(x) \leq \mu_A(y),\ \gamma_A(x) \geq \gamma_A(y))$.

(ii) $(\forall x, y \in L)$ $(\mu_A(x \cdot y) \leq \mu_A(x) \wedge \mu_A(y),\ \gamma_A(x \cdot y) \geq \gamma_A(x) \vee \gamma_A(y))$.

(iii) $(\forall x, y \in L)$ $(\mu_A(x + y) \geq \mu_A(x) \vee \mu_A(y),\ \gamma_A(x + y) \leq \gamma_A(x) \wedge \gamma_A(y))$.

Theorem 3.4. *An intuitionistic fuzzy sublattice $A = \langle L, \mu_A, \gamma_A \rangle$ of L is an intuitionistic fuzzy ideal* (resp. *filter*) *of L if and only if $A = \langle L, \mu_A, \gamma_A \rangle$ satisfies any one of the conditions in Proposition 3.2* (resp. *Proposition 3.3*).

PROOF. Straightforward. □

Let $A = \langle L, \mu_A, \gamma_A \rangle$ be an IFS in L and let $\alpha, \beta \in [0,1]$ with $\alpha+\beta \leq 1$. Then the set

$$L_A^{(\alpha,\beta)} := \{x \in L \mid \mu_A(x) \geq \alpha,\ \gamma_A(x) \leq \beta\}$$

is called an (α, β)*-level subset* of $A = \langle L, \mu_A, \gamma_A \rangle$.

Theorem 3.5. *Let $A = \langle L, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy ideal* (resp. *filter*) *of L. Then $L_A^{(\alpha,\beta)}$ is an ideal* (resp. *filter*) *of L for every $(\alpha, \beta) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\gamma_A)$ with $\alpha + \beta \leq 1$.*

PROOF. Let $x, y \in L_A^{(\alpha,\beta)}$. Then $\mu_A(x) \geq \alpha$, $\gamma_A(x) \leq \beta$, $\mu_A(y) \geq \alpha$, $\gamma_A(y) \leq \beta$ which imply that

$$\mu_A(x + y) \wedge \mu_A(x \cdot y) \geq \mu_A(x) \wedge \mu_A(y) \geq \alpha,$$

$$\gamma_A(x + y) \vee \gamma_A(x \cdot y) \leq \gamma_A(x) \vee \gamma_A(y) \leq \beta.$$

Thus $x+y, x \cdot y \in L_A^{(\alpha,\beta)}$, that is, $L_A^{(\alpha,\beta)}$ is a sublattice of L. Let $x \in L$ and $y \in L_A^{(\alpha,\beta)}$ be such that $x \leq y$ (resp. $y \leq x$). Then $\mu_A(x) \geq \mu_A(y) \geq \alpha$ and $\gamma_A(x) \leq \gamma_A(y) \leq \beta$. It follows that $x \in L_A^{(\alpha,\beta)}$ so that $L_A^{(\alpha,\beta)}$ is an ideal (resp. filter) of L. □

Theorem 3.6. *Let $A = \langle L, \mu_A, \gamma_A \rangle$ be an IFS in L such that $L_A^{(\alpha,\beta)}$ is an ideal* (resp. *filter*) *of L for every $(\alpha, \beta) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\gamma_A)$ with $\alpha + \beta \leq 1$. Then $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy ideal* (resp. *filter*) *of L.*

PROOF. Let $x, y \in L$ and let $A(x) = (\alpha_1, \beta_1)$ and $A(y) = (\alpha_2, \beta_2)$, i.e., $\mu_A(x) = \alpha_1$, $\gamma_A(x) = \beta_1$, $\mu_A(y) = \alpha_2$, $\gamma_A(y) = \beta_2$. Then $x \in L_A^{(\alpha_1,\beta_1)}$ and $y \in L_A^{(\alpha_2,\beta_2)}$. We may assume that $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$, i.e., $\alpha_1 \leq \alpha_2$

and $\beta_1 \geq \beta_2$ without loss of generality. It follows that $L_A^{(\alpha_2,\beta_2)} \subseteq L_A^{(\alpha_1,\beta_1)}$ so that $x, y \in L_A^{(\alpha_1,\beta_1)}$. Since $L_A^{(\alpha_1,\beta_1)}$ is a sublattice of L, we have $x+y \in L_A^{(\alpha_1,\beta_1)}$ and $x \cdot y \in L_A^{(\alpha_1,\beta_1)}$. Thus

$$\mu_A(x+y) \wedge \mu_A(x \cdot y) \geq \alpha_1 = \alpha_1 \wedge \alpha_2 = \mu_A(x) \wedge \mu_A(y),$$

$$\gamma_A(x+y) \vee \gamma_A(x \cdot y) \leq \beta_1 = \beta_1 \vee \beta_2 = \gamma_A(x) \vee \gamma_A(y),$$

which shows that $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy sublattice of L. Let $x, y \in L$ be such that $x \leq y$ (resp. $y \leq x$). Assume that $\mu_A(x) < \mu_A(y)$ and $\gamma_A(x) > \gamma_A(y)$ and let $\alpha_0 := \frac{1}{2}(\mu_A(x) + \mu_A(y))$ and $\beta_0 := \frac{1}{2}(\gamma_A(x) + \gamma_A(y))$. Then $\mu_A(x) < \alpha_0 < \mu_A(y)$ and $\gamma_A(x) > \beta_0 > \gamma_A(y)$. It follows that $y \in L_A^{(\alpha_0,\beta_0)}$ and $x \notin L_A^{(\alpha_0,\beta_0)}$. This is a contradiction. Hence we have the following three cases:

- $\mu_A(x) \geq \mu_A(y)$, $\gamma_A(x) > \gamma_A(y)$,
- $\mu_A(x) < \mu_A(y)$, $\gamma_A(x) \leq \gamma_A(y)$,
- $\mu_A(x) \geq \mu_A(y)$, $\gamma_A(x) \leq \gamma_A(y)$.

If the first case is valid, then $y \in L_A^{(\mu_A(y),\beta_1)}$ and $x \notin L_A^{(\mu_A(y),\beta_1)}$ for every $\beta \in [0,1]$ with $\gamma_A(y) < \beta_1 < \gamma_A(x)$. This is a contradiction. Similarly the second case induces a contradiction, and therefore the third case only is valid. This completes the proof. □

Definition 3.7. Let $A = \langle L, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy sublattice of L. Then $A = \langle L, \mu_A, \gamma_A \rangle$ is said to be an *intuitionistic fuzzy convex* if for every interval $[a, b] \subseteq L$, we have

$$(\forall x \in [a,b])\, (\mu_A(x) \geq \mu_A(a) \wedge \mu_A(b),\ \gamma_A(x) \leq \gamma_A(a) \vee \gamma_A(b)).$$

Proposition 3.8. *Every intuitionistic fuzzy ideal/filter is an intuitionistic fuzzy convex sublattice.*

PROOF. Straightforward. □

Theorem 3.9. *Let $A = \langle L, \mu_A, \gamma_A \rangle$ be an intuitionistic fuzzy sublattice of L. Then $A = \langle L, \mu_A, \gamma_A \rangle$ is intuitionistic fuzzy convex if and only* *if for every $(\alpha,\beta) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\gamma_A)$ with $\alpha+\beta \le 1$, the (α,β)-level subset $L_A^{(\alpha,\beta)}$ is a convex sublattice of L.*

PROOF. Suppose $A = \langle L, \mu_A, \gamma_A\rangle$ is an intuitionistic fuzzy convex sublattice of L and let $[a,b]$ be any interval contained in $L_A^{(\alpha,\beta)}$ where $(\alpha,\beta) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\gamma_A)$ with $\alpha+\beta \le 1$. Then $\mu_A(a) \ge \alpha$, $\gamma_A(a) \le \beta$, $\mu_A(b) \ge \alpha$, $\gamma_A(b) \le \beta$, which imply that

$$\mu_A(a) \wedge \mu_A(b) \ge \alpha,\ \gamma_A(a) \vee \gamma_A(b) \le \beta.$$

Since $A = \langle L, \mu_A, \gamma_A\rangle$ is intuitionistic fuzzy convex, it follows that

$$\mu_A(x) \ge \mu_A(a) \wedge \mu_A(b) \ge \alpha,\ \gamma_A(x) \le \gamma_A(a) \vee \gamma_A(b) \le \beta$$

for all $x \in [a,b]$ so that $x \in L_A^{(\alpha,\beta)}$. Since $L_A^{(\alpha,\beta)}$ is a sublattice of L (see Theorem 3.5), we conclude that $L_A^{(\alpha,\beta)}$ is a convex sublattice of L.

Conversely assume that $L_A^{(\alpha,\beta)}$ is a convex sublattice of L for every $(\alpha,\beta) \in \mathrm{Im}(\mu_A) \times \mathrm{Im}(\gamma_A)$ with $\alpha+\beta \le 1$. Let $[a,b]$ be any interval of L. If we set $\mu_A(a) \wedge \mu_A(b) = \alpha$ and $\gamma_A(a) \vee \gamma_A(b) = \beta$, then $a \in L_A^{(\alpha,\beta)}$ and $b \in L_A^{(\alpha,\beta)}$. Since $L_A^{(\alpha,\beta)}$ is a convex sublattice of L, we have $x \in L_A^{(\alpha,\beta)}$ for all $x \in [a,b]$. Thus

$$\mu_A(x) \ge \alpha = \mu_A(a) \wedge \mu_A(b),\ \gamma_A(x) \le \beta = \gamma_A(a) \vee \gamma_A(b)$$

for all $x \in [a,b]$. Since $A = \langle L, \mu_A, \gamma_A\rangle$ is an intuitionistic fuzzy sublattice of L (see Theorem 3.6), we conclude that $A = \langle L, \mu_A, \gamma_A\rangle$ is an intuitionistic fuzzy convex sublattice of L. □

Theorem 3.10. *Let $\{A_i = \langle L, \mu_{A_i}, \gamma_{A_i}\rangle \mid i \in \Lambda\}$ be a family of intuitionistic fuzzy convex sublattices of L. Then $\cap A_i = \langle L, \mu_{\cap A_i}, \gamma_{\cap A_i}\rangle$ is an intuitionistic fuzzy convex sublattice of L.*

PROOF. Obviously $\cap A_i = \langle L, \mu_{\cap A_i}, \gamma_{\cap A_i}\rangle$ is an intuitionistic fuzzy sublattice of L. Let $[a,b]$ be any interval in L. Then

$$\begin{aligned}\mu_{\cap A_i}(x) &= \wedge \mu_{A_i}(x) \ge \wedge[\mu_{A_i}(a) \wedge \mu_{A_i}(b)]\\ &= (\wedge \mu_{A_i}(a)) \wedge (\wedge \mu_{A_i}(b)) = \mu_{\cap A_i}(a) \wedge \mu_{\cap A_i}(b),\end{aligned}$$

$$\begin{aligned}\gamma_{\cap A_i}(x) &= \vee\gamma_{A_i}(x) \leq \vee[\gamma_{A_i}(a) \vee \gamma_{A_i}(b)] \\ &= (\vee\gamma_{A_i}(a)) \vee (\vee\gamma_{A_i}(b)) = \gamma_{\cap A_i}(a) \vee \gamma_{\cap A_i}(b)\end{aligned}$$

for all $x \in [a,b]$. Thus $\cap A_i = \langle L, \mu_{\cap A_i}, \gamma_{\cap A_i}\rangle$ is an intuitionistic fuzzy convex sublattice of L. □

Theorem 3.11. *If $A = \langle L, \mu_A, \gamma_A\rangle$ is an intuitionistic fuzzy ideal* (resp. *filter*) *of L, then the upper α-level cut $U(\mu_A;\alpha)$ of μ_A and the lower α-level cut $L(\gamma_A;\alpha)$ of γ_A are ideals* (resp. *filters*) *of L for every $\alpha \in \text{Im}(\mu_A) \cap \text{Im}(\gamma_A) \cap [0, 0.5]$.*

PROOF. Let $\alpha \in \text{Im}(\mu_A) \cap \text{Im}(\gamma_A) \cap [0, 0.5]$ and let $x, y \in U(\mu_A;\alpha)$ (resp. $x, y \in L(\gamma_A;\alpha)$). Then $\mu_A(x) \geq \alpha$ and $\mu_A(y) \geq \alpha$ (resp. $\gamma_A(x) \leq \alpha$ and $\gamma_A(y) \leq \alpha$), and so

$$\mu_A(x+y) \wedge \mu_A(x \cdot y) \geq \mu_A(x) \wedge \mu_A(y) \geq \alpha$$

$$(\text{resp. } \gamma_A(x+y) \vee \gamma_A(x \cdot y) \leq \gamma_A(x) \vee \gamma_A(y) \leq \alpha).$$

Thus $x+y, x\cdot y \in U(\mu_A;\alpha)$ (resp. $x+y, x\cdot y \in L(\gamma_A;\alpha)$), and so $U(\mu_A;\alpha)$ (resp. $L(\gamma_A;\alpha)$) is a sublattice of L. Now let $x \in L$ and $y \in U(\mu_A;\alpha)$ be such that $x \leq y$. Then $\mu_A(x) \geq \mu_A(y) \geq \alpha$ and so $x \in U(\mu_A;\alpha)$. Finally let $x \in L(\gamma_A;\alpha)$ and $y \in L$ be such that $x \leq y$. Then $\gamma_A(x) \leq \gamma_A(y) \leq \alpha$ and therefore $x \in L(\gamma_A;\alpha)$. This completes the proof. □

Theorem 3.12. *If $A = \langle L, \mu_A, \gamma_A\rangle$ is an IFS in L such that the nonempty sets $U(\mu_A;\alpha)$ and $L(\gamma_A;\alpha)$ are ideals* (resp. *filters*) *of L for all $\alpha \in [0, 0.5]$, then $A = \langle L, \mu_A, \gamma_A\rangle$ is an intuitionistic fuzzy ideal* (resp. *filter*) *of L.*

PROOF. For any $\alpha \in [0, 0.5]$, assume that $U(\mu_A;\alpha) \neq \emptyset$ and $L(\gamma_A;\alpha) \neq \emptyset$ are ideals (resp. filters) of L. Let $x, y \in L$. We put $\alpha_1 := \mu_A(x) \wedge \mu_A(y)$ and $\alpha_2 := \gamma_A(x) \vee \gamma_A(y)$. Then $x, y \in U(\mu_A;\alpha_1) \cap L(\gamma_A;\alpha_2)$, which implies that $x+y, x\cdot y \in U(\mu_A;\alpha_1) \cap L(\gamma_A;\alpha_2)$ so that

$$\mu_A(x+y) \wedge \mu_A(x \cdot y) \geq \alpha_1 = \mu_A(x) \wedge \mu_A(y),$$

$$\gamma_A(x+y) \vee \gamma_A(x \cdot y) \leq \alpha_2 = \gamma_A(x) \vee \gamma_A(y).$$

Let $x, y \in L$ be such that $x \leq y$. If $\mu_A(x) < \mu_A(y)$ (resp. $\mu_A(x) > \mu_A(y)$), then $\mu_A(x) < \alpha_3 < \mu_A(y)$ (resp. $\mu_A(x) > \alpha_4 > \mu_A(y)$) for some $\alpha_3 \in (0, 0.5)$ (resp. $\alpha_4 \in (0, 0.5)$). Hence $y \in U(\mu_A; \alpha_3)$ and $x \notin U(\mu_A; \alpha_3)$ (resp. $y \in U(\mu_A; \alpha_4)$ and $x \notin U(\mu_A; \alpha_4)$). This is a contradiction. Assume that $\gamma_A(x) > \gamma_A(y)$ (resp. $\gamma_A(x) < \gamma_A(y)$). Then there exists $\beta_1 \in (0, 0.5)$ (resp. $\beta_2 \in (0, 0.5)$) such that $\gamma_A(x) > \beta_1 > \gamma_A(y)$ (resp. $\gamma_A(x) < \beta_2 < \gamma_A(y)$). It follows that $y \in L(\gamma_A; \beta_1)$ and $x \notin L(\gamma_A; \beta_1)$ (resp. $y \in L(\gamma_A; \beta_2)$ and $x \notin L(\gamma_A; \beta_2)$), a contradiction. Hence $\gamma_A(x) \leq \gamma_A(y)$ (resp. $\gamma_A(x) \geq \gamma_A(y)$). Consequently, $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy ideal (resp. filter) of L. □

Corollary 3.13. *Let K be an ideal* (resp. *filter*) *of L. If fuzzy sets μ_A and γ_A in L are defined by*

$$\mu_A(x) := \begin{cases} \alpha_0 & \text{if } x \in K, \\ \alpha_1 & \text{if } x \in L \setminus K, \end{cases} \qquad \gamma_A(x) := \begin{cases} \beta_0 & \text{if } x \in K, \\ \beta_1 & \text{if } x \in L \setminus K, \end{cases}$$

where $0 \leq \alpha_1 < \alpha_0$, $0 \leq \beta_0 < \beta_1$ and $\alpha_i + \beta_i \leq 1$ for $i = 0, 1$, then $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy ideal (resp. *filter*) *of L and $U(\mu_A; \alpha_0) = K = L(\gamma_A; \beta_0)$.*

Theorem 3.14. *Let Ω be a nonempty finite subset of $[0, 0.5]$. If $\{K_\alpha \mid \alpha \in \Omega\}$ is a collection of ideals* (resp. *filters*) *of L such that*

(i) $L = \bigcup_{\alpha \in \Omega} K_\alpha$,

(ii) $(\forall \alpha, \beta \in \Omega)\ (\alpha > \beta \Leftrightarrow K_\alpha \subset K_\beta)$,

then an IFS $A = \langle L, \mu_A, \gamma_A \rangle$ in L defined by $\mu_A(x) = \bigvee\{\alpha \in \Omega \mid x \in K_\alpha\}$ and $\gamma_A(x) = \bigwedge\{\alpha \in \Omega \mid x \in K_\alpha\}$ is an intuitionistic fuzzy ideal (resp. *filter*) *of L.*

Proof. According to Theorem 3.12, it is sufficient to show that the nonempty sets $U(\mu_A; \alpha)$ and $L(\gamma_A; \beta)$ are ideals (resp. filters) of L, where $\alpha + \beta \leq 1$. We show that $U(\mu_A; \alpha) = K_\alpha$. Note that

$$\begin{aligned} x \in U(\mu_A;\alpha) &\Longleftrightarrow \mu_A(x) \geq \alpha \\ &\Longleftrightarrow \bigvee\{\delta \in \Omega \mid x \in K_\delta\} \geq \alpha \\ &\Longleftrightarrow \exists \delta_0 \in \Omega,\ x \in K_{\delta_0},\ \delta_0 \geq \alpha \\ &\Longleftrightarrow x \in K_\alpha \quad (\text{since } K_{\delta_0} \subseteq K_\alpha). \end{aligned}$$

Thus $U(\mu_A; \alpha) = K_\alpha$. Now, we prove that $L(\gamma_A; \beta) \neq \emptyset$ is an ideal (resp. filter) of L. We have

$$\begin{aligned} x \in L(\gamma_A;\beta) &\Longleftrightarrow \gamma_A(x) \leq \beta \\ &\Longleftrightarrow \bigwedge\{\delta \in \Omega \mid x \in K_\delta\} \leq \beta \\ &\Longleftrightarrow \exists \delta_0 \in \Omega,\ x \in K_{\delta_0},\ \delta_0 \leq \beta \\ &\Longleftrightarrow x \in \bigcup_{\delta \leq \beta} K_\delta \end{aligned}$$

and hence $L(\gamma_A; \beta) = \bigcup_{\delta \leq \beta} K_\delta$, which is an ideal (resp. filter) of L. This completes the proof. □

4. Relations

Let $\alpha \in [0, 1]$ be fixed and let $IFI(L)$ (resp. $IFF(L)$) be the family of all intuitionistic fuzzy ideals (resp. filters) of L. For any $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle L, \mu_B, \gamma_B\rangle$ from $IFI(L)$ (resp. $IFF(L)$) we define two binary relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ on $IFI(L)$ (resp. $IFF(L)$) as follows:

$$(A, B) \in \mathfrak{U}^\alpha \Longleftrightarrow U(\mu_A; \alpha) = U(\mu_B; \alpha)$$

and

$$(A, B) \in \mathfrak{L}^\alpha \Longleftrightarrow L(\gamma_A; \alpha) = L(\gamma_B; \alpha).$$

These two relations $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$ are equivalence relations. Hence $IFI(L)$ (resp. $IFF(L)$) can be divided into the equivalence classes of $\mathfrak{U}^\alpha$ and $\mathfrak{L}^\alpha$, denoted by $[A]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha}$ for any $A = \langle L, \mu_A, \gamma_A\rangle \in IFI(L)$ (resp. $IFF(L)$), respectively. The corresponding quotient sets will be denoted by $IFI(L)/\mathfrak{U}^\alpha$ and $IFI(L)/\mathfrak{L}^\alpha$, (resp. $IFF(L)/\mathfrak{U}^\alpha$ and $IFF(L)/\mathfrak{L}^\alpha$), respectively.

For the family $I(L)$ (resp. $F(L)$) of all ideals (resp. filters) of L we define two maps U_α and L_α from $IFI(L)$ (resp. $IFF(L)$) to $I(L) \cup \{\emptyset\}$ (resp. $F(L) \cup \{\emptyset\}$) by putting

$$U_\alpha(A) = U(\mu_A; \alpha) \quad \text{and} \quad L_\alpha(A) = L(\gamma_A; \alpha)$$

for each $A = \langle L, \mu_A, \gamma_A \rangle \in IFI(L)$ (resp. $IFF(L)$).

It is not difficult to see that these maps are well-defined.

Lemma 4.1. *For any $\alpha \in (0,1)$ the maps U_α and L_α are surjective.*

PROOF. Note that $\mathbf{0}_\sim = \langle L, \mathbf{0}, \mathbf{1} \rangle \in IFI(L)$ (resp. $IFF(L)$) and $U_\alpha(\mathbf{0}_\sim) = L_\alpha(\mathbf{0}_\sim) = \emptyset$ for any $\alpha \in (0,1)$. Moreover for any $K \in I(L)$ (resp. $F(L)$) we have $K_\sim = \langle L, \chi_K, \bar{\chi}_K \rangle \in IFI(L)$ (resp. $IFF(L)$), $U_\alpha(K_\sim) = U(\chi_K; \alpha) = K$ and $L_\alpha(K_\sim) = L(\bar{\chi}_K; \alpha) = K$. Hence U_α and L_α are surjective. □

Theorem 4.2. *For any $\alpha \in (0,1)$ the sets $IFI(L)/\mathfrak{U}^\alpha$ and $IFI(L)/\mathfrak{L}^\alpha$ are equipotent to $I(L) \cup \{\emptyset\}$.*

PROOF. Let $\alpha \in (0,1)$. Putting $U^*_\alpha([A]_{\mathfrak{U}^\alpha}) = U_\alpha(A)$ and $L^*_\alpha([A]_{\mathfrak{L}^\alpha}) = L_\alpha(A)$ for any $A = (\mu_A, \gamma_A) \in IFI(L)$, we obtain two maps

$$U^*_\alpha : IFI(L)/\mathfrak{U}^\alpha \to I(L) \cup \{\emptyset\} \quad \text{and} \quad L^*_\alpha : IFI(L)/\mathfrak{L}^\alpha \to I(L) \cup \{\emptyset\}.$$

If $U(\mu_A; \alpha) = U(\mu_B; \alpha)$ and $L(\gamma_A; \alpha) = L(\gamma_B; \alpha)$ for some $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ from $IFI(L)$ (resp. $IFF(L)$), then $(A, B) \in \mathfrak{U}^\alpha$ and $(A, B) \in \mathfrak{L}^\alpha$, whence $[A]_{\mathfrak{U}^\alpha} = [B]_{\mathfrak{U}^\alpha}$ and $[A]_{\mathfrak{L}^\alpha} = [B]_{\mathfrak{L}^\alpha}$, which means that U^*_α and L^*_α are injective.

To show that the maps U^*_α and L^*_α are surjective, let $K \in I(L)$. Then for $K_\sim = \langle \chi_K, \bar{\chi}_K \rangle \in IFI(L)$ we have $U^*_\alpha([K_\sim]_{\mathfrak{U}^\alpha}) = U(\chi_K; \alpha) = K$ and $L^*_\alpha([K_\sim]_{\mathfrak{L}^\alpha}) = L(\bar{\chi}_K; \alpha) = K$. Also $\mathbf{0}_\sim = \langle L, \mathbf{0}, \mathbf{1} \rangle \in IFI(L)$. Moreover $U^*_\alpha([\mathbf{0}_\sim]_{\mathfrak{U}^\alpha}) = U(\mathbf{0}; \alpha) = \emptyset$ and $L^*_\alpha([\mathbf{0}_\sim]_{\mathfrak{L}^\alpha}) = L(\mathbf{1}; \alpha) = \emptyset$. Hence U^*_α and L^*_α are surjective. □

Similarly, we have

Theorem 4.3. *For any $\alpha \in (0,1)$ the sets $IFF(L)/\mathfrak{U}^\alpha$ and $IFF(L)/\mathfrak{L}^\alpha$ are equipotent to $F(L) \cup \{\emptyset\}$.*

Now for any $\alpha \in [0,1]$ we define a new relation $\mathfrak{R}^\alpha$ on $IFI(L)$ (resp. $IFF(L)$) by putting:

$$(A,B) \in \mathfrak{R}^\alpha \iff U(\mu_A;\alpha) \cap L(\gamma_A;\alpha) = U(\mu_B;\alpha) \cap L(\gamma_B;\alpha),$$

where $A = \langle L, \mu_A, \gamma_A \rangle$ and $B = \langle L, \mu_B, \gamma_B \rangle$. Obviously $\mathfrak{R}^\alpha$ is an equivalence relation.

Lemma 4.4. *The map $I_\alpha : IFI(L) \to I(L) \cup \{\emptyset\}$ defined by*

$$I_\alpha(A) = U(\mu_A;\alpha) \cap L(\gamma_A;\alpha),$$

where $A = \langle L, \mu_A, \gamma_A \rangle$, is surjective for any $\alpha \in (0,1)$.

Proof. If $\alpha \in (0,1)$ is fixed, then for $\mathbf{0}_\sim = \langle L, \mathbf{0}, \mathbf{1} \rangle \in IFI(L)$ we have

$$I_\alpha(\mathbf{0}_\sim) = U(\mathbf{0};\alpha) \cap L(\mathbf{1};\alpha) = \emptyset\,,$$

and for any $K \in I(L)$ there exists $K_\sim = \langle L, \chi_K, \bar{\chi}_K \rangle \in IFI(L)$ such that $I_\alpha(K_\sim) = U(\chi_K;\alpha) \cap L(\bar{\chi}_K;\alpha) = K$. □

Similarly, we get

Lemma 4.5. *The map $F_\alpha : IFF(L) \to F(L) \cup \{\emptyset\}$ defined by*

$$F_\alpha(A) = U(\mu_A;\alpha) \cap L(\gamma_A;\alpha),$$

where $A = \langle L, \mu_A, \gamma_A \rangle$, is surjective for any $\alpha \in (0,1)$.

Theorem 4.6. *For any $\alpha \in (0,1)$ the quotient set $IFI(L)/\mathfrak{R}^\alpha$ is equipotent to $I(L) \cup \{\emptyset\}$.*

Proof. Let $I^*_\alpha : IFI(L)/\mathfrak{R}^\alpha \to I(L) \cup \{\emptyset\}$, where $\alpha \in (0,1)$, be defined by the formula:

$$I^*_\alpha([A]_{\mathfrak{R}^\alpha}) = I_\alpha(A) \quad \text{for each} \quad [A]_{\mathfrak{R}^\alpha} \in IFI(L)/\mathfrak{R}^\alpha.$$

If $I^*_\alpha([A]_{\mathfrak{R}^\alpha}) = I^*_\alpha([B]_{\mathfrak{R}^\alpha})$ for some $[A]_{\mathfrak{R}^\alpha}$, $[B]_{\mathfrak{R}^\alpha} \in IFI(L)/\mathfrak{R}^\alpha$, then

$$U(\mu_A;\alpha) \cap L(\gamma_A;\alpha) = U(\mu_B;\alpha) \cap L(\gamma_B;\alpha),$$

which implies $(A,B) \in \mathfrak{R}^\alpha$ and, in the consequence, $[A]_{\mathfrak{R}^\alpha} = [B]_{\mathfrak{R}^\alpha}$. Thus I^*_α is injective. It is also onto because $I^*_\alpha(\mathbf{0}_\sim) = I_\alpha(\mathbf{0}_\sim) = \emptyset$ for $\mathbf{0}_\sim = \langle L, \mathbf{0}, \mathbf{1}\rangle \in IFI(L)$, and $I^*_\alpha(K_\sim) = I_\alpha(K) = K$ for $K \in I(L)$ and $K_\sim = \langle L, \chi_K, \bar{\chi}_K\rangle \in IFI(L)$. □

Similarly, we obtain

Theorem 4.7. *For any $\alpha \in (0,1)$ the quotient set $IFF(L)/\mathfrak{R}^\alpha$ is equipotent to $F(L) \cup \{\emptyset\}$.*

5. Operations on intuitionistic fuzzy ideals/filters

Definition 5.1. Let $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle L, \mu_B, \gamma_B\rangle$ be IFSs in L. We define operations $A+B = \langle L, \mu_{A+B}, \gamma_{A+B}\rangle$, $A{\cdot}B = \langle L, \mu_{A\cdot B}, \gamma_{A\cdot B}\rangle$, $A \oplus B = \langle L, \mu_{A\oplus B}, \gamma_{A\oplus B}\rangle$, and $A \odot B = \langle L, \mu_{A\odot B}, \gamma_{A\odot B}\rangle$, respectively, as follows:

- $\mu_{A+B}(z) = \bigvee_{z=x+y} [\mu_A(x) \wedge \mu_B(y)]$, $\gamma_{A+B}(z) = \bigwedge_{z=x+y} [\gamma_A(x) \vee \gamma_B(y)]$,
- $\mu_{A\cdot B}(z) = \bigvee_{z=x\cdot y} [\mu_A(x) \wedge \mu_B(y)]$, $\gamma_{A\cdot B}(z) = \bigwedge_{z=x\cdot y} [\gamma_A(x) \vee \gamma_B(y)]$,
- $\mu_{A\oplus B}(z) = \bigvee_{z\le x+y} [\mu_A(x) \wedge \mu_B(y)]$, $\gamma_{A\oplus B}(z) = \bigwedge_{z\le x+y} [\gamma_A(x) \vee \gamma_B(y)]$,
- $\mu_{A\odot B}(z) = \bigvee_{x\cdot y\le z} [\mu_A(x) \wedge \mu_B(y)]$, $\gamma_{A\odot B}(z) = \bigwedge_{x\cdot y\le z} [\gamma_A(x) \vee \gamma_B(y)]$.

Note that Definition 5.1 implies that $A \subseteq A{+}A$, $A \subseteq A{\cdot}A$, $A \subseteq A{\oplus}A$, $A \subseteq A{\odot}A$, $A{+}B \subseteq A{\oplus}B$, and $A{\cdot}B \subseteq A{\oplus}B$. Moreover if L is distributive then $A \oplus B \subseteq A + B$ and $A \odot B \subseteq A \cdot B$ for all IFSs $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle L, \mu_B, \gamma_B\rangle$ in L.

Lemma 5.2. *Let $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle L, \mu_B, \gamma_B\rangle$ be IFSs in L such that $A \subseteq A \oplus B$, i.e., $\mu_A \le \mu_{A\oplus B}$ and $\gamma_A \ge \gamma_{A\oplus B}$. If (α_1, β_1)*

and (α_2, β_2) *are intuitionistic upper bounds of* $A = \langle L, \mu_A, \gamma_A \rangle$ *and* $B = \langle L, \mu_B, \gamma_B \rangle$ *respectively, then* $(\alpha_1, \beta_1) \leq (\alpha_2, \beta_2)$*, that is,* $\alpha_1 \leq \alpha_2$ *and* $\beta_1 \geq \beta_2$.

PROOF. Suppose that $(\alpha_1, \beta_1) \not\leq (\alpha_2, \beta_2)$. Then $\alpha_1 > \alpha_2$ or $\beta_1 < \beta_2$. If $\alpha_1 > \alpha_2$, then $\bigvee_{y \in L} \mu_B(y) < \bigvee_{y \in L} \mu_A(y)$, and so $\bigvee_{y \in L} \mu_B(y) < \mu_A(z)$ for some $z \in L$. It follows that

$$\mu_{A \oplus B}(z) = \bigvee_{z \leq x+y} [\mu_A(x) \wedge \mu_B(y)] \leq \bigvee_{z \leq x+y} \mu_B(y) \leq \bigvee_{y \in L} \mu_B(y) < \mu_A(z),$$

which is a contradiction. If $\beta_1 < \beta_2$, then $\bigwedge_{y \in L} \gamma_B(y) > \bigwedge_{y \in L} \gamma_A(y)$, which implies that there exists $z \in L$ such that $\bigwedge_{y \in L} \gamma_B(y) > \gamma_A(z)$. Therefore

$$\gamma_{A \oplus B}(z) = \bigwedge_{z \leq x+y} [\gamma_A(x) \vee \gamma_B(y)] \geq \bigwedge_{z \leq x+y} \gamma_B(y) \geq \bigwedge_{y \in L} \gamma_B(y) > \gamma_A(z).$$

This is impossible, and therefore we have the desired result. □

Corollary 5.3. *Let* $A = \langle L, \mu_A, \gamma_A \rangle$ *and* $B = \langle L, \mu_B, \gamma_B \rangle$ *be IFSs in* L *such that* $A \subseteq A \oplus B$ *and* $B \subseteq A \oplus B$*. If* (α_1, β_1) *and* (α_2, β_2) *are intuitionistic upper bound of* $A = \langle L, \mu_A, \gamma_A \rangle$ *and* $B = \langle L, \mu_B, \gamma_B \rangle$ *respectively, then* $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$.

Lemma 5.4. *For any IFSs* $A = \langle L, \mu_A, \gamma_A \rangle$ *and* $B = \langle L, \mu_B, \gamma_B \rangle$ *in* L *with the same intuitionistic upper bound, we have* $A \subseteq A \oplus B$ *and* $B \subseteq A \oplus B$*, that is,* $\mu_A \leq \mu_{A \oplus B}$, $\gamma_A \geq \gamma_{A \oplus B}$, $\mu_B \leq \mu_{A \oplus B}$, $\gamma_B \geq \gamma_{A \oplus B}$.

PROOF. Assume that $A = \langle L, \mu_A, \gamma_A \rangle$ and $B = \langle L, \mu_B, \gamma_B \rangle$ attain their intuitionistic upper bound. Let $x_0, y_0 \in L$ be such that

$$\bigvee_{x \in L} \mu_A(x) = \mu_A(x_0), \quad \bigvee_{y \in L} \mu_B(y) = \mu_B(y_0);$$

and let $u_0, v_0 \in L$ be such that

$$\bigwedge_{x \in L} \gamma_A(x) = \gamma_A(u_0), \quad \bigwedge_{y \in L} \gamma_B(y) = \gamma_B(v_0).$$

Then $\mu_A(x_0) = \mu_B(y_0)$ and $\gamma_A(u_0) = \gamma_B(v_0)$ by assumption. For any $z \in L$, we get

$$\begin{aligned}\mu_{A\oplus B}(z) &= \bigvee_{z\le x+y} [\mu_A(x) \wedge \mu_B(y)] \\ &\ge \mu_A(z) \wedge \mu_B(y_0) \qquad [\because z \le z + y_0] \\ &= \mu_A(z) \qquad [\because \mu_A(z) \le \bigvee_{x\in L} \mu_A(x) = \mu_B(y_0)].\end{aligned}$$

Now for any $z \in L$ we have

$$\begin{aligned}\gamma_{A\oplus B}(z) &= \bigwedge_{z\le x+y} [\gamma_A(x) \vee \gamma_B(y)] \\ &\le \gamma_A(z) \vee \gamma_B(v_0) \qquad [\because z \le z + v_0] \\ &= \gamma_A(z) \qquad [\because \gamma_A(z) \ge \bigwedge_{x\in L} \gamma_A(x) = \gamma_B(v_0)].\end{aligned}$$

Hence $A \subseteq A \oplus B$. Similarly we obtain $B \subseteq A \oplus B$. Now we suppose that $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle L, \mu_B, \gamma_B\rangle$ do not attain their intuitionistic upper bound. Let $\bigvee_{x\in L} \mu_A(x) = \bigvee_{y\in L} \mu_B(y) = \alpha$ and $\bigwedge_{x\in L} \gamma_A(x) = \bigwedge_{y\in L} \gamma_B(y) = \beta$. Since $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle L, \mu_B, \gamma_B\rangle$ do not attain their intuitionistic upper bound, $\mu_A(z) < \alpha$ and $\gamma_A(z) > \beta$ for all $z \in L$. Then there exist $y_0, u_0 \in L$ such that $\mu_B(y_0) > \mu_A(z)$ and $\gamma_B(u_0) < \gamma_A(z)$. But $z \le z + y_0$ and $z \le z + u_0$ and so

$$\mu_{A\oplus B}(z) = \bigvee_{z\le x+y} [\mu_A(x) \wedge \mu_B(y)] \ge \mu_A(z) \wedge \mu_B(y_0) = \mu_A(z),$$

$$\gamma_{A\oplus B}(z) = \bigwedge_{z\le x+y} [\gamma_A(x) \vee \gamma_B(y)] \le \gamma_A(z) \vee \gamma_B(u_0) = \gamma_A(z).$$

Hence $A \subseteq A \oplus B$. Similarly one can verify that $B \subseteq A \oplus B$. This completes the proof. □

Lemma 5.5. *Let $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle L, \mu_B, \gamma_B\rangle$ be IFSs in L with the same intuitionistic upper bound (α, β). If $B \not\subseteq A \oplus B$, then $A \subseteq A \oplus B$.*

PROOF. If $B \not\subseteq A \oplus B$, then $\mu_B \not\leq \mu_{A\oplus B}$ or $\gamma_B \not\geq \gamma_{A\oplus B}$. Assume that $\mu_B \not\leq \mu_{A\oplus B}$. Then there exist $u_0, v_0 \in L$ such that

$$\mu_B(u_0) > \mu_{A\oplus B}(u_0) = \bigvee_{u_0 \leq x+y} [\mu_A(x) \wedge \mu_B(y)] \geq \mu_A(x) \wedge \mu_B(u_0) = \mu_A(x),$$

$$\gamma_B(v_0) < \gamma_{A\oplus B}(v_0) = \bigwedge_{v_0 \leq x+y} [\gamma_A(x) \vee \gamma_B(y)] \leq \gamma_A(x) \vee \gamma_B(v_0) = \gamma_A(x).$$

Thus

$$\mu_B(u_0) \geq \bigvee_{x \in L} \mu_A(x) = \alpha \geq \mu_B(u_0),$$

$$\gamma_B(v_0) \leq \bigwedge_{x \in L} \gamma_A(x) = \beta \leq \gamma_B(v_0),$$

and so $\mu_B(u_0) = \alpha$ and $\gamma_B(v_0) = \beta$. This shows that $B = \langle L, \mu_B, \gamma_B \rangle$ attains its supremum. It follows from Lemma 5.4 that $A \subseteq A \oplus B$. □

Using Lemmas 5.4 and 5.5, we have the following theorem.

Theorem 5.6. *Let $A = \langle L, \mu_A, \gamma_A \rangle$ and $B = \langle L, \mu_B, \gamma_B \rangle$ be IFSs in L with the same intuitionistic upper bound. Then exactly one of $A = \langle L, \mu_A, \gamma_A \rangle$ and $B = \langle L, \mu_B, \gamma_B \rangle$ is contained in $A \oplus B = \langle L, \mu_{A\oplus B}, \gamma_{A\oplus B} \rangle$ if and only if exactly one of $A = \langle L, \mu_A, \gamma_A \rangle$ and $B = \langle L, \mu_B, \gamma_B \rangle$ attains the intuitionistic upper bound.*

Theorem 5.7. *An IFS $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy sublattice of L if and only if $A + A = A$ and $A \cdot A = A$, that is, $\mu_{A+A} = \mu_A$, $\gamma_{A+A} = \gamma_A$, $\mu_{A \cdot A} = \mu_A$, $\gamma_{A \cdot A} = \gamma_A$.*

PROOF. ($\Rightarrow$) Let $z \in L$. For every $x, y \in L$ with $z = x + y$, we have

$$\mu_A(z) = \mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y),\ \gamma_A(z) = \gamma_A(x+y) \leq \gamma_A(x) \vee \gamma_A(y),$$

and so

$$\mu_A(z) \geq \bigvee_{z=x+y} [\mu_A(x) \wedge \mu_A(y)] = \mu_{A+A}(z),$$

$$\gamma_A(z) \leq \bigwedge_{z=x+y} [\gamma_A(x) \vee \gamma_A(y)] = \gamma_{A+A}(z).$$

Hence $A + A \subseteq A$, and so $A + A = A$. Now let $z \in L$. For every $x, y \in L$ such that $z = x \cdot y$, we obtain

$$\mu_A(z) = \mu_A(x \cdot y) \geq \mu_A(x) \wedge \mu_A(y),\ \gamma_A(z) = \gamma_A(x \cdot y) \leq \gamma_A(x) \vee \gamma_A(y).$$

It follows that

$$\mu_A(z) \geq \bigvee_{z=x\cdot y} [\mu_A(x) \wedge \mu_A(y)] = \mu_{A\cdot A}(z),$$

$$\gamma_A(z) \leq \bigwedge_{z=x\cdot y} [\gamma_A(x) \vee \gamma_A(y)] = \gamma_{A\cdot A}(z)$$

so that $A \cdot A \subseteq A$, and hence $A \cdot A = A$.

($\Leftarrow$) For any $x, y \in L$, we have

$$\begin{aligned}
&\mu_A(x + y) \wedge \mu_A(x \cdot y) = \mu_{A+A}(x + y) \wedge \mu_{A\cdot A}(x \cdot y) \\
&= \Big(\bigvee_{x+y=a+b} [\mu_A(a) \wedge \mu_A(b)]\Big) \wedge \Big(\bigvee_{x\cdot y=c\cdot d} [\mu_A(c) \wedge \mu_A(d)]\Big) \\
&\geq (\mu_A(x) \wedge \mu_A(y)) \wedge (\mu_A(x) \wedge \mu_A(y)) \\
&= \mu_A(x) \wedge \mu_A(y),
\end{aligned}$$

$$\begin{aligned}
&\gamma_A(x + y) \vee \gamma_A(x \cdot y) = \gamma_{A+A}(x + y) \vee \gamma_{A\cdot A}(x \cdot y) \\
&= \Big(\bigwedge_{x+y=a+b} [\gamma_A(a) \vee \gamma_A(b)]\Big) \vee \Big(\bigwedge_{x\cdot y=c\cdot d} [\gamma_A(c) \vee \gamma_A(d)]\Big) \\
&\leq (\gamma_A(x) \vee \gamma_A(y)) \vee (\gamma_A(x) \vee \gamma_A(y)) \\
&= \gamma_A(x) \vee \gamma_A(y).
\end{aligned}$$

Hence $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy sublattice of L. □

Theorem 5.8. *An IFS $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy ideal of L if and only if $A \oplus A = A$, that is, $\mu_{A\oplus A} = \mu_A$ and $\gamma_{A\oplus A} = \gamma_A$.*

Proof. Assume that $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy ideal of L and let $z \in L$. Taking $x, y \in L$ such that $z \leq x + y$ induces that

$$\mu_A(z) \geq \mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y),\ \gamma_A(z) \leq \gamma_A(x+y) \leq \gamma_A(x) \vee \gamma_A(y).$$

It follows that

$$\mu_A(z) \geq \bigvee_{z\leq x+y} [\mu_A(x) \wedge \mu_A(y)] = \mu_{A\oplus A}(z),$$

$$\gamma_A(z) \le \bigwedge_{z \le x+y} [\gamma_A(x) \vee \gamma_A(y)] = \gamma_{A\oplus A}(z)$$

so that $A \oplus A \subseteq A$, and hence $A \oplus A = A$. Now assume that $A \oplus A = A$ for any IFS $A = \langle L, \mu_A, \gamma_A \rangle$ in L and let $x, y \in L$. Then

$$\begin{aligned} &\mu_A(x+y) \wedge \mu_A(x \cdot y) = \mu_{A\oplus A}(x+y) \wedge \mu_{A\oplus A}(x \cdot y) \\ &\quad = \Big(\bigvee_{x+y \le a+b} [\mu_A(a) \wedge \mu_A(b)]\Big) \wedge \Big(\bigvee_{x \cdot y \le c+d} [\mu_A(c) \wedge \mu_A(d)]\Big) \\ &\quad \ge (\mu_A(x) \wedge \mu_A(y)) \wedge (\mu_A(x) \wedge \mu_A(y)) \qquad [\because x \cdot y \le x+y] \\ &\quad = \mu_A(x) \wedge \mu_A(y), \end{aligned}$$

$$\begin{aligned} &\gamma_A(x+y) \vee \gamma_A(x \cdot y) = \gamma_{A\oplus A}(x+y) \vee \gamma_{A\oplus A}(x \cdot y) \\ &\quad = \Big(\bigwedge_{x+y \le a+b} [\gamma_A(a) \vee \gamma_A(b)]\Big) \vee \Big(\bigwedge_{x \cdot y \le c+d} [\gamma_A(c) \vee \gamma_A(d)]\Big) \\ &\quad \le (\gamma_A(x) \vee \gamma_A(y)) \vee (\gamma_A(x) \vee \gamma_A(y)) \qquad [\because x \cdot y \le x+y] \\ &\quad = \gamma_A(x) \vee \gamma_A(y). \end{aligned}$$

This shows that $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy sublattice of L. Let $z_1, z_2 \in L$ be such that $z_1 \le z_2$. Then

$$\begin{aligned} \mu_A(z_2) &= \mu_{A\oplus A}(z_2) = \bigvee_{z_2 \le x_2+y_2} [\mu_A(x_2) \wedge \mu_A(y_2)] \\ &\le \bigvee_{z_1 \le x_1+y_1} [\mu_A(x_1) \wedge \mu_A(y_1)] = \mu_{A\oplus A}(z_1) = \mu_A(z_1), \end{aligned}$$

$$\begin{aligned} \gamma_A(z_2) &= \gamma_{A\oplus A}(z_2) = \bigwedge_{z_2 \le x_2+y_2} [\gamma_A(x_2) \vee \gamma_A(y_2)] \\ &\ge \bigwedge_{z_1 \le x_1+y_1} [\gamma_A(x_1) \vee \gamma_A(y_1)] = \gamma_{A\oplus A}(z_1) = \gamma_A(z_1), \end{aligned}$$

and so $A = \langle L, \mu_A, \gamma_A \rangle$ is intuitionistic antimonotonic. Therefore $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy ideal of L. □

Theorem 5.9. *An IFS $A = \langle L, \mu_A, \gamma_A \rangle$ is an intuitionistic fuzzy filter of L if and only if $A \odot A = A$, that is, $\mu_{A\odot A} = \mu_A$ and $\gamma_{A\odot A} = \gamma_A$.*

PROOF. The proof is similar to the proof of Theorem 5.8. □

Theorem 5.10. *If $A = \langle L, \mu_A, \gamma_A \rangle$ and $B = \langle L, \mu_B, \gamma_B \rangle$ are intuitionistic fuzzy ideals of L with the same intuitionistic upper bound, then $A \oplus B = \langle L, \mu_{A\oplus B}, \gamma_{A\oplus B} \rangle$ is an intuitionistic fuzzy ideal of L generated by $A = \langle L, \mu_A, \gamma_A \rangle$ and $B = \langle L, \mu_B, \gamma_B \rangle$, that is, it is the least intuitionistic fuzzy ideal containing $A = \langle L, \mu_A, \gamma_A \rangle$ and $B = \langle L, \mu_B, \gamma_B \rangle$.*

PROOF. We first show that

$$\mu_{A\oplus B}(x+y) \geq \mu_{A\oplus B}(x) \wedge \mu_{A\oplus B}(y), \tag{5.1}$$

$$\mu_{A\oplus B}(x \cdot y) \geq \mu_{A\oplus B}(x) \wedge \mu_{A\oplus B}(y), \tag{5.2}$$

$$\gamma_{A\oplus B}(x+y) \leq \gamma_{A\oplus B}(x) \vee \gamma_{A\oplus B}(y), \tag{5.3}$$

$$\gamma_{A\oplus B}(x \cdot y) \leq \gamma_{A\oplus B}(x) \vee \gamma_{A\oplus B}(y). \tag{5.4}$$

Suppose that $\mu_{A\oplus B}(x+y) < \mu_{A\oplus B}(x) \wedge \mu_{A\oplus B}(y)$ for some $x, y \in L$ and let $\mu_{A\oplus B}(x+y) = \alpha_0$. Then $\mu_{A\oplus B}(x) > \alpha_0$ and $\mu_{A\oplus B}(y) > \alpha_0$, which imply that there exist $a, b, c, d \in L$ such that $x \leq a+b$, $\mu_A(a) \wedge \mu_B(b) > \alpha_0$, $y \leq c+d$, $\mu_A(c) \wedge \mu_B(d) > \alpha_0$. Since $x+y \leq a+c+b+d$, it follows that

$$\begin{aligned} \mu_{A\oplus B}(x+y) &= \bigvee_{x+y\leq u+v} [\mu_A(u) \wedge \mu_B(v)] \\ &\geq \mu_A(a+c) \wedge \mu_B(b+d) \\ &\geq [\mu_A(a) \wedge \mu_A(c)] \wedge [\mu_B(b) \wedge \mu_B(d)] \\ &= [\mu_A(a) \wedge \mu_B(b)] \wedge [\mu_A(c) \wedge \mu_B(d)] > \alpha_0 \end{aligned}$$

which is a contradiction. If (5.2) is not valid, then there exist $x_1, y_1 \in L$ such that $\mu_{A\oplus B}(x_1 \cdot y_1) < \mu_{A\oplus B}(x_1) \wedge \mu_{A\oplus B}(y_1)$. Setting $\mu_{A\oplus B}(x_1 \cdot y_1) = \alpha_1$, then there are $a_1, b_1 \in L$ such that $x_1 \leq a_1 + b_1$ and $\mu_A(a_1) \wedge \mu_B(b_1) > \alpha_1$. Thus

$$\begin{aligned} \mu_{A\oplus B}(x_1 \cdot y_1) &= \bigvee_{x_1\cdot y_1\leq u+v} [\mu_A(u) \wedge \mu_B(v)] \\ &\geq \mu_A(a_1) \wedge \mu_B(b_1) \qquad [\because x_1 \cdot y_1 \leq x_1 \leq a_1 + b_1] \\ &> \alpha_1, \end{aligned}$$

which is impossible. Therefore

$$\mu_{A\oplus B}(x+y)\wedge\mu_{A\oplus B}(x\cdot y)\geq\mu_{A\oplus B}(x)\wedge\mu_{A\oplus B}(y)$$

for all $x, y\in L$. Now assume that (5.3) is not valid. Then $\gamma_{A\oplus B}(x+y) > \gamma_{A\oplus B}(x)\vee\gamma_{A\oplus B}(y)$ for some $x, y\in L$. Let $\gamma_{A\oplus B}(x+y)=\beta_0$. Then $\gamma_{A\oplus B}(x)<\beta_0$ and $\gamma_{A\oplus B}(y)<\beta_0$. It follows that there exist $a, b, c, d\in L$ such that $x\leq a+b$, $\gamma_A(a)\vee\gamma_B(b)<\beta_0$, $y\leq c+d$, $\gamma_A(c)\vee\gamma_B(d)<\beta_0$ so that

$$\begin{aligned}\gamma_{A\oplus B}(x+y) &= \bigwedge_{x+y\leq u+v}[\gamma_A(u)\vee\gamma_B(v)]\\ &\leq \gamma_A(a+c)\vee\gamma_B(b+d)\\ &\leq [\gamma_A(a)\vee\gamma_A(c)]\vee[\gamma_B(b)\vee\gamma_B(d)]\\ &= [\gamma_A(a)\vee\gamma_B(b)]\vee[\gamma_A(c)\vee\gamma_B(d)] < \beta_0.\end{aligned}$$

This is a contradiction. Finally if (5.4) is false, then $\gamma_{A\oplus B}(x_1\cdot y_1) > \gamma_{A\oplus B}(x_1)\vee\gamma_{A\oplus B}(y_1)$ for some $x_1, y_1\in L$. Let $\gamma_{A\oplus B}(x_1\cdot y_1)=\beta_1$. Then there exist $u_1, v_1\in L$ such that $x_1\leq u_1+v_1$ and $\gamma_A(u_1)\vee\gamma_B(v_1)<\beta_1$. Since $x_1\cdot y_1\leq x_1\leq u_1+v_1$, it follows that

$$\gamma_{A\oplus B}(x_1\cdot y_1)=\bigwedge_{x_1\cdot y_1\leq s+t}[\gamma_A(s)\vee\gamma_B(t)]\leq\gamma_A(u_1)\vee\gamma_B(v_1)<\beta_1,$$

which is a contradiction. Hence

$$\gamma_{A\oplus B}(x+y)\vee\gamma_{A\oplus B}(x\cdot y)\leq\gamma_{A\oplus B}(x)\vee\gamma_{A\oplus B}(y)$$

for all $x, y\in L$. Now let $z_1, z_2\in L$ be such that $z_1\leq z_2$. Then

$$\begin{aligned}\mu_{A\oplus B}(z_1) &= \bigvee_{z_1\leq x_1+y_1}[\mu_A(x_1)\wedge\mu_B(y_1)]\\ &\geq \bigvee_{z_2\leq x_2+y_2}[\mu_A(x_2)\wedge\mu_B(y_2)]=\mu_{A\oplus B}(z_2),\end{aligned}$$

$$\begin{aligned}\gamma_{A\oplus B}(z_1) &= \bigwedge_{z_1\leq x_1+y_1}[\gamma_A(x_1)\vee\gamma_B(y_1)]\\ &\leq \bigwedge_{z_2\leq x_2+y_2}[\gamma_A(x_2)\vee\gamma_B(y_2)]=\gamma_{A\oplus B}(z_2).\end{aligned}$$

Consequently $A \oplus B = \langle L, \mu_{A\oplus B}, \gamma_{A\oplus B}\rangle$ is an intuitionistic fuzzy ideal of L. Let $C = \langle L, \mu_C, \gamma_C\rangle$ be an intuitionistic fuzzy ideal of L containing $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle L, \mu_B, \gamma_B\rangle$. For every $z \in L$ we get

$$\mu_{A\oplus B}(z) = \bigvee_{z\leq x+y} [\mu_A(x)\wedge\mu_B(y)] \leq \bigvee_{z\leq x+y} [\mu_C(x)\wedge\mu_C(y)] = \mu_{C\oplus C}(z) = \mu_C(z),$$

$$\gamma_{A\oplus B}(z) = \bigwedge_{z\leq x+y} [\gamma_A(x)\vee\gamma_B(y)] \geq \bigvee_{z\leq x+y} [\gamma_C(x)\vee\gamma_C(y)] = \gamma_{C\oplus C}(z) = \gamma_C(z),$$

and so $A \oplus B = \langle L, \mu_{A\oplus B}, \gamma_{A\oplus B}\rangle$ is contained in $C = \langle L, \mu_C, \gamma_C\rangle$. By means of Lemma 5.4, $A\oplus B = \langle L, \mu_{A\oplus B}, \gamma_{A\oplus B}\rangle$ contains $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle L, \mu_B, \gamma_B\rangle$. This completes the proof. □

Theorem 5.11. *If $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle L, \mu_B, \gamma_B\rangle$ are intuitionistic fuzzy filters of L with the same intuitionistic upper bound, then $A \oplus B = \langle L, \mu_{A\oplus B}, \gamma_{A\oplus B}\rangle$ is an intuitionistic fuzzy filter of L generated by $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle L, \mu_B, \gamma_B\rangle$, that is, it is the least intuitionistic fuzzy filter containing $A = \langle L, \mu_A, \gamma_A\rangle$ and $B = \langle L, \mu_B, \gamma_B\rangle$.*

PROOF. The proof is similar to the proof of Theorem 5.10. □

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