Honam Mathematical J. **27**(2005), No. 1, pp. 51–68

SOLUTION AND STABILITY OF A GENERAL POPOVICIU FUNCTIONAL EQUATION

Eun Hwi Lee and Sang Won Lee

Abstract. In this paper we solve a generalized Popoviciu functional equation and investigate the stability of this equation.

1. Introduction

In 1940, S. M. Ulam [See, 23] posed the following question on the stability of homomorphisms: Given a metric group $(G, +, d)$, a number $\epsilon > 0$ and a mapping $f : G \to G$ which satisfies the inequality

$$d(f(x+y), f(x)+f(y)) < \epsilon$$

for all $x, y \in X$, does there exist an automorphism $a : G \to G$ and a constant $k > 0$, depending only on G, such that for all $x \in G$

$$d(f(x), a(x)) < k\epsilon?$$

This question became a source of the stability theory in the spirit of Hyers-Ulam.

The case of approximately additive mappings was solved by D. H. Hyers [1] under the assumption that X and Y are Banach spaces. In 1978, Th. M. Rassias [15] generalized the result of Hyers as follows:

Let $f : X \to Y$ be a mapping between Banach spaces and let $0 \leq p < 1$ be fixed. If f satisfies the inequality

$$||f(x+y) - f(x) - f(y)|| \leq \theta(||x||^p + ||y||^p)$$

Received September 13, 2004. Accepted February 7, 2005.

2000 Mathematics Subject Classification : 39B72.

Key words and phrases : stability, popoviviu functional equation.

for some $\theta \geq 0$ and for all $x, y \in X$, then there exists a unique additive mapping $A : X \to Y$ such that

$$||A(x) - f(x)|| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$. If, in addition, $f(tx)$ is continuous in t for each fixed $x \in X$, then A is linear.

Later, many Rassias type theorems concerning the stability of different functional equations were obtained by numerous authors (see, for instance, [1-23]).

E. H. Lee, Y. W. Lee and S. H. Park [10] solved Popoviciu functional equation

$$\begin{aligned} &3f\left(\frac{x-y-z}{3}\right) + f(x) + f(y) + f(z) \\ &= 2\left[f\left(\frac{x-y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{x-z}{2}\right)\right] \end{aligned}$$

and proved the stability of this equation.

In this paper we deal with the generalized Popoviciu functional equation

$$\begin{aligned} &mf\left(\frac{x-y-z}{3}\right) + f(x) + f(y) + f(z) \\ &= n\left[f\left(\frac{x-y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{x-z}{2}\right)\right]. \end{aligned} \tag{1.1}$$

where $m, n \in R - \{0\}$. In section 2 in this paper we solve the functional equation (1.1). In section 3 we investigate the stability of the functional equation (1.1).

2. Solution of the functional equation (1.1)

It is well known that if X and Y are real linear spaces, then a function $f : X \to Y$ is a solution of the Jensen functional equation $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$ if and only if there exist an element $B \in Y$ and an additive mapping $A : X \to Y$ such that $f(x) = A(x) + B$ for all $x \in X$. Assume that $m = 3(n-1)$, where $m, n \in R - \{0\}$.. Given a mapping $f : X \to Y$, consider the following equation

$$\begin{aligned} Df(x,y,z) &:= mf\left(\frac{x-y-z}{3}\right) + f(x) + f(y) + f(z) \\ &\quad - n\left[f\left(\frac{x-y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{x-z}{2}\right)\right] \\ &= 0 \end{aligned} \tag{2.1}$$

for all $x, y, z \in X$.

Lemma 2.1. *Let X and Y be real linear spaces. A function $f : X \to Y$ satisfies $Df(x,y,z) = 0$ and $f(-x) = -f(x)$ for all $x, y, z \in X$ if and only if f is additive. Furthermore $f(x) = 0$ for all $x \in X$ when $n \neq 2$.*

Proof. *Necessity.* Note that $f(0) = 0$ since $f(-x) = -f(x)$ for all $x \in X$.

(I) If $n = 0$, then $m = -3$. From (2.1) we get

$$3f\left(\frac{x-y-z}{3}\right) = f(x) + f(y) + f(z) \tag{2.2}$$

for all $x, y, z \in X$. Putting $z = -y$ in (2.2) we have

$$3f\left(\frac{x}{3}\right) = f(x) \tag{2.3}$$

for all $x \in X$. Putting $x = z = 0$ in (2.2) and applying (2.3) we have $f(y) = 0$ for all $y \in X$. Thus f is additive.

(II) If $n = 1$, then $m = 0$. From (2.1) we get

$$f(x) + f(y) + f(z) = f\left(\frac{x-y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{x-z}{2}\right) \tag{2.4}$$

for all $x, y, z \in X$. Putting $z = y$ in (2.4) we have $f(x) + f(y) = 2f\left(\frac{x-y}{2}\right)$ for all $x, y \in X$. Since $f(0) = 0$, $f(x) = 0$ for all $x \in X$. Thus f is additive.

(III) The case $0 \neq n \neq 1$. Putting $z = -y$ in (2.1) we have

$$mf\left(\frac{x}{3}\right) + f(x) = n\left[f\left(\frac{x-y}{2}\right) + f\left(\frac{x+y}{2}\right)\right] \tag{2.5}$$

for all $x, y \in X$. Replacing y by x in (2.5) and using $m = 3(n-1)$ we deduce

$$3f(x) = f(3x) \tag{2.6}$$

for all $x \in X$. Replacing x by $3y$ in (2.5) and using (2.6) we get

$$f(2y) = 2f(y) \tag{2.7}$$

for all $y \in X$. Putting $z = x - y$ in (2.1) we get

$$f(x) + f(y) + f(x-y) = n\left[f\left(\frac{x-y}{2}\right) + f\left(\frac{x}{2}\right) + f\left(\frac{y}{2}\right)\right] \tag{2.8}$$

for all $x, y \in X$. From (2.7) and (2.8) we have $f(x) = 0$) for all $x \in X$ when $n \neq 2$. Assume that $n = 2$. Then $m = 3$. From (2.5) we get

$$3f\left(\frac{x}{3}\right) + f(x) = 2\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right)\right] \tag{2.9}$$

for all $x, y \in X$. From (2.6), (2.7) and (2.9) we deduce

$$f(2x) = f(x+y) + f(x-y) \tag{2.10}$$

for all $x, y \in X$. Replacing $x + y$ by z and $x - y$ by w in (2.10) we have $f(z+w) = f(z) + f(w)$ for all $z, w \in X$. *Sufficiency.* This is obvious. □

Lemma 2.2. *Let X and Y be real linear spaces. If a function $f : X \to Y$ satisfies $Df(x,y,z) = 0$, $f(0) = 0$ and $f(-x) = f(x)$ for all $x, y, z \in X$ if and only if f is quadratic. Furthermore, $f(x) = 0$ for all $x \in X$ when $n \neq 4$.*

Proof. We claim that $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in X$. Putting $z = x$ in (2.1) we have

$$mf\left(\frac{y}{3}\right) + 2f(x) + f(y) = n\left[f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right)\right] \tag{2.11}$$

for all $x, y \in X$. Putting $y = 0$ and $x = 2x$ in (2.11) we get

$$f(2x) = nf(x) \tag{2.12}$$

for all $x \in X$. Putting $x = 0$ in (2.11) and using (2.12) we obtain

$$mf\left(\frac{y}{3}\right) + f(y) = 2f(y) \tag{2.13}$$

for all $y \in X$. From (2.11), (2.12) and (2.13) we deduce

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2.14}$$

for all $x, y \in X$. From (2.12) and (2.14) we deduce $f(x) = 0$ for all $x \in X$ if $n \neq 4$. □

Theorem 2.3. *Let X and Y be real linear spaces. If a function $f : X \to Y$ satisfies (2.1) for all $x, y, z \in X$, then there exist an element $B \in Y$, an additive mapping $A : X \to Y$, and a quadratic function $Q : X \to Y$ such that*

$$f(x) = Q(x) + A(x) + B \tag{2.15}$$

for all $x \in X$, where $A(x) = 0$ if $n \neq 2$ and $Q(x) = 0$ if $n \neq 4$.

Proof. Let $A(x) := \frac{1}{2}(f(x) - f(-x))$ for all $x \in X$. Then $A(-x) = -A(x)$ and $DA(x, y, z) = 0$ for all $x, y, z \in X$. By Lemma 1, A is additive. Let $B := f(0)$ and $Q(x) := \frac{1}{2}(f(x) + f(-x)) - f(0)$ for all $x \in X$. Then $Q(0) = 0$, $Q(-x) = Q(x)$ and $DQ(x, y, z) = 0$ for all $x, y, z \in X$. By Lemma 2, Q is quadratic. Clearly, we have $f(x) = Q(x) + A(x) + B$ for all $x \in X$. □

3. Stability of the functional equation (1.1)

Let R_+ denote the set of nonnegative real numbers. Recall that a function $H : R_+ \times R_+ \times R_+ \to R_+$ is homogeneous of degree $p > 0$ if it satisfies $H(tu, tv, tw) = t^p H(u, v, w)$ for all nonnegative real numbers

t, u, v and w. Throughout this section X and Y will be a real normed linear space and a real Banach space, respectively. We may assume that H is monotonically increasing in each variable and homogeneous of degree p. Assume that $m = 3(n-1)$, where $n, m \in R - \{0\}$. If $n = 1$. Then $m = 0$ and

$$Df(x,y,z) = f(x)+f(y)+f(z)-\left[f\left(\frac{x-y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{x-z}{2}\right)\right]$$

for all $x, y, z \in X$.

Theorem 3.1. *Assume that $n = 1$, $\delta \geq 0$ and $p \in (0, \infty) \setminus \{1\}$. Let a function $f : X \to Y$ satisfy the following inequality*

$$||Df(x,y,z)|| \leq \delta + H(||x||, ||y||, ||z||) \tag{3.1}$$

for all $x, y, z \in X$. Furthermore, assume $f(0) = 0$ and $\delta = 0$ in (3.1) for the case of $p > 1$. Then there exists a unique zero additive mapping $A : X \to Y$ such that

$$||A(x) - f(x) + f(0)|| \leq \delta + H(||x||, 0, 0)\frac{1}{2^{1-p}-1} \quad (for\ p < 1)$$

and

$$||A(x) - f(x)|| \leq H(||x||, 0, 0)\frac{1}{1-2^{1-p}} \quad (for\ p > 1)$$

for all $x \in X$.

Proof. Putting $z = y$ in (3.1) we get

$$||Df(x,y,y)|| = ||f(x) + f(y) - 2f\left(\frac{x-y}{2}\right)|| \leq \delta + H(||x||, ||y||, ||y||)$$

for all $x, y \in X$. Letting $y = 0$, we have

$$||f(x) - 2f(\frac{x}{2})|| \leq \delta + H(||x||, 0, 0)$$

for all $x \in X$. The other proof is similar to Hyers-Ulam-Rassias stability Theorem in [6]. □

Theorem 3.2. *Assume that $n \neq 1$, $\delta \geq 0$ and $0 < p < 1$. If a function $f : X \to Y$ satisfies*

$$||Df(x,y,z)|| \leq \delta + H(||x||, ||y||, ||z||) \tag{3.2}$$

for all $x, y, z \in X$, then there exist a unique quadratic function $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$||f(x) - Q(x) - A(x) - f(0)|| \leq \alpha + \beta H(||x||, ||x||, ||x||), \tag{3.3}$$

$$\left|\left|\frac{f(x) + f(-x)}{2} - f(0) - Q(x)\right|\right| \leq \alpha_1 + \beta_1 H(||x||, ||x||, ||x||), \tag{3.4}$$

and

$$\left|\left|\frac{f(x) - f(-x)}{2} - A(x)\right|\right| \leq \alpha_2 + \beta_2 H(||x||, ||x||, ||x||) \tag{3.5}$$

for all $x \in X$ where $\alpha_1 = \frac{7\delta}{6}$, $\alpha_2 = \frac{3\delta}{2m}$, $\beta_1 = \frac{2^{p-1}+3}{4-2^p}$, $\beta_2 = \frac{3}{m(3^{1-p}-1)}$, $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$.

Proof. Let $f_a(x) := \frac{f(x)-f(-x)}{2}$ and $f_q(x) := \frac{f(x)+f(-x)}{2} - f(0)$ for all $x \in X$. Then $f_a(0) = f_q(0) = 0$, $f_a(-x) = -f_a(x)$, $f_q(-x) = f_q(x)$,

$$||Df_q(x,y,z)|| \leq \delta + H(||x||, ||y||, ||z||), \tag{3.6}$$

and

$$||Df_a(x,y,z)|| \leq \delta + H(||x||, ||y||, ||z||) \tag{3.7}$$

for all $x, y, z \in X$. Putting $y = x$ and $z = -x$ in (3.7) we get

$$\left|\left|mf_a\left(\frac{x}{3}\right) + f_a(x) - nf_a(x)\right|\right| \leq \delta + H(||x||, ||x||, ||x||) \tag{3.8}$$

for all $x \in X$. Replacing x by $3x$ in (3.8) and dividing its result by m we get

$$\left|\left|f_a(x) - \frac{f_a(3x)}{3}\right|\right| \leq \frac{1}{m}(\delta + 3^p H(||x||, ||x||, ||x||)) \tag{3.9}$$

for all $x \in X$ since $m = 3(n-1)$. Using (3.9) we have

(3.10)

$$\left|\left|\frac{f_a(3^n x)}{3^n} - \frac{f_a(3^{n+1}x)}{3^{n+1}}\right|\right| = \frac{1}{3^n}\left|\left|f_a(3^n x) - \frac{f_a(3\cdot 3^n x)}{3}\right|\right| \leq \frac{1}{m}(3^{-n}\delta + 3^{n(p-1)}3^p H(||x||,||x||,||x||))$$

for all $x \in X$ and for all positive integers n. From(3.10) we have

$$\left|\left|\frac{f_a(3^m x)}{3^m} - \frac{f_a(3^n x)}{3^n}\right|\right| \leq \frac{1}{m}\left(\frac{\delta}{3^m}\sum_{k=0}^{n-m-1} 3^{-k} + 3^{(p-1)m}3^p H(||x||,||x||,||x||)\sum_{k=0}^{n-m-1} 3^{(p-1)k}\right)$$

for all $x \in X$ and for all positive integers m and n with $m < n$. This shows that $\left\{\frac{f_a(3^n x)}{3^n}\right\}$ is a Cauchy sequence for all $x \in X$. Consequently, we can define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} \frac{f_a(3^n x)}{3^n} \tag{3.11}$$

for all $x \in X$. Since $f_a(-x) = -f_a(x)$ for all $x \in X$, we have $A(-x) = -A(x)$ for all $x \in X$. Also, we get

$$\begin{aligned}||DA(x,y,z)|| &= \lim_{n\to\infty} 3^{-n}||Df_a(3^n x, 3^n y, 3^n z)|| \\ &\leq \lim_{n\to\infty}(3^{-n}\delta + 3^{(p-1)n}H(||x||,||y||,||z||)) \\ &= 0\end{aligned}$$

for all $x, y, z \in X$. By Lemma 1, it follows that A is additive.

From (3.9) and (3.10), we have

(3.12)

$$\begin{aligned}\left|\left|\frac{f_a(3^n x)}{3^n} - f_a(x)\right|\right| &\leq \frac{1}{m}\left(\delta\sum_{k=0}^{n-1}3^{-k} + 3^p H(||x||,||x||,||x||)\sum_{k=0}^{n-1}3^{(p-1)k}\right) \\ &\leq \frac{3\delta}{2m} + \frac{3}{m(3^{1-p}-1)}H(||x||,||x||,||x||)\end{aligned}$$

for all $x \in X$ and for all positive integers n. Taking limit (3.12) in as $n \to \infty$, we get (3.5).

Now, let $A' : X \to Y$ be another additive mapping satisfying (3.5). Then we have

$$\begin{aligned}||A(x) - A'(x)|| &= 3^{-n}||A(3^n x) - A'(3^n x)|| \\ &\leq 3^{-n}(||A(3^n x) - f_a(3^n x)|| + ||A'(3^n x) - f_a(3^n x)||) \\ &\leq \frac{3\delta}{3^n m} + \frac{6}{m(3^{1-p}-1)} 3^{(p-1)n} H(||x||, ||x||, ||x||)\end{aligned}$$

for all $x \in X$ and all positive integers n. Since

$$\lim_{n\to\infty}\left(\frac{3\delta}{3^n m} + \frac{6}{m(3^{1-p}-1)} 3^{(p-1)n} H(||x||, ||x||, ||x||)\right) = 0,$$

we can conclude that $A(x) = A'(x)$ for all $x \in X$.

Putting $z = x$ in (3.6) we get

$$\left|\left|mf_q\left(\frac{y}{3}\right) + 2f_q(x) + f_q(y) - n\left[f_q\left(\frac{x+y}{2}\right) + f_q\left(\frac{y-x}{2}\right)\right]\right|\right| \leq \delta + H(||x||, ||y||, ||x||) \tag{3.13}$$

for all $x, y \in X$. Putting $y = 0$ in (3.13) we get

$$\left|\left|2f_q(x) - 2nf_q\left(\frac{x}{2}\right)\right|\right| \leq \delta + H(||x||, 0, ||x||) \tag{3.14}$$

for all $x \in X$. Replacing y by x in (3.13) we get

$$\left|\left|mf_q\left(\frac{x}{3}\right) + 3f_q(x) - nf_q(x)\right|\right| \leq \delta + H(||x||, ||x||, ||x||) \tag{3.15}$$

for all $x \in X$. Putting $x = 0$ and $y = x$ in (3.13) we get

$$\left|\left|mf_q\left(\frac{x}{3}\right) + f_q(x) - 2nf_q\left(\frac{x}{2}\right)\right|\right| \leq \delta + H(0, ||x||, 0) \tag{3.16}$$

for all $x \in X$. From (3.14) and (3.16) we have

$$\left|\left|mf_q\left(\frac{x}{3}\right) - f_q(x)\right|\right| \leq 2\delta + H(||x||, 0, ||x||) + H(0, ||x||, 0) \tag{3.17}$$

for all $x \in X$. From (3.14), (3.15) and (3.17) we deduce

$$\left|\left|f_q(x) - \frac{f_q(2x)}{4}\right|\right| \leq \frac{7}{8}\delta + 4^{-1}(2^{p-1} + 3)H(||x||, ||x||, ||x||) \tag{3.18}$$

for all $x \in X$. Using (3.18) we have

$$\begin{aligned} &\left\|\frac{f_q(2^n x)}{4^n} - \frac{f_q(2^{n+1}x)}{4^{n+1}}\right\| \\ &= \frac{1}{4^n}\left\|f_q(2^n x) - \frac{f_q(2\cdot 2^n x)}{4}\right\| \\ &\le \frac{7}{8}\delta 4^{-n} + 2^{n(p-2)}4^{-1}(2^{p-1}+3)H(\|x\|,\|x\|,\|x\|) \end{aligned} \tag{3.19}$$

for all $x \in X$ and for all positive integers n. From (3.19) we have

$$\begin{aligned} \left\|\frac{f_q(2^m x)}{4^m} - \frac{f_q(2^n x)}{4^n}\right\| &\le \frac{7}{8}\delta 4^{-m}\sum_{k=0}^{n-m-1} 4^{-k} \\ &+ 2^{m(p-2)}4^{-1}(2^{p-1}+3)H(\|x\|,\|x\|,\|x\|)\sum_{k=0}^{n-m-1} 2^{(p-2)k} \end{aligned}$$

for all $x \in X$ and for all positive integers m and n with $m < n$. This shows that $\left\{\frac{f_q(2^n x)}{4^n}\right\}$ is a Cauchy sequence for all $x \in X$. Consequently, we can define a mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{f_q(2^n x)}{4^n} \tag{3.20}$$

for all $x \in X$. Since $f_q(-x) = f_q(x)$ for all $x \in X$, we have $Q(-x) = Q(x)$ for all $x \in X$ and $Q(0) = 0$. Also, we get

$$\begin{aligned} \|DQ(x,y,z)\| &= \lim_{n\to\infty} \frac{1}{4^n}\|Df_q(2^n x, 2^n y, 2^n z)\| \\ &\le \lim_{n\to\infty}\left(\frac{\delta}{4^n} + 2^{(p-2)n}H(\|x\|,\|y\|,\|z\|)\right) \\ &= 0 \end{aligned}$$

for all $x, y, z \in X$. By Lemma 2, it follows that Q is quadratic.

From (3.18) and (3.19), we have

$$\begin{aligned} &\left\|\frac{f_q(2^n x)}{4^n} - f_q(x)\right\| \\ &\le \frac{7}{8}\delta\sum_{k=0}^{n-1} 4^{-k} + 4^{-1}(2^{p-1}+3)H(\|x\|,\|x\|,\|x\|)\sum_{k=0}^{n-1} 2^{(p-2)k} \end{aligned} \tag{3.21}$$

for all $x \in X$ and for all positive integers n. Taking limit in (3.21) as $n \to \infty$, we get (3.4).

Now, let $Q' : X \to Y$ be another quadratic mapping satisfying (3.4). Then we have

$$\begin{aligned}||Q(x) - Q'(x)|| &= 4^{-n}||Q(2^n x) - Q'(2^n x)|| \\ &\leq 4^{-n}(||Q(2^n x) - f_q(2^n x)|| + ||Q'(2^n x) - f_q(2^n x)||) \\ &\leq 4^{-n}\left(\frac{7\delta}{3} + \frac{2^p + 6}{4 - 2^p} H(||x||, ||x||, ||x||)2^{pn}\right)\end{aligned}$$

for all $x \in X$ and all positive integers n. Since

$$\lim_{n\to\infty} 4^{-n}\left(\frac{7\delta}{3} + \frac{2^p + 6}{4 - 2^p} H(||x||, ||x||, ||x||)2^{np}\right) = 0$$

we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. □

Theorem 3.3. *Assume that $n \neq 1$ and $2 < p$. If a function $f : X \to Y$ satisfies $f(0) = 0$ and*

$$||Df(x, y, z)|| \leq H(||x||, ||y||, ||z||) \tag{3.22}$$

for all $x, y, z \in X$, then there exist a unique quadratic function $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$||f(x) - Q(x) - A(x)|| \leq (k_1 + k_2)H(||x||, ||x||, ||x||), \tag{3.23}$$

$$\left|\left|\frac{f(x) + f(-x)}{2} - Q(x)\right|\right| \leq k_1 H(||x||, ||x||, ||x||), \tag{3.24}$$

and

$$\left|\left|\frac{f(x) - f(-x)}{2} - A(x)\right|\right| \leq k_2 H(||x||, ||x||, ||x||) \tag{3.25}$$

for all $x \in X$ where $k_1 = \frac{2^{p-1}+3}{2^p-4}$ and $k_2 = \frac{3}{m(1-3^{1-p})}$.

Proof. Let $f_a(x) := \frac{f(x)-f(-x)}{2}$ and $f_q(x) := \frac{f(x)+f(-x)}{2}$ for all $x \in X$. Then $f_a(0) = f_q(0) = 0$, $f_a(-x) = -f_a(x)$, $f_q(-x) = f_q(x)$,

$$||Df_q(x, y, z)|| \leq H(||x||, ||y||, ||z||), \tag{3.26}$$

and

$$\|Df_a(x,y,z)\| \le H(\|x\|,\|y\|,\|z\|) \tag{3.27}$$

for all $x,y,z \in X$. Putting $y = x$ and $z = -x$ in (3.27) we deduce

$$\left\|3f_a\left(\frac{x}{3}\right) - f_a(x)\right\| \le \frac{3}{m}H(\|x\|,\|x\|,\|x\|) \tag{3.28}$$

for all $x \in X$ since $m = 3(n-1)$. Using (3.28) we have

(3.29)

$$\begin{aligned}\left\|3^n f_a(3^{-n}x) - 3^{n+1}f_a(3^{-(n+1)}x)\right\| &= 3^n\left\|f_a(3^{-n}x) - 3f_a\left(\frac{3^{-n}x}{3}\right)\right\| \\ &\le \frac{3}{m}3^{n(1-p)}H(\|x\|,\|x\|,\|x\|)\end{aligned}$$

for all $x \in X$ and for all positive integers n. From (3.29) we have

$$\|3^m f_a(3^{-m}x) - 3^n f_a(3^{-n}x)\| \le \frac{3}{m}3^{(1-p)m}H(\|x\|,\|x\|,\|x\|)\sum_{k=0}^{n-m-1}3^{(1-p)k}$$

for all $x \in X$ and for all positive integers m and n with $m < n$. This shows that $\{3^n f_a(3^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Consequently, we can define a mapping $A : X \to Y$ by

$$A(x) := \lim_{n\to\infty} 3^n f_a(3^{-n}x) \tag{3.30}$$

for all $x \in X$. The other proof of $A(x)$ is similar to the corresponding part of the proof in Theorem 5

Putting $z = x$ in (3.26) we get

$$\begin{aligned}&\left\|mf_q\left(\frac{y}{3}\right) + 2f_q(x) + f_q(y) - n\left[f_q\left(\frac{x+y}{2}\right) + f_q\left(\frac{y-x}{2}\right)\right]\right\| \\ &\le H(\|x\|,\|y\|,\|x\|)\end{aligned} \tag{3.31}$$

for all $x,y \in X$. Putting $y = 0$ in (3.31) we get

$$\left\|2f_q(x) - 2nf_q\left(\frac{x}{2}\right)\right\| \le H(\|x\|,0,\|x\|) \tag{3.32}$$

for all $x \in X$. Putting $y = x$ in (3.31) we get

$$\left\|mf_q\left(\frac{x}{3}\right) + 3f_q(x) - nf_q(x)\right\| \le H(\|x\|,\|x\|,\|x\|) \tag{3.33}$$

for all $x \in X$. Putting $x = 0$ and $y = x$ in (3.31) we get

$$\left|\left|mf_q\left(\frac{x}{3}\right) + f_q(x) - 2nf_q\left(\frac{x}{2}\right)\right|\right| \leq H(0, ||x||, 0) \tag{3.34}$$

for all $x \in X$. From (3.32) and (3.34) we deduce

$$\left|\left|mf_q\left(\frac{x}{3}\right) - f_q(x)\right|\right| \leq H(||x||, 0, ||x||) + H(0, ||x||, 0) \tag{3.35}$$

for all $x \in X$. From (3.32), (3.33) and (3.35) we deduce

$$||f_q(x) - 4f_q(2^{-1}x)|| \leq (2^{-1} + 3 \cdot 2^{-p})H(||x||, ||x||, ||x||) \tag{3.36}$$

for all $x \in X$. Using (3.36) we have

$$\begin{aligned} &||4^n f_q(2^{-n}x) - 4^{n+1} f_q(2^{-(n+1)}x)|| \\ &= 4^n||f_q(2^{-n}x) - 4f_q(2^{-1} \cdot 2^{-n}x)|| \\ &\leq (2^{-1} + 3 \cdot 2^{-p})2^{(2-p)n}H(||x||, ||x||, ||x||) \end{aligned} \tag{3.37}$$

for all $x \in X$ and for all positive integers n. From (3.37) we have

$$\begin{aligned} &||4^m f_q(2^{-m}x) - 4^n f_q(2^{-n}x)|| \\ &\leq (2^{-1} + 3 \cdot 2^{-p})2^{(2-p)m}H(||x||, ||x||, ||x||) \sum_{k=0}^{n-m-1} 2^{(2-p)k} \end{aligned}$$

for all $x \in X$ and for all positive integers m and n with $m < n$. This shows that $\{4^n \cdot f_q(2^{-n}x)\}$ is a Cauchy sequence for all $x \in X$. Consequently, we can define a mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n \cdot f_q(2^{-n}x) \tag{3.38}$$

for all $x \in X$. The other proof of $Q(x)$ is similar to the corresponding part of the proof in Theorem 5 □

If we refer to the process of the proofs in Theorem 5 and Theorem 6, then we have the following theorem.

Theorem 3.4. *Assume that $n \neq 1$ and $1 < p < 2$. If a function $f : X \to Y$ satisfies $f(0) = 0$ and*

$$||Df(x, y, z)|| \leq H(||x||, ||y||, ||z||)$$

for all $x, y, z \in X$, then there exist a unique quadratic function $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$||f(x) - Q(x) - A(x)|| \leq (k_1 + k_2)H(||x||, ||x||, ||x||),$$

$$\left|\left|\frac{f(x) + f(-x)}{2} - Q(x)\right|\right| \leq k_1 H(||x||, ||x||, ||x||),$$

and

$$\left|\left|\frac{f(x) - f(-x)}{2} - A(x)\right|\right| \leq k_2 H(||x||, ||x||, ||x||)$$

for all $x \in X$ where $k_1 = \frac{2^{p-1}+3}{4-2^p}$ and $k_2 = \frac{3}{m(1-3^{1-p})}$

Define a function $H : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ by $H(a, b, c) = (a^p + b^p + c^p)\theta$ where $\theta \geq 0$ and $p \in (0, \infty) \setminus \{1\}$. Then H is homogeneous of degree p. Thus we have the following corollaries.

Corollary 3.5. *Assume that $n = 1$, $\delta \geq 0$, $\theta \geq 0$ and $p \in (0, \infty)\setminus\{1\}$. Let a function $f : X \to Y$ satisfy the following inequality*

$$||Df(x, y, z)|| \leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$. Furthermore, assume $f(0) = 0$ and $\delta = 0$ in (1) for the case of $p > 1$. Then there exists a unique zero additive mapping $A : X \to Y$ such that

$$||A(x) - f(x) + f(0)|| \leq \delta + \frac{1}{2^{1-p} - 1}\theta||x|| \quad (for\ p < 1)$$

and

$$||A(x) - f(x)|| \leq \frac{1}{1 - 2^{1-p}}\theta||x|| \quad (for\ p > 1)$$

for all $x \in X$.

Corollary 3.6. *Assume that $n \neq 1$, $\delta \geq 0$, $\theta \geq 0$ and $0 < p < 1$. If a function $f : X \to Y$ satisfies*

$$||Df(x,y,z)|| \leq \delta + \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$, then there exist a unique quadratic function $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$||f(x) - Q(x) - A(x) - f(0)|| \leq \alpha + 3\beta\theta||x||^p,$$

$$\left|\left|\frac{f(x)+f(-x)}{2} - f(0) - Q(x)\right|\right| \leq \alpha_1 + 3\beta_1\theta||x||^p$$

and

$$\left|\left|\frac{f(x)-f(-x)}{2} - A(x)\right|\right| \leq \alpha_2 + 3\beta_2\theta||x||^p$$

for all $x \in X$ where $\alpha_1 = \frac{7\delta}{6}$, $\alpha_2 = \frac{3\delta}{2m}$, $\beta_1 = \frac{2^{p-1}+3}{4-2^p}$, $\beta_2 = \frac{3}{m(3^{1-p}-1)}$, $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$.

Corollary 3.7. *Assume that $n \neq 1$, $\theta \geq 0$ and $2 < p$. If a function $f : X \to Y$ satisfies $f(0) = 0$ and*

$$||Df(x,y,z)|| \leq \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$, then there exist a unique quadratic function $Q : X \to Y$ and a unique additive mapping $A : X \to Y$ such that

$$||f(x) - Q(x) - A(x)|| \leq 3(k_1 + k_2)\theta||x||^p,$$

$$\left|\left|\frac{f(x)+f(-x)}{2} - Q(x)\right|\right| \leq 3k_1\theta||x||^p$$

and

$$\left|\left|\frac{f(x)-f(-x)}{2} - A(x)\right|\right| \leq 3k_2\theta||x||^p$$

for all $x \in X$ where $k_1 = \frac{2^{p-1}+3}{2^p-4}$ and $k_2 = \frac{3}{m(1-3^{1-p})}$.

Corollary 3.8. *Assume that* $n \neq 1$, $\theta \geq 0$ *and* $1 < p < 2$. *If a function* $f : X \to Y$ *satisfies* $f(0) = 0$ *and*

$$||Df(x, y, z)|| \leq \theta(||x||^p + ||y||^p + ||z||^p)$$

for all $x, y, z \in X$, *then there exist a unique quadratic function* $Q : X \to Y$ *and a unique additive mapping* $A : X \to Y$ *such that*

$$||f(x) - Q(x) - A(x)|| \leq 3(k_1 + k_2)\theta||x||^p,$$

$$\left|\left|\frac{f(x) + f(-x)}{2} - Q(x)\right|\right| \leq 3k_1\theta||x||^p$$

and

$$\left|\left|\frac{f(x) - f(-x)}{2} - A(x)\right|\right| \leq 3k_2\theta||x||^p$$

for all $x \in X$ *where* $k_1 = \frac{2^{p-1}+3}{4-2^p}$ *and* $k_2 = \frac{3}{m(1-3^{1-p})}$.

References

[1] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. 27 (1941), 222-224.

[2] D. H. Hyers, G. Isac, and Th. M. Rassisas, *Stability of Functional Equations in Several Variables*, Birkhäuser, Boston/Basel/Berlin (1998).

[3] D. H. Hyers and Th. M. Rassisas, *Approximate homomorphisms*, Aequ. Math. 44 (1992), 125-153.

[4] K. W. Jun, G. H. Kim and Y. W. Lee, *Stability of generalized gamma and beta functional equations*, Aequ. Math. 60 (2000), 15-24.

[5] S. -M. Jung, *Hyers-Ulam-Rassias stability of functional equations*, Dynam. Systems Appl.6 (1997), 541-566.

[6] S. -M. Jung, *Hyers-Ulam-Rassias stability of Jensen's equation and its appliaction*, Proc. Amer. Math. Soc. 126 (1998), 3137-3143.

[7] S. -M. Jung, *On the Hyers-Ulam stability of the functional equations that have the quadratic property*, J. Math. Anal. Appl. 222 (1998), 126-137.

[8] S. -M. Jung, *On the Hyers-Ulam -Rassias stability of a quadratic functional equation*, J. Math. Anal. Appl. 232 (1999), 384-393.

[9] S. H. Lee, *Stability of a Jensen type functional Equations with three variables*, Korean J. Comput and Appl. Math. 9 (2002), 389-399.

[10] E. H. Lee, Y. W. Lee and S. H. Park, *Stability of a Jensen functional equation with three variables*, J. Appl Math and Computing 10 (2002), 283-295.

[11] Y. H. Lee and K. W. Jun, *A generalization of the Hyers-Ulam-Rassias stability of Jensen's Equation*, J. Math. Anal. Appl. 238 (1999), 305-315.

[12] Y. W. Lee, *The stability of derivatins on Banach algebras*, Bull. Institute of Math. Academia Sinica. 28 (2000), 113-116.

[13] Y. W. Lee, *On the stability of a quadratic Jensen type functional Equation*, J. Math. Anal. Appl. 270 (2002), 590-601.

[14] Y. W. Lee and S. H. Park, *On the Hyers-Ulam-Rassias stability of a quadratic functional equation*, Korean J. Comput and Appl. Math. 9 (2002), 371-380.

[15] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978), 297-300.

[16] Th. M. Rassias, *On a problem of S. M. Ulam and the asymptotic stability of the Cauchy functional equation with applications*, in : General Inequalities 7, MFO, Oberwolfach, Birkhäuser Verlag, Basel ISNM Vol · 123 (1997), 297-309.

[17] Th. M. Rassias,*On the stability of the quadratic functional equation and its applications*, Studia, Univ. Babes-Bolyai XLIII(3) (1998), 89-124.

[18] Th. M. Rassias, *The problem of S. M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. 246 (2000), 352-378.

[19] Th. M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. 251 (2000), 264-284.

[20] Th. M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Applican. Math. 62 (2000), 23-130.

[21] Th. M. Rassias and P. Semrl,*On the behavior of mapping that do not satisfy Hyers-Ulam stability*, Proc. Amer. Math. Soc. 114 (1992), 989-993.

[22] T. Trif, *Hyers-Ulam-Rassias stability of a Jensen type functional Equation*, J. Math. Anal. Appl. 250 (2000), 579-588.

[23] S. M. Ulam, *"Problems in Modern Mathematics"*, Proc. Chap. VI, Wiley, New York 1964.

[24] S. M. Jung, *"On the Hyers-Ulam-Rassias stability of approximately additive mappings"*, J. Math. Anal. Appal. 204(1996), 221-226.

[25] Z. Gajda, *"On stability of the Cauchy equation on semigroups"*, Aequationes Math. 36(1988), 76-79.

[26] D. Găvruta, *"A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings"*,J. Math. Anal. Appl. 184(1994), 431-436.

Eun Hwi Lee and Sang Won Lee
School of Information Technology and Engineering
Jeonju University
Jeonju 560-759
E-mail address : ehl@ jj.ac.kr