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DISCRETE SIMULTANEOUS ℓ_1^m-APPROXIMATION

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Abstract. The aim of this work is to generalize L_1−approximation in order to apply them to a discrete approximation. In L_1−approximation, we use the norm given by

$$||f||_1 = \int |f| d\mu$$

where μ a non-atomic positive measure. In this paper, we go to the other extreme and consider measure μ which is purely atomic. In fact we shall assume that μ has exactly m atoms.

For any ℓ-tuple $b^1, \cdots, b^\ell \in \mathbb{R}^m$, we defined the $\ell_1^m(w)$-norn , and consider $s^* \in S$ such that, for any $b^1, \cdots, b^\ell \in \mathbb{R}^m$,

$$\min_{s \in S} \max_{1 \le i \le \ell} ||b^i - s||_w,$$

where S is a n-dimensional subspace of $\mathbb{R}^m$. The s^* is called the Chebyshev center or a discrete simultaneous ℓ_1^m-approximation from the finite dimensional subspace.

Introduction

Let $W = \{w | w = (w_1, \cdots, w_m), w_i > 0, i = 1, \cdots, m\}$. We say that $w \in W$ is a weight. On $\mathbb{R}^m$ we define the $\ell_1^m(w)$-norm given by

$$||x||_w = \sum_{i=1}^{m} |x_i| w_i,$$

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where $x = (x_1, \cdots, x_m) \in \mathbb{R}^m$. In this paper we study the theoretical problem of best simultaneous ℓ_1^m-approximation from a finite dimensional subspace.

Let S be an n-dimensional subspace of $\mathbb{R}^m$. Given $w \in W$ and $B = \{b^1, \cdots, b^\ell\} \subset \mathbb{R}^m$, we consider

$$\min_{s \in S} \max_{1 \le i \le \ell} ||b^i - s||_w := d(B, S).$$

Each $s^* \in S$ attaining the minimum is called a discrete simultaneous ℓ_1^m-approximation to $\{b^1, \cdots, b^\ell\}$. Moreover $Ball(s^*, d(B, S))$ with center s^* and radius $d(B, S)$ is the smallest ball containing the compact set B, and so s^* is called the Chebyshev center for B.

In the previous papers, we considered a one-sided L_1-approximation for a element [8] and a one-sided best simultaneous L_1-approximation for an ℓ-tuple, a compact or a bounded set where the norm was given by

$$||f||_1 = \int_X |f| d\mu$$

with μ a non-atomic positive measure.

In this paper we go to the other extreme and consider measure μ which are purely atomic. In fact we shall assume that μ has exactly m atoms. This corresponds to approximation in the normed linear space $\mathbb{R}^m$. Moreover, Pinkus[8] showed many characterizations for a discrete ℓ_1^m-approximation to $B = \{b\}$. But we study about a discrete simultaneous ℓ_1^m-approximation to a bounded set B, thus we show results which extend some of the earlier work by A.M.Pinkus.

Let $conv\{b^1, \cdots, b^\ell\}$ be the smallest convex set containing $b^1, \cdots, b^\ell \in \mathbb{R}^m$. Then, for any $s \in S$, we have

$$\begin{aligned} \max_{i=1,\cdots,\ell} ||b^i - s||_w &\leq \max_{\text{conv}\{b^1,\cdots,b^\ell\}} ||\sum_{i=1}^{\ell} a_i b^i - s||_w \\ &\leq \max_{\text{conv}\{b^1,\cdots,b^\ell\}} \sum_{i=1}^{\ell} ||a_i b^i - s||_w \\ &\leq \max_{i=1,\cdots,\ell} ||b^i - s||_w, \end{aligned}$$

where $\sum_{i=1}^{\ell} a_i = 1, \ a_i \geq 0$.

Remark 0.0.1. In the above inequalities, we can show that s^* is a discrete simultaneous ℓ_1^m-approximation for $\{b^1, \cdots, b^\ell\}$ from S if and only if s^* is a discrete simultaneous ℓ_1^m-approximation for $conv\{b^1, \cdots, b^\ell\}$ from S.

Now let us consider the existence of a discrete simultaneous ℓ_1^m-approximation. We know the following lemma.

Lemma 0.0.2. [8] *Suppose S is a finite-dimensional subspace of a normed linear space X. Then, for any compact subset $B \subset X$, there exists a best simultaneous approximation from S.*

Since $dimS$=n, we have $S = \text{span}\{s^1, \cdots, s^n\}$ for some linearly independent $s^i \in \mathbb{R}^m, i = 1, \cdots, n$. If we let A denote $m \times n$ matrix whose j-th column is the vector s^j and B denote $m \times \ell$ matrix whose j-th column is the vector b^j then we have the following form

$$\min_{\alpha \in \mathbb{R}^n} \max_a \{||Ba - A\alpha||_w \mid a = (a_1, \cdots, a_\ell), a_i \geq 0, \sum_{i=1}^{\ell} a_i = 1\}.$$

1. Characterization

We present a characterization theorem. The theorem is based on the one-sided Gateaux derivatives.

Lemma 1.0.3. [8] *Let x, y on a normed linear space X and let*

$$r(t) = \frac{||x + ty|| - ||x||}{t}$$

Then, on $(0, \infty)$, $r(t)$ is a non-decreasing function of t and is bounded below.

For any x, y in a normed linear space X, let

$$\tau_+(x, y) = \lim_{t \to 0^+} \frac{||x + ty|| - ||x||}{t}.$$

Theorem 1.0.4. [8] *Let S be a linear subspace of a normed linear space X and $x \in X \backslash S$. Then s^* is a best approximation to x if and only if $\tau_+(x - s^*, s) \geq 0$ for all $s \in S$.*

On the basis of lemma and theorem, we will consider a discrete simultaneous ℓ_1^m-approximation. So we have a remark.

Remark 1.0.5. For any $x, y \in \mathbb{R}^m$, $r(t)_w$ is denoted by

$$r(t)_w = \frac{||x + ty||_w - ||x||_w}{t}.$$

Then $r(t)_w$ is a non-decreasing function of t and is bounded below on $(0, \infty)$.

For any $x, y \in \mathbb{R}^m$, let

$$\tau_+(x, y)_w = \lim_{t \to 0^+} \frac{||x + ty||_w - ||x||_w}{t}.$$

The next result is the Theorem 1.5 in [7].

Theorem 1.0.6. *Suppose that $B = conv\{b^1, \cdots, b^\ell\} \subset \mathbb{R}^m \backslash S$. Then $s^* \in S$ is a discrete simultaneous ℓ_1^m-approximation to B if and only if there exist $b_1, \cdots, b_p$ in B and positive real numbers $\lambda_1, \cdots, \lambda_p$ with $\sum_{i=1}^p \lambda_i = 1$ for some $1 \leq p \leq n+1$ such that*
(1) $||b_i - s^*||_w = \min_{s \in S} \max_{b^i} ||b^i - s||_w$.
(2) $\sum_{i=1}^p \lambda_i ||b_i - s^*||_w \leq \sum_{i=1}^p \lambda_i ||b_i - s||_w$ for any $s \in S$.

Define a set $S^\perp(s^*)$ by

$$S^\perp(s^*) = \{x \in \mathbb{R}^m | \tau_+(x - s^*, s)_w \geq 0 \quad \text{for every} s \in S\}$$

and

$$P_S(x) = \{s \in S | \ ||x - s||_w = d(x, S)\}.$$

Our first characterization theorem now follows.

Theorem 1.0.7. *Suppose that $B = conv\{b^1, \cdots, b^\ell\} \subset \mathbb{R}^m \backslash S$. If $s^* \in S$ is a discrete simultaneous ℓ_1^m-approximation to B then there exists b^* in B such that $\tau_+(b^* - s^*, s)_w \geq 0$ for every $s \in S$.*

Proof. Suppose that s^* is a discrete simultaneous ℓ_1^m-approximation to B. If $p = 1$, the theorem is trivial, by Theorem 1.0.6. Assume that $p \geq 2$. Then, for each $s \in S$, there exist $x^* \in \mathbb{R}^{m*}$ with $||x^*||_w = 1$ and an element $b^* \in B$ such that

$$\operatorname{Re} x^*(s^* - s) \geq 0 \qquad \cdots (1)$$

and $x^*(b^* - s^*) = d(B, S)$. Since s^* is a discrete simultaneous ℓ_1^m-approximation to B, $d(B, S) \geq ||b^* - s^*||_w$ and $||b^* - s^*||_w \geq x^*(b^* - s^*) = d(B, S)$ [6]. Thus,

$$x^*(b^* - s^*) = ||b^* - s^*||_w. \qquad \cdots (2)$$

By (1) and (2), s^* is a ℓ_1^m-approximation to b^* from S. By Theorem 1.0.4. $\tau_+(b^* - s^*, s)_w \geq 0$ for every $s \in S$. Hence, if $s^* \in S$ is a discrete simultaneous ℓ_1^m-approximation to B then $S^\perp(s^*) \neq \emptyset$. □

Specially, let $B = \{b\}$. Then, by the above theorem, s^* is a discrete ℓ_1^m-approximation to B if and only if $\tau_+(b - s^*, s)_w \geq 0$ for all $s \in S$. But next example shows that the converse of the Theorem 1.0.7. does not hold if B is not singleton.

Example 1.0.8. Let $S = \{(x, 5x) : x \in \mathbb{R}\}$ on $\mathbb{R}^2$ and $B = \lambda(-1, 0) + (1 - \lambda)(0, 2), \lambda \in [0, 1]$. Assume that $w = (1, 3)$. Then, $d(B, S) = \frac{11}{3}$, $s^* = (\frac{1}{6}, \frac{5}{6})$ is a discrete ℓ_1^m-approximation to B and s^* is a best ℓ_1^m-approximation for $b^* = (-\frac{7}{12}, \frac{5}{6})$. Each $b = (b_1, b_2) \in B$, $s = (\frac{b_2}{5}, b_2)$ is a discrete ℓ_1^m-approximation to b. But s^* is unique. Thus the converse of the Theorem 1.0.7. does not hold.

2. The Main Result

In this section, using the above theorems, we can immediately establish the following property of the discrete simultaneous ℓ_1^m-approximation.

Theorem 2.0.9. *Let $B = conv\{b^1, \cdots, b^\ell\} \subset \mathbb{R}^m \backslash S$ and $s^* \in S$. Then the following statements are equivalent :*

(1) There exists b^* in B such that $\tau_+(b^* - s^*, s)_w \geq 0$ for every $s \in S$ and $||b^* - s^*||_w = d(B, S)$.

(2) $s^* \in S$ is a discrete simultaneous ℓ_1^m-approximation to B.

PROOF. Suppose that there exists b^* in B such that $\tau_+(b^* - s^*, s)_w \geq 0$ for every $s \in S$ and $||b^* - s^*||_w = d(B, S)$. Then $s^* \in S$ is a ℓ_1^m-approximation for b^*. Since $||b^* - s^*||_w = d(B, S)$, $s^* \in S$ is a discrete simultaneous ℓ_1^m-approximation for B by Theorem 1.0.6.

Suppose that $s^* \in S$ is a discrete simultaneous ℓ_1^m-approximation to B. Then there exists b^* in B such that $\tau_+(b^* - s^*, s)_w \geq 0$ for every $s \in S$ and $||b^* - s^*||_w = d(B, S)$ by Theorem 1.0.7. □

By example 1.0.8, if we choose $w = (5, 1)$ then there exist many discrete simultaneous ℓ_1^m-approximation for some bounded set B, so the space S is not a unicity space.

Finally, we show some research themes. Given B and a discrete simultaneous ℓ_1^m-approximation s^* for B, when is s^* the unique discrete simultaneous ℓ_1^m-approximation for B?. Moreover, we set

$$S(b) = \{s \in S : s \geq b\}$$

and $S(B) = \cap_{b \in B} S(b)$ where by $s \geq b$ we mean that $s_i \geq b_i$ for each $i = 1, \cdots, m$. Assuming $S(B) \neq \emptyset$, we will consider the problem

$$\min_{s \in S(B)} \max_{1 \leq i \leq \ell} ||b^i - s||_w.$$

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