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DETERMINATION OF PERMUTATION GRAPHS*

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Abstract. A permutation graph is the graph of inversions in a permutation. Here we determine whether a given labelled graph is a permutation graph or not and when a graph is a permutation graph we find the associated permutation. We also characterize all the 2-regular permutation graphs.

1. Introduction

For a given a permutation $\pi \in S_n$, the *permutation graph* $G_\pi = (V, E)$ is defined as $V = \{1, 2, \ldots, n\}$ and $E = \{(i, j) \,:\, i < j,\, \pi^{-1}(i) > \pi^{-1}(j)\}$.

A *comparability graph* is an undirected graph admitting a transitive orientation of its edges. It is well known that a graph G is a permutation graph if and only if G and its complement $\bar{G}$ are both comparability graphs [5].

Recognizing that a given graph is a particular graph in mind has been an important and interesting problem to many mathematicians and computer scientists. Recognizing interval graphs, or bipartite graphs, or planar graphs are among them. For permutation graphs, not many recognition algorithms are known. A linear time algorithm for finding a transitive orientation of a comparability graph is given in [4], so it gives

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the recognition algorithm of a permutation graph. Later, in [3], they showed that when a given graph G is not a permutation graph, they can determine G or $\bar{G}$ is not of comparability in linear time.

In this paper, we consider labelled graphs and give some conditions for a labelled graph G to be a permutation graph. The conditions depend only on G not on $\bar{G}$. When a graph G satisfies the conditions we find the corresponding permutation π so that $G = G_\pi$.

In section 2, the definition and some properties of permutation graphs are introduced. In section 3, we deal with the problem of determining if a given graph is a permutation graph. In section 4, we find all the 2-regular permutation graphs.

2. Permutation Graphs

For a permutation $\pi \in S_n$ defined as

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix},$$

we simply write this permutation as $\pi = \pi(1)\pi(2)\cdots\pi(n)$ in one line notation. The matching diagram of π is obtained by writing $1, 2, \cdots, n$ in a horizontal row and the sequence $\pi(i), i = 1, 2, \ldots, n$, in another row below it, and by drawing n straight line segments matching the points, i.e., 1 and 1, 2 and 2, and so on. The undirected graph $G_\pi = (V, E)$, $V = \{1, 2, \ldots, n\}$, constructed from the matching diagram by the manner that two vertices i and j are connected if the corresponding line segments intersect in the matching diagram of π, is called the *permutation graph* associated with π.

Equivalently, given a permutation $\pi = \pi(1)\pi(2)\cdots\pi(n) \in S_n$, the *permutation graph* $G_\pi = (V, E)$ is defined as $V = \{1, 2, \ldots, n\}$ and

$$E = \{\{i, j\} : (i - j)(\pi^{-1}(i) - \pi^{-1}(j)) < 0\}.$$

Notice that the position of the number i in the sequence π is $\pi^{-1}(i)$. Even though a permutation graph is not a directed graph, we write an edge $(i,j) \in E$ with $i < j$, i.e., as an ordered pair instead of writing $\{i,j\} \in E$ for our convenience:

$$E = \{(i,j) \,:\, i < j,\, \pi^{-1}(i) > \pi^{-1}(j)\}.$$

With this notation, E becomes a relation which is irreflexive and antisymmetric. That means G_π for any $\pi \in S_n$ is a simple graph. The edge set of the permutation graph G_π can be equivalently defined as

$$E = \{\ (\pi(j), \pi(i)) \,:\, i < j \text{ and } \pi(i) > \pi(j)\ \}.$$

Example. If $\pi = 23154$, then $G_\pi = (V, E)$, where $V = \{1,2,3,4,5\}$ and $E = \{(1,2),(1,3),(4,5)\}$. □

Some particular examples of permutation graphs are: the empty graph $\emptyset = G_{id}$, the complete graph $K_n = G_\pi$, where $\pi = n\, n-1 \cdots 1$, and the complete bipartite graph $K_{m,m} = G_\sigma$, where $\sigma = m\,(m+1) \cdots$ $\cdots (2m)\, 1\, 2 \cdots m$.

Given a permutation $\pi \in S_n$, the number of k's greater than i which is positioned before i in the sequence π is denoted by $\text{inv}(i)$. That is, $\text{inv}(i)$ is the cardinality of the set $\{\, k \,:\, i < k,\, \pi^{-1}(i) > \pi^{-1}(k)\}$. For each number $i = 1, 2, \ldots, n$, it holds that $0 \le \text{inv}(i) \le n - i$. The permutation graph $G_\pi = (V, E)$ represents the inversions of π. It is clear that

$$|E| = \sum_{i=1}^{n} \text{inv}(i).$$

Example. If $\pi = 6257143$, then $G_\pi = (V, E)$, where $V = \{1,2,3,4,5,6,7\}$ and $E = \{(1,2),(1,5),(1,6),(1,7),(2,6),(3,4),(3,5),(3,6),(3,7),(4,5),$ $(4,6),(4,7),(5,6)\}$. So $\text{inv}(1) = 4$, $\text{inv}(2) = 1$, $\text{inv}(3) = 4$, $\text{inv}(4) = 3$, $\text{inv}(5) = 1$, $\text{inv}(6) = \text{inv}(7) = 0$ and $|E| = \sum_{i=1}^{n} \text{inv}(i) = 13$. □

Given π,

$$I(\pi) = (\text{inv}(1), \text{inv}(2), \dots, \text{inv}(n))$$

is called the inversion table of π.

Proposition 2.1. [6] *Let*

$$\begin{aligned} T_n &= \{(a_1, a_2, \dots, a_n) : 0 \le a_i \le n - i\} \\ &= [0, n-1] \times [0, n-2] \times \cdots \times [0,1] \times [0,0], \end{aligned}$$

where $[0,k] = \{0, 1, \dots, k\}$. *Then* $I : S_n \longrightarrow T_n,\quad \pi \longmapsto I(\pi)$ *is a bijection.*

Due to this proposition, even when a sequence of nonnegative integers is given as an inversion table, we can associate a permutation graph.

Now we consider some of the properties of permutation graphs. From the definition of permutation graphs, it is immediate that if $\pi \in S_n$ satisfies that $\pi^{-1}(i) > \pi^{-1}(j) > \pi^{-1}(k)$ for some triple $i < j < k$, then the permutation graph G_π contains a triangle, i.e., K_3. Also if $\pi \in S_n$ satisfies that for some quadruple $i < j < k < l$, $\pi^{-1}(p) < \pi^{-1}(q)$ for $p = k, l$ and $q = i, j$, then the permutation graph G_π contains a 4-cycle.

Theorem 2.2. *Given a permutation graph* G_π, *the complement graph* $\bar{G}_\pi$ *of* G_π *is also a permutation graph. In fact,* $\bar{G}_\pi = G_\sigma$, *where* $\sigma(i) = \pi(n - i + 1)$ *for* $i = 1, 2, \dots, n$.

Proof. Given i, j, let $i = \pi(k)$ and $j = \pi(l)$ for some k, l. Suppose $\pi^{-1}(i) > \pi^{-1}(j)$, i.e., $k > l$. Then the following equalities

$$\sigma^{-1}(i) = \sigma^{-1}(\pi(k)) = \sigma^{-1}(\sigma(n-k+1)) = n - k + 1,$$

$$\sigma^{-1}(j) = \sigma^{-1}(\pi(l)) = \sigma^{-1}(\sigma(n-l+1)) = n - l + 1$$

imply $\sigma^{-1}(i) < \sigma^{-1}(j)$.

Similarly, it can be shown that if $\pi^{-1}(i) < \pi^{-1}(j)$, then $\sigma^{-1}(i) > \sigma^{-1}(j)$.

Hence, an edge (i, j) is in the graph G_π if and only if it is not in G_σ, meaning that $\bar{G}_\pi = G_\sigma$. □

Theorem 2.3. *G_π and $G_{\pi^{-1}}$ are isomorphic.*

Proof. Consider the bijection $\psi_\pi : G_\pi \to G_{\pi^{-1}}$ defined by $\psi_\pi(i) = \pi^{-1}(i)$. If (i, j), where $i < j$, is an edge of G_π then $\pi^{-1}(i) > \pi^{-1}(j)$ holds. Now $\pi^{-1}(i) > \pi^{-1}(j)$ and $\pi(\pi^{-1}(i)) = i < j = \pi(\pi^{-1}(j))$ imply that $(\pi^{-1}(i), \pi^{-1}(j))$ is an edge in $G_{\pi^{-1}}$. Hence $\psi_\pi : G_\pi \to G_{\pi^{-1}}$ is a graph isomorphism. □

3. Determination of permutation graphs

For a graph $G = (V, E)$ on n vertices, $V = [n] = \{1, 2, \ldots, n\}$, E stands for the edge set $E = \{ (i, j) : i < j,\ i \text{ is connected to } j\}$. Now we consider the following conditions on the edges of the graph:

(P1): E is transitive, i.e., if $(i, j) \in E$ and $(j, k) \in E$ holds then $(i, k) \in E$.

(P2): If $(i, k) \in E$ and $i < j < k$ for some j, then it must hold that $(i, j) \in E$ or $(j, k) \in E$.

Theorem 3.1. *If $G = (V, E)$ is a permutation graph, where E is written as a set of ordered pairs as before, then the edge set E satisfies the two conditions (P1) and (P2).*

Proof. Suppose that (i, j) and (j, k) are in E, that is, they are edges of G. Then, since G is a permutation graph, we have that $i < j < k$ and $\pi^{-1}(i) > \pi^{-1}(j) > \pi^{-1}(k)$, which means that $(i, k) \in E$.

Assume that $(i, k) \in E$, and $i < j < k$ for some j. If $(i, j) \notin E$ and $(j, k) \notin E$, then it must hold that $\pi^{-1}(i) < \pi^{-1}(j) < \pi^{-1}(k)$, which contradicts to $(i, k) \in E$. □

The above theorem provides some conditions that the edge set of a permutation graph should satisfy. In fact, a graph with edge set satisfying these conditions is a permutation graph.

Theorem 3.2. *Suppose that the edge set E, written as a set of ordered pairs of vertices, of a labelled graph $G = (V, E)$ with $V = [n]$ satisfies conditions (P1) and (P2). Then G is a permutation graph, that is, there exists a permutation $\pi \in S_n$ such that $G = G_\pi$.*

Proof. Let a given graph satisfy the conditions. We will construct a permutation by some procedure and show that the associated permutation graph is exactly the original one.

With the edge set as an input, the following algorithm produces a permutation.

Algorithm

1. Let $U_{11} = \{i \in V : (1, i) \in E\}$ and $U_{12} = V - E_{11}$, and then record

$$U_{11} \quad 1 \quad U_{12}$$

   If U_{11} or U_{12} is empty, then drop the empty set from the record.
2. Let a and b be the smallest number in the set U_{11} and U_{12}, respectively. Let $U_{a1} = \{i \in U_{11} : (a, i) \in E\}$, $U_{a2} = U_{11} - U_{a1}$ and $U_{b1} = \{j \in U_{12} : (b, j) \in E\}$, $U_{b2} = U_{12} - U_{b1}$, and then record

$$U_{a1} \quad a \quad U_{a2} \quad 1 \quad U_{b1} \quad b \quad U_{b2}$$

   If any set of the above is empty then drop it.
3. Apply the same process for each nonempty set as in (2) until no nonempty set is left.
4. The resulting numbers

$$i_1, i_2, \ldots, i_n$$

   is the desired permutation π, where $\pi(j) = i_j$.

In the above algorithm the principle is that in each set find the smallest number first and then separate this set into two sets, one, the set of vertices which are connected to the smallest one, and the other, not connected to it.

Now we prove that the permutation graph associated with the permutation π obtained from the algorithm procedure is $G_\pi = G$. For this it is enough to show if $(j,k) \in E$, then k proceeds j, and if $(p,q) \notin E$, then p proceeds q in the output of the algorithm.

From the first step of the algorithm, it is clear that all vertices i proceeds 1 if and only if $(1,i) \in E$. Suppose that $(j,k) \in E$, $1 < j < k$, and the resulting permutation is $\pi = \cdots j \cdots k \cdots$. If at some stage, $j \in U_{l1}$ and $k \in U_{l2}$ for some $l < j < k$. Then $(l,j) \in E$ and $(l,k) \notin E$. But $(j,k) \in E$ must imply that $(l,k) \in E$ by the condition (E1). This contradicts to $(l,k) \notin E$. So for all $l < j < k$, either $j,k \in U_{l1}$ or $j,k \in U_{l2}$. At some stage the set $\{j,k\}$ will be recorded and then $U_{j1} = \{k\}$ and $U_{j2} = \emptyset$. As a result, $\cdots k \cdots j \cdots$ will be recorded. This contradicts to our assumption.

Now assume that $(p,q) \notin E$, $1 < p < q$, and $\pi = \cdots q \cdots p \cdots$. If for any $r < p < q$, $q \in U_{r1}$ and $p \in U_{r2}$, then $(r,q) \in E$ and $(r,p) \notin E$. This together with $(p,q) \notin E$ contradicts to the condition (E2). So for all $r < p < q$, either $p,q \in U_{r1}$ or $p,q \in U_{r2}$. By the same argument as before there will be some stage where $U_{p1} = \emptyset$, $U_{p2} = \{q\}$ and so q should be written after p in the permutation. We have a contradiction.

□

Example. Let $G = (V,E)$ be a graph on 7 vertices and let

$$E = \{(1,2),(1,3),(1,5),(2,3),(2,5),(4,5),(6,7)\}.$$

Then E satisfies the two conditions in the theorem. Applying the algorithm, since $\{2,3,5\}$ is the set of vertices connected to 1 we get as the result of the first step,

$$\{2,3,5\} \quad 1 \quad \{4,6,7\}.$$

3 and 5 are the only vertices from the set $\{2,3,5\}$ which are connected to the smallest vertex 2 in this set, and no vertex from $\{4,6,7\}$ is connected to 4. So we obtain as a result of the second step,

$$\{3,5\} \quad 2 \quad 1 \quad 4 \quad \{6,7\}.$$

From the third step,

$$3 \ \{5\} \ 2 \ 1 \ 4 \ \{7\} \ 6.$$

Finally, we get the permutation

$$\pi : 3521476$$

It is easy to check that $G = G_\pi$. □

Since we know that when a labelled graph satisfies the conditions it is a permutation graph, we can use the inversion table to find the corresponding permutation using Proposition 2.1. For example, let a given graph be the graph in the above example. Then the inversion table is

$$I(\pi) = (\mathrm{inv}(1), \mathrm{inv}(2), \ldots, \mathrm{inv}(7)) = (3,2,0,1,0,1,0).$$

The following shows how to get the permutation corresponding to the inversion table. First write

$$7$$

Next look at 6. $\mathrm{inv}(6) = 1$ means that there is one greater number before 6. So we write

$$7 \quad 6$$

Now $\mathrm{inv}(5) = 0$. There is no greater number before 5. Write

$$5 \quad 7 \quad 6$$

inv(4) = 1 means there is one greater number before 4. So write

$$5 \quad 4 \quad 7 \quad 6$$

In this way we get the permutation

$$3\,5\,2\,1\,4\,7\,6.$$

4. 2-regular permutation graphs

Sometimes we can tell if a given labelled graph is a permutation graph or not by simply investigating the degree sequence $(d_1, d_2, \dots, d_n)$, $d_i = \deg(i)$, of the graph. For example, a graph with degree sequence $(1,2,1)$ can not be a permutation graph. But if this graph is given another labelling so that the degree sequence can be changed to $(2,1,1)$, then it could be a permutation graph.

In general as in [5], different from our definition, a graph G is called a permutation graph if G is isomorphic to G_π for some permutation π. In other words, G is regarded as a permutation graph when there is a proper labelling for G so that G with this labelling satisfies the edge set conditions. For example, if a 2-path G is labelled as $2-1-3$, then $G = G_\pi$ for $\pi = 2\,3\,1$. So any 2-path can be regarded as a permutation graph in the sense of [5].

Not all graphs are permutation graphs. Some graphs with any labelling do not satisfy the edge set conditions. For example, an n-cycle with $n \geq 5$ is not a permutation graph. We will prove this.

Lemma 4.1. *Let G_π be a permutation graph with given degree sequence $(d_1, d_2, \dots, d_n)$. Then*

(1) $\pi^{-1}(1) = d_1 + 1$.

(2) $\pi^{-1}(n) = n - d_n$.

Proof. Since $d_1 = \deg(1) = \text{inv}(1)$ and so $\pi(d_1+1) = 1$. Since $d_n = \deg(n)$ is the number of i's which appear after n in π, and so $\pi(n-d_n) = n$. □

Lemma 4.1 says that the position of 1 and n in a permutation are determined from the degrees. For example, using this knowledge we can show that there is a permutation graph with $(1,2,2,1)$ as the degree sequence. Since $\deg(1) = \deg(4) = 1$, $\pi(2) = 1$ and $\pi(3) = 4$ by the lemma. Since $\deg(2) = 2$, $\pi(4) = 2$, and then $\pi(1) = 3$. So we have $\pi = 3142$ and G_π is the path $1-3-2-4$. Certainly this graph has the given degree sequence.

Notice that $\deg(i)$ in G_π is the number of j's greater than i which is before i plus the number of k's less than i which is after i in π.

By investigating the degree sequence, we can prove the following theorems. The proofs are given in [2].

Theorem 4.2. [2] *The only permutations which determine a star on n vertices are $\pi : 2\,3 \ldots n\,1$ and $\sigma : n\,1\,2 \ldots n-1$. That means when a star is labelled so that either 1 or n as the root, it is a permutation graph.*

Theorem 4.3. [2] *There are exactly two labellings for which a path on n vertices is a permutation graph.*

A 2-regular graph means that every vertex has degree 2. Now consider the problem that which 2-regular graph can be a permutation graph.

Lemma 4.4. [1] *A permutation $\pi \in S_n$ defines a disconnected permutation graph if and only if there is a number $1 \le i \le n-1$ such that $\{1,2,\ldots,i\} = \{\pi(1),\pi(2),\ldots,\pi(i)\}$ and $\{i+1,i+2,\ldots,n\} = \{\pi(i+1),\pi(i+2),\ldots,\pi(n)\}$.*

A proof of this lemma can be found in [1].

Theorem 4.5. *An n-cycle is a permutation n-cycle if and only if it is either C_3 with $\pi = 3\,2\,1$ or C_4 with $\sigma = 3\,4\,1\,2$.*

Proof. Notice that an n-cycle is connected and every vertex has degree 2. If $n = 3$, by lemma $\pi^{-1}(1) = 3$ and $\pi^{-1}(3) = 1$. So $\pi = 3\,2\,1$ and $G_\pi = C_3$.

Let $n = 4$. Then $\pi^{-1}(1) = 3$ and $\pi^{-1}(4) = 2$. From $\deg(2) = \deg(3) = 2$, we have $\pi = 3\,4\,1\,2$ and so $G_\pi = C_4$.

If $n = 5$, $\pi^{-1}(1) = 3 = \pi^{-1}(5)$ means there is no permutation for which 5-cycle is a permutation graph.

Let $n \geq 6$. Since $\pi^{-1}(1) = 3$, both 2 and 3 cannot be before 1 in π by Lemma 4.4. If $\pi(1) = 2$, then $\deg(2) = 1$ in G_π. If $\pi(2) = 2$, since $\pi(1) \neq 3$, $\pi(1) = k$ for some $k \geq 4$, in which case, k is connected to $1, 2, 3$ and so $\deg(k) \geq 3$. Let $\pi(1) = 3$. Since $\pi(2) \neq 2$ and $\deg(2) = 2$, $\pi^{-1}(2) = 4$. Since $\pi(2) = 4$ would produce a disconnected permutation graph by Lemma 4.4, $\pi(2) = j$ for some $j \geq 5$. That means, j is connected to $1, 2, 4$, and so $\deg(j) \geq 3$. If $\pi(2) = 3$, then $\pi(1) \geq 4$ and $\deg(2) = 2$ imply $\pi(4) = 2$, in which case, $\deg(3) = 3$.

If both 2 and 3 appear after 1 in π, then $\pi(1) = l \geq 4$ and so l is connected to $1, 2, 3$, i.e., $\deg(l) \geq 3$.

Hence, there is no permutation n-cycle for $n \geq 5$. □

Here we can conclude that for $n \geq 6$, a disjoint union of C_3's and C_4's is a graph isomorphic to a permutation graph and a 2-regular graph containing an n-cycle for $n \geq 5$ with any labelling can not be a permutation graph.

For example, a 2-regular graph on 7 vertices is a permutation graph when it consists of a C_3 and a C_4. In this case, the proper labelling is the following: either C_3 is defined on the vertex set $\{1, 2, 3\}$ and C_4 is defined on $\{4, 5, 6, 7\}$, or C_4 is on $\{1, 2, 3, 4\}$ and C_3 is on $\{5, 6, 7\}$. The corresponding permutation is either $\pi = 3\,2\,1\,6\,7\,5\,4$ or $\sigma = 3\,4\,1\,2\,7\,6\,5$.

Corollary 4.6. *A permutation 2-regular graph is a disjoint union of 3-cycles and 4-cycles (with a proper labelling).*

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