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RESULTS ON HYPERK-ALGEBRAS OF ORDER 3

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Abstract. In this paper, by considering the notion of hyper$K$-algebras of order 3 (which satisfies the simple or normal condition), we state and prove some theorems which determine the relationships between (weak) hyper$K$-ideals and positive implicative hyper$K$-ideals of type 1,...,8.

1. Introduction

The study of BCK-algebras was initiated by Y. Imai and K. Iséki[4] in 1966 as a generalization of the concept of set-theoretic difference and propositional calculi. Since then a great deal of literature has been produced on the theory of BCK-algebras. In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. The hyperstructure theory (called also multialgebras)was introduced in 1934 by F. Marty [7] at the 8th congress of Scandinavian Mathematiciens. Around the 40's, several authors worked on hypergroups, especially in France and in the United States, but also in Italy, Russia and Japan. Over the following decades, many important results appeared, but above all since the 70's onwards the most luxuriant flourishing of hyperstructures has been seen. Hyperstructures have many applications to several sectors of both pure and applied sciences. In [6], Y. B. Jun et al. applied the hyperstructures to BCK-algebras, and introduced the notion of a

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hyper BCK-algebra which is a generalization of BCK-algebra. In [2] R. A. Borzooei et al. applied the hyperstructures to BCK-algebras and introduced the concept of hyper$K$-algebra which is a generalization of BCK-algebra and hyper BCK-algebra. Now, in this note we get some results on hyper$K$-algebras of order 3, as mentioned in the abstract.

2. Preliminaries

Definition 2.1.[2] Let H be a nonempty set and "$\circ$" be a hyperoperation on H, that is "$\circ$" is a function from $H \times H$ to $P^*(H) = P(H) - \{\emptyset\}$. Then H is called a *hyperK-algebra* if and only if it contains a constant "0" and satisfies the following axioms:

(HK1) $(x \circ z) \circ (y \circ z) < x \circ y$

(HK2) $(x \circ y) \circ z = (x \circ z) \circ y$

(HK3) $x < x$

(HK4) $x < y, y < x \implies x = y$

(HK5) $0 < x$,

for all $x, y, z \in H$, where $x < y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A < B$ is defined by $\exists a \in A$, $\exists b \in B$ such that $a < b$.

Note that if $A, B \subseteq H$, then by $A \circ B$ we mean the subset $\bigcup\limits_{a \in A, b \in B} a \circ b$ of H.

Theorem 2.2.[2] Let H be a hyper$K$-algebra. Then for all $x, y, z \in H$ and for all nonempty subsets A and B of H the following hold:

(i) $x \circ y < z \Leftrightarrow x \circ z < y$,

(ii) $(x \circ z) \circ (x \circ y) < y \circ z$,

(iii) $x \circ y < x$,

(iv) $A \subseteq B$ implies $A < B$,

(v) $x \in x \circ 0$,

(vi) $0 \in 0 \circ x$,

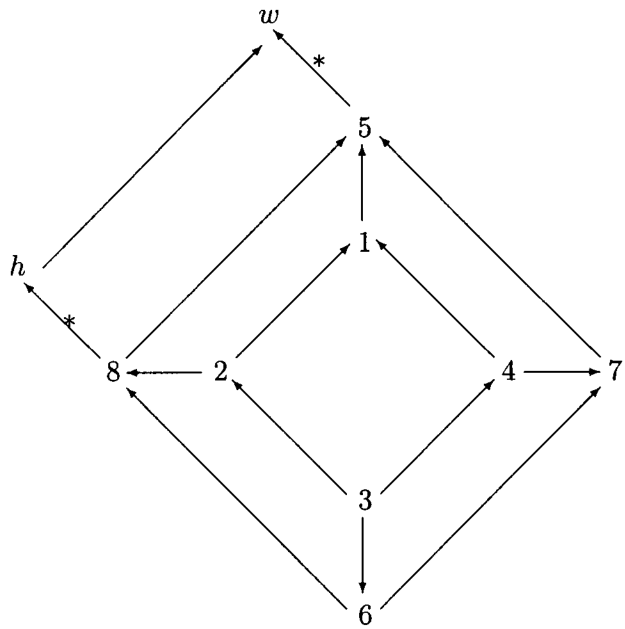
(vii) $A \subseteq A \circ 0$,

(viii) If $a \neq 0$, then $0 \notin a \circ 0$.

Definition 2.3.[1,3] Let I be a nonempty subset of hyper$K$-algebra H and $0 \in I$. Then, I is called a *weak hyperK-ideal* of H if $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$, *hyperK-ideal* of H if $x \circ y < I$ and $y \in I$ imply $x \in I$, *positive implicative hyperK-ideal of type 1* if $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply $x \circ z \subseteq I$, *positive implicative hyperK-ideal of type 2* if $(x \circ y) \circ z < I$ and $y \circ z \subseteq I$ imply $x \circ z \subseteq I$, *positive implicative hyperK-ideal of type 3* if $(x \circ y) \circ z < I$ and $y \circ z <$ imply $x \circ z \subseteq I$, *positive implicative hyperK-ideal of type 4* if $(x \circ y) \circ z \subseteq I$ and $y \circ z < I$ imply $x \circ z \subseteq I$, *positive implicative hyperK-ideal of type 5* if $(x \circ y) \circ z \subseteq I$ and $y \circ z \subseteq I$ imply $x \circ z < I$, *positive implicative hyperK-ideal of type 6* if $(x \circ y) \circ z < I$ and $y \circ z < I$ imply $x \circ z < I$, *positive implicative hyperK-ideal of type 7* if $(x \circ y) \circ z \subseteq I$ and $y \circ z < I$ imply $x \circ z < I$, *positive implicative hyperK-ideal of type 8* if $(x \circ y) \circ z < I$ and $y \circ z \subseteq I$ imply $x \circ z < I$, for all $x, y, z \in H$.

Theorem 2.4.[1] Let H be a hyper$K$-algebra. Then the following diagram shows the relationships between all of types of positive implicative hyper$K$-ideals and (weak) hyper$K$-ideals of H:

Definition 2.5. Let H be a hyper$K$-algebra. An element $a \in H$ is called a *left(resp. right) scalar* if $|a \circ x| = 1$(resp. $|x \circ a| = 1$) for all $x \in H$. If $a \in H$ is both left and right scalar, we say that a is a *scalar element.* Let I be a nonempty subset of H. Then we say that I satisfies the *closed condition* if $x < y$ and $y \in I$ implies that $x \in I$, for all $x, y \in H$.



h= hyper$K$-ideal, w= weak hyper$K$-ideal, $*$: when 0 is a scalar element of H.

3. Main results

Note. Throughout this paper we let always H be a hyper$K$-algebra of order 3 and we use the set $\{0, a, b\}$ for showing the elements of H. Moreover, we let always $I_1 = \{0, a\}$ and $I_2 = \{0, b\}$.

Theorem 3.1. Any proper positive implicative hyper$K$-ideal of type 5 is a positive implicative hyper$K$-ideal of type 7.

Proof. Let I be a proper positive implicative hyper$K$-ideal of type 5 of H. Then $I \in \{I_1, I_2\}$. Let $I = I_1$. We must prove that I_1 is a positive implicative hyper$K$-ideal of type 7. Let $(x \circ y) \circ z \subseteq I_1$ and $y \circ z < I_1$, but $x \circ z \not< I_1$, by the contrary. We claim that $x = b$. Since if $x \in \{0, a\}$ then by Theorem 2.3(iii), $x \circ z < x \in \{0, a\} = I$ which is a contradiction. Moreover, we claim that

$$b \circ z = x \circ z = \{b\} \quad , \quad (1)$$

Since if $b \circ z \cap \{0,a\} \neq \emptyset$ then $x \circ z = b \circ z < \{0,a\} = I_1$ which is a contradiction by the hypothesis. Also, $z \neq b$. Since if $z = b$ then $0 \in b \circ b = b \circ z = \{b\}$ which is impossible. Now if $y \circ z \subseteq I_1$, since I_1 is a positive implicative hyper$K$-ideal of type 5 then $x \circ z < I_1$ which is a contradiction by the hypothesis. Hence $y \circ z \not\subseteq I_1$ and so $b \in y \circ z$. Thus,

$$y \circ z = \{b\} \text{ or } \{0,b\} \text{ or } \{a,b\} \text{ or } \{0,a,b\} \quad , \quad (2)$$

Moreover, $0 \notin b \circ a$ because if $0 \in b \circ a$ then $b < a$ and so $x \circ z = b \circ z = \{b\} < a \in I_1$ which is a contradiction. Now we consider the three following cases for y;

Case 1. Let $y = 0$. By Theorem 2.3(iv), (HK2) and (1),

$$b \in b \circ 0 = b \circ y = (b \circ z) \circ y = (x \circ z) \circ y = (x \circ y) \circ z \subseteq I_1 = \{0,a\}$$

which is impossible.

Case 2. Let $y = a$. By (HK2) and (1),

$$b \circ a = b \circ y = (b \circ z) \circ y = (x \circ z) \circ y = (x \circ y) \circ z \subseteq I_1 = \{0,a\}$$

Since $0 \notin b \circ a$ then $b \circ a = \{a\}$. Now if $z = 0$, since by Theorem 2.2(viii) and (v), $0 \notin a \circ 0$ and $a \in a \circ 0$ then by (2), $a \circ 0 = y \circ z = \{a,b\}$. But in this case,

$$(b \circ a) \circ 0 = a \circ 0 = \{a,b\} \neq \{a\} = b \circ a = (b \circ z) \circ a = (b \circ 0) \circ a$$

which is a contradiction by (HK2). If $z = a$ then

$$\{b\} = b \circ z = b \circ a = b \circ y = (b \circ z) \circ y = (x \circ z) \circ y = (x \circ y) \circ z \subseteq I_1$$

which is impossible.

Case 3. Let $y = b$. If $z = 0$, since $0 \notin b \circ 0$ then by (2), $b \circ 0 = y \circ z = \{b\}$ or $\{a,b\}$. If $b \circ 0 = \{b\}$, since $\{b\} = b \circ 0 = y \circ z < I_1 = \{0,a\}$ then $b < a$ and so $0 \in b \circ a$ which is a contradiction. If $b \circ 0 = \{a,b\}$ then

$\{a,b\} = b \circ 0 = b \circ z = \{b\}$, which is impossible. If $z = a$, since $0 \notin b \circ a$ then by (2), $b \circ a = y \circ z = \{b\}$ or $\{a,b\}$ and so similar to the proof of above case we get a contradiction.

Therefore, any proper positive implicative hyper$K$-ideal of type 5 is a positive implicative hyper$K$-ideal of type 7.

Corollary 3.2. Let $I \subseteq H$. Then I is a proper positive implicative hyper$K$-ideal of type 5 if and only if I is a proper positive implicative hyper$K$-ideal of type 7.

Proof. The proof follows by Theorems 3.1 and 2.4.

Corollary 3.3 Any proper positive implicative hyper$K$-ideal of type 8 is a positive implicative hyper$K$-ideal of type 7.

Proof. The proof follows by Corollary 3.2 and Theorem 2.4.

Definition 3.4.[9] We say that H satisfies the *normal condition* if one of the conditions $a < b$ or $b < a$ holds. If no one of these conditions hold, then we say that H satisfies the *simple condition.*

Note: Clearly the conditions $a < b$ and $b < a$ can not hold simultaneously, because $a < b$ and $b < a$ imply that $a = b$ which is impossible. Clearly H always satisfies the normal condition or simple condition.

Lemma 3.5.[9] Let H satisfy the simple condition. Then

(i) $a \circ b \neq \{b\}$ and $b \circ a \neq \{a\}$.

(ii) $a \circ 0 = \{a\}$ and $b \circ 0 = \{b\}$.

Theorem 3.6. There is at least one proper positive implicative hyper$K$-ideal of type 5 in H

Proof. Let I_1 not be a positive implicative hyper$K$-ideal of type 5. Then we prove that I_2 is a positive implicative hyper$K$-ideal of type 5. The proof of the other case is similar. Since I_1 is not a positive implicative hyper$K$-ideal of type 5 then there exist $x_1, y_1, z_1 \in H$ such that $(x_1 \circ y_1) \circ z_1 \subseteq I_1$ and $y_1 \circ z_1 \subseteq I_1$ but $x_1 \circ z_1 \not< I_1$. Since $x_1 \circ z_1 \not< I_1$

then by the similar way in the proof of Theorem 3.1, we get that

$$x_1 = b, b \circ z_1 = \{b\} \text{ and } 0 \notin b \circ a \ , \quad (1)$$

Moreover $y_1 \notin \{0, b\}$ because if $y_1 = 0$ then by Theorem 2.2(vii),

$$\{b\} = b \circ z_1 \subseteq (b \circ z_1) \circ 0 = (b \circ 0) \circ z_1 = (x_1 \circ y_1) \circ z_1 \subseteq I_1 = \{0, a\}$$

which is impossible. If $y_1 = b$ then $\{b\} = b \circ z_1 = y_1 \circ z_1 \subseteq I_1 = \{0, a\}$, which is impossible. Hence $y_1 = a$. On the other hand, $z_1 \notin \{a, b\}$. Since if $z_1 = a$ then $\{b\} = b \circ z_1 = b \circ a$ and so

$$\{b\} = b \circ a = (b \circ a) \circ a = (x_1 \circ y_1) \circ z_1 \subseteq I_1 = \{0, a\}$$

which is impossible. If $z_1 = b$ then $0 \in b \circ b = b \circ z_1 = \{b\}$, which is impossible. Hence $z_1 = 0$. Since $0 \notin b \circ a$ and

$$b \circ a \subseteq (b \circ a) \circ 0 = (x_1 \circ y_1) \circ z_1 \subseteq I_1 = \{0, a\}$$

then we conclude that

$$b \circ a = \{a\} \ , \quad (2)$$

Now, we show that $x \circ z < I_2$, for all $x, z \in H$. Let there exist $x_2, z_2 \in H$ such that $x_2 \circ z_2 \nless I_2$, by the contrary. Then by the similar way in the proof of (1), we get that $0 \notin a \circ b$. Now since $0 \notin a \circ b$ and $0 \notin b \circ a$ then H satisfy the simple condition. Hence, by the Theorem 3.5(i), $b \circ a \neq \{a\}$, which is a contribution by the relation (2). Therefore, I_2 is a positive implicative hyper$K$-ideal of type 5.

Corollary 3.7. If $I_1(I_2)$ is not a positive implicative hyper$K$-ideal of type 5, then $I_2(I_1)$ is a positive implicative hyper$K$-ideal of types 5,6,7,8

Proof. Let $I_1(I_2)$ not be a positive implicative hyper$K$-ideal of type 5. Then by the proof of Theorem 3.6, $x \circ z < I_2(I_1)$ for any $x, z \in H$. Therefore, $I_2(I_1)$ is a positive implicative hyper$K$-ideal of type 5,6,7,8.

Theorem 3.8. Any proper weak hyper$K$-ideal of H is a positive implicative hyper$K$-ideal of type 5.

Proof. Let I be a proper weak hyper$K$-ideal of H. Then $I \in \{I_1, I_2\}$. Without loss of generality, let $I = I_1$. The proof of case $I = I_2$ is similar. Now, let for $x, y, z \in H$, $(x \circ y) \circ z \subseteq I_1$ and $y \circ z \subseteq I_1$ but $x \circ z \not< I_1$. Then by the similar way in the proof of Theorems 3.1 and 3.6, we get that

$$x = b, b \circ z = \{b\}, y = a \ , \quad (1)$$

Thus,

$$b \circ a = (b \circ z) \circ a = (b \circ a) \circ z = (x \circ y) \circ z \subseteq I_1$$

Since, $a \in I_1$ and I_1 is a weak hyper$K$-ideal of H then $b \in I_1$, which is impossible. Therefore, $x \circ z < I_1$ and so I_1 is a positive implicative hyper$K$-ideal of type 5.

Theorem 3.9. Let I be a proper positive implicative hyper$K$-ideal of type 5 and satisfy the closed condition. Then I is a weak hyper$K$-ideal of H.

Proof. Let I be a proper positive implicative hyper$K$-ideal of type 5. Then $I \in \{I_1, I_2\}$. Without loss of generality, let $I = I_1$. The proof of case $I = I_2$ is similar. Let $x \circ y \subseteq I_1$ and $y \in I_1$ but $x \notin I_1$, for $x, y \in H$. Since $x \notin I_1$, then $x = b$. Since $y \in I_1$ then $y = 0$ or $y = a$. If $y = 0$ then

$$b \in b \circ 0 = b \circ y = x \circ y \subseteq I_1 = \{0, a\}$$

which is impossible. Hence $y = a$. Moreover, we claim that $b \not< a$. If $b < a$, since I satisfies the closed condition and $a \in I_1$ then $b \in I_1$, which is impossible. Hence $0 \notin b \circ a = x \circ y \subseteq I_1 = \{0, a\}$ and this means that

$$b \circ a = \{a\} \ , \quad (1)$$

Also ,we claim that $a \circ 0 = \{a\}$. Let $a \circ 0 \neq \{a\}$, by the contrary. Since by Theorem 2.2, $a \in a \circ 0$ and $0 \notin a \circ 0$, then

$$a \circ 0 = \{a, b\} \ , \quad (2)$$

Moreover, since by Theorem 2.2, $b \in b \circ 0$ and $0 \notin b \circ 0$ then $b \circ 0 = \{b\}$ or $\{a, b\}$. If $b \circ 0 = \{b\}$, then by (1), (2) and (HK2),

$$\{a\} = b \circ a = (b \circ 0) \circ a = (b \circ a) \circ 0 = a \circ 0 = \{a, b\}$$

which is impossible. If $b \circ 0 = \{a, b\}$, then by (1),(2) and (HK2),

$$0 \in a \circ a \subseteq a \circ a \cup b \circ a = \{a, b\} \circ a = (b \circ 0) \circ a = (b \circ a) \circ 0 = a \circ 0 = \{a, b\}$$

which is impossible. Therefore, $a \circ 0 = \{a\}$. Now $(b \circ a) \circ 0 = a \circ 0 = \{a\} \subseteq I_1$ and $a \circ 0 = \{a\} \subseteq I_1$ but $b \circ 0 \not< I_1$ (Since if $b \circ 0 < I_1$, then there exists $c \in I_1$ such that $b \circ 0 < c$. Hence by the Theorem 2.3(i), $b \circ c < 0$ and so there exists $d \in b \circ c$ such that $d < 0$. Then by (HK4) and (HK5), $d = 0$. Hence $0 \in b \circ c$ and so $b < c$. Since I_1 satisfies the closed condition and $c \in I_1$, then we get that $b \in I_1$, which is impossible). Hence I_1 is a not positive implicative hyper$K$-ideal of type 5, which is a contradiction. Therefore I_1 is a weak hyper$K$-ideal of H.

Note. In the following example, we shows that the closed condition is necessary in Theorem 3.9.

Example 3.10. Let $H = \{0, a, b\}$. Then the following table shows a hyper$K$-algebra structure on H;

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{0, a\}$	$\{0, b\}$

It is clear that $I_1 = \{0, a\}$ is a positive implicative hyper$K$-ideal of type 5, but it is not a weak hyper$K$-ideal of H (Since $b \circ a \subseteq I_1$ and $a \in I_1$, but $b \notin I_1$). Moreover, I_1 doesn't satisfy the closed condition (Since $b < a$ and $a \in I_1$, but $b \notin I_1$).

Corollary 3.11. Let I be a nonempty subset of H and satisfy the closed condition. Then I is a proper weak hyper$K$-ideal of H if and only if I is a proper positive implicative hyper$K$-ideal of type 5(7).

Proof. The proof follows by Theorems 2.4, 3.1, 3.8 and 3.9.

Theorem 3.12. Let H satisfy the simple condition. Then I_1 and I_2 are weak hyper$K$-ideals of H.

Proof. First we proves the theorem for the case $I = I_1$, but the proof of case $I = I_2$ is similar. Let $x \circ y \subseteq I_1$ and $y \in I_1$, but $x \notin I_1$ for $x, y \in H$. Then $x = b$. Since $y \in I_1 = \{0, a\}$ then $y = 0$ or a. If $y = 0$ then $b \in b \circ 0 = x \circ y \subseteq I_1$, which is impossible. If $y = a$ then $b \circ a = x \circ y \subseteq I_1 = \{0, a\}$. Since by Lemma 3.5(i), $b \circ a \neq \{a\}$ then $b \circ a = \{0, a\}$. Since H satisfies the simple condition then $b \not< a$ and so $0 \notin b \circ a$, which is impossible. Therefore, $x \in I_1$ and so I_1 is a weak hyper$K$-ideal of H.

Note. The following example shows that the simple condition is necessary in Theorem 3.12.

Example 3.13. (i) Let $H = \{0, a, b\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, b\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

It is clear that H doesn't satisfy the simple condition, and I_2 is not a weak hyper$K$-ideal of H.

(ii) Let $H = \{0, a, b\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0, b\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0\}$

It is clear that H doesn't satisfy the simple condition, and I_1 and I_2 are not weak hyper$K$-ideals of H.

Theorem 3.14. Let H satisfy the simple condition. Then H has at least one proper positive implicative hyper$K$-ideal of type 4.

Proof. Without loss of generality, let I_1 not be a positive implicative hyper$K$-ideal of type 4. We must prove that I_2 is a positive implicative hyper$K$-ideal of type 4. By the hypothesis there exist $x, y, z \in H$ such that $(x \circ y) \circ z \subseteq I_1$ and $y \circ z < I_1$, but $x \circ z \not\subseteq I_1$. Hence $b \in x \circ z$. We claim that $y \neq 0$ because if $y = 0$ then by Theorem 2.2(v),

$$b \in x \circ z \subseteq (x \circ 0) \circ z = (x \circ y) \circ z \subseteq I_1$$

which is impossible. Now, we consider three following cases for x:

Case 1. Let $x = 0$. Then by Theorem 2.2(vi),

$$b \in x \circ z = 0 \circ z \subseteq (0 \circ y) \circ z = (x \circ y) \circ z \subseteq I_1$$

which is impossible.

Case 2. Let $x = a$. Then,

(2-1): If $z = 0$ then by Lemma 3.5(ii), $x \circ z = a \circ 0 = \{a\} \subseteq I_1$, which is a contradiction.

(2-2): If $z = a$ then we consider two cases for y. If $y = a$, since by Lemma 3.5, $b \in b \circ a$ (since $0 \notin b \circ a$ and $b \circ a \neq \{a\}$, then $b \in b \circ a$) hence

$$b \in b \circ a \subseteq (x \circ z) \circ a = (x \circ z) \circ y = (x \circ y) \circ z \subseteq I_1$$

which is impossible. If $y = b$, since by Lemma 3.5, $a \in a \circ b$ then

$$b \in x \circ z = a \circ a \subseteq (a \circ b) \circ a = (x \circ y) \circ z \subseteq I_1$$

which is impossible.

(2-3): If $z = b$ since $b \in x \circ z = a \circ b$ and by Lemma 3.5, $a \in a \circ b$ then $a \circ b = \{a, b\}$. Now, if $y = a$ then

$$b \in b \circ a \subseteq a \circ a \cup b \circ a = \{a, b\} \circ a = (a \circ b) \circ a = (a \circ a) \circ b = (x \circ y) \circ z \subseteq I_1$$

which is impossible. If $y = b$ then

$$b \in x \circ z = a \circ b \subseteq a \circ b \cup b \circ b = \{a, b\} \circ b = (a \circ b) \circ b = (x \circ y) \circ z \subseteq I_1$$

which is impossible.

Case 3. Let $x = b$. Then,

(3-1): If $z = 0$ then we consider two cases for y. If $y = a$ then by Lemma 3.5 and Theorem 2.2(vii),

$$b \in b \circ a \subseteq (b \circ a) \circ 0 = (x \circ y) \circ z \subseteq I_1$$

which is impossible. If $y = b$ then $\{b\} = b \circ 0 = y \circ z < I_1 = \{0, a\}$ and so $b < a$, which is impossible.

(3-2): If $z = a$ then we consider two cases for y. If $y = a$ since $b \in x \circ z = b \circ a$ then

$$b \in x \circ z = b \circ a \subseteq (b \circ a) \circ a = (x \circ y) \circ z \subseteq I_1$$

which is impossible.

If $y = b$ then $b \circ a = y \circ z < I_1$ and $b \circ a = x \circ z \not\subseteq I_1$. Hence $b \circ a = \{a, b\}$. Since

$$a \circ b \cup b \circ b = \{a, b\} \circ b = (b \circ a) \circ b = (b \circ b) \circ a = (x \circ y) \circ z \subseteq I_1$$

Then $a \circ b \subseteq I_1$ and $b \circ b \subseteq I_1$. Since $0 \notin a \circ b$, then $a \circ b = \{a\}$. Moreover, since $0 \in b \circ b$ then $b \circ b = \{0\}$ or $\{0, a\}$.

(3-2-1): If $b \circ b = \{0\}$ then,

$$0 \circ a = (b \circ b) \circ a = (b \circ a) \circ b = \{a, b\} \circ b = a \circ b \cup b \circ b = \{a\} \cup \{0\} = \{0, a\}$$

Moreover, since $(b \circ 0) \circ b = (b \circ b) \circ 0$ then $\{0\} = b \circ b = 0 \circ 0$. Also, since $0 \in 0 \circ b$ and $0 \in a \circ a$ then

$$0 \circ b = \{0\} \text{ or } \{0, a\} \text{ or } \{0, b\} \text{ or } \{0, a, b\}$$

$$a \circ a = \{0\} \text{ or } \{0, a\} \text{ or } \{0, b\} \text{ or } \{0, a, b\}$$

Therefore, by Lemma 3.5 and above results, we get the following table:

$\circ$	0	a	b
0	$\{0\}$	$\{0,a\}$	$\{0\}$ or $\{0,a\}$ or $\{0,b\}$ or $\{0,a,b\}$
a	$\{a\}$	$\{0\}$ or $\{0,a\}$ or $\{0,b\}$ or $\{0,a,b\}$	$\{a\}$
b	$\{b\}$	$\{a,b\}$	$\{0\}$

Now, in these cases we can check that I_2 is a positive implicative hyper$K$-ideal of type 4.

(3-2-2): If $b \circ b = \{0,a\}$then by the similar way in the proof of Case (3-2-1) and by some modification we get the following table:

$\circ$	0	a	b
0	$\{0\}$ or $\{0,a\}$	$\{0\}$ or $\{0,a\}$	$\{0\}$ or $\{0,a\}$
a	$\{a\}$	$\{0,a\}$	$\{a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,b\}$

Moreover, in these cases we can check that I_2 is a positive implicative hyper$K$-ideal of type 4.

(3-3): If $z = b$ then we consider two cases for y. If $y = a$, since $b \in x \circ z = b \circ b$ and $b \in b \circ a$, then

$$b \in b \circ b \subseteq (b \circ a) \circ b = (x \circ y) \circ z \subseteq I_1$$

which is impossible. If $y = b$ then by hypothesis $(b \circ b) \circ b = (x \circ y) \circ z \subseteq I_1$ and $b \circ b = y \circ z < I_1$ but $b \circ b = x \circ z \not\subseteq I_1$. Hence $b \circ b = \{0,a\}$ or $\{0,a,b\}$. If $b \circ b = \{0,a,b\}$ then

$$b \in b \circ b \subseteq b \circ b \cup a \circ b \cup 0 \circ b = \{0,a,b\} \circ b = (b \circ b) \circ b \subseteq I_1$$

which is impossible. Then $b \circ b = \{0,a\}$. Now, in this case,

$$a \circ b \cup 0 \circ b = \{0,a\} \circ b = (b \circ b) \circ b \subseteq I_1$$

Then $a \circ b \subseteq I_1$ and $0 \circ b \subseteq I_1$. Since $0 \notin a \circ b$ then $a \circ b = \{a\}$. Moreover, since $0 \in 0 \circ b$ then $0 \circ b = \{0\}$ or $\{0,a\}$. Now, since $b \in b \circ a$ and $0 \notin b \circ a$

then $b \circ a = \{b\}$ or $\{a, b\}$.

(3-3-1): If $b \circ a = \{b\}$ then,

$$0 \circ a \cup a \circ a = \{0, a\} \circ a = (b \circ b) \circ a = (b \circ a) \circ b = b \circ b = \{0, a\}$$

Hence, $0 \circ a \subseteq \{0, a\}$ and $a \circ a \subseteq \{0, a\}$. Since $0 \in 0 \circ a$ and $0 \in a \circ a$, then $0 \circ a = \{0\}$ or $\{0, a\}$ and $a \circ a = \{0\}$ or $\{0, a\}$. If $a \circ a = \{0\}$ then

$$0 \circ 0 = (a \circ a) \circ 0 = (a \circ 0) \circ a = a \circ a \subseteq \{0, a\}$$

Also, if $a \circ a = \{0, a\}$, then

$$0 \circ 0 \subseteq 0 \circ 0 \cup a \circ 0 = \{0, a\} \circ a = (a \circ a) \circ 0 = (a \circ 0) \circ a = a \circ a \subseteq \{0, a\}$$

Hence in both cases $0 \circ 0 \subseteq \{0, a\}$. Therefore, by Lemma 3.5 and above results we get the following table for H:

$\circ$	0	a	b
0	$\{0\}$ or $\{0, a\}$	$\{0\}$ or $\{0, a\}$	$\{0\}$ or $\{0, a\}$
a	$\{a\}$	$\{0\}$ or $\{0, a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0, a\}$

But in these cases we can check that I_1 is a positive implicative hyper$K$-ideal of type 4, which is a contradiction.

(3-3-2): If $b \circ a = \{a, b\}$ then by the similar way in the proof of Case (3-3-1) and by some modification we get the following table for H:

$\circ$	0	a	b
0	$\{0\}$ or $\{0, a\}$	$\{0\}$ or $\{0, a\}$	$\{0\}$ or $\{0, a\}$
a	$\{a\}$	$\{0\}$ or $\{0, a\}$	$\{a\}$
b	$\{b\}$	$\{a, b\}$	$\{0, a\}$

Now, in these cases we can check that I_2 is a positive implicative hyper$K$-ideal of type 4.

Therefore, we prove that I_2 is a positive implicative hyper$K$-ideal of type 4.

Note. The following example shows that the simple condition is necessary in Theorem 3.14.

Example 3.15. Let $H = \{0, a, b\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0, a\}$
b	$\{b\}$	$\{a\}$	$\{0, b\}$

It is clear that H doesn't satisfy the simple condition, and I_1 and I_2 are not positive implicative hyper$K$-ideals of type 4.

Lemma 3.16. Let H satisfy the simple condition. Then $I_1(I_2)$ is a positive implicative hyper$K$-ideal of type 6 if and only if $b \circ a = \{b\}(a \circ b = \{a\})$.

Proof.($\Rightarrow$) Let I_1 be a positive implicative hyper$K$-ideal of type 6 and on the contrary let $b \circ a \neq \{b\}$. Since H satisfies the simple condition, then $0 \notin b \circ a$. Moreover, by Lemma 3.5, $b \circ a \neq \{a\}$. Thus we have $b \circ a = \{a, b\}$. So $(b \circ a) \circ 0 = \{a, b\} \circ 0 = \{a, b\} < I_1$ and $a \circ 0 = \{a\} < I_1$. Since I_1 is of type 6, then by Lemma 3.5 we conclude that $\{b\} = b \circ 0 < I_1$ and so $b < a$, which is a contradiction. Hence $b \circ a = \{b\}$.

For the case of I_2, the proof is similar.

($\Leftarrow$) Let $b \circ a = \{b\}$. On the contrapositive we show that if there are $x, z \in H$ such that $x \circ z \not< I_1$, then for all $y \in H, y \circ z \not< I_1$ or $(x \circ y) \circ z \not< I_1$. Since $x \circ z \not< I_1$, we conclude $x \circ z = \{b\}$. Since $b \circ a = \{b\}$ and by Lemma 3.5, $b \circ 0 = \{b\}$, thus the only cases which $x \circ z \not< I_1$ are $x = b, z = a$ and $x = b, z = 0$. In the first case we show that $y \circ a \not< I_1$ or $(b \circ y) \circ a \not< I_1$, for all $y \in H$. If $y = 0$, then

$$(b \circ y) \circ a = (b \circ 0) \circ a = \{b\} \circ a = \{b\} \not< I_1$$

If $y = a$, then

$$(b \circ y) \circ a = (b \circ a) \circ a = \{b\} \circ a = \{b\} \not< I_1$$

If $y = b$, then

$$y \circ a = b \circ a = \{b\} \not< I_1$$

The proof of the second case is similar. Therefore we prove that I_1 is a positive implicative hyper$K$-ideal of type 6. By the similar way, we can prove that I_2 is a positive implicative hyper$K$-ideal of type 6.

Theorem 3.17. Let H satisfy the simple condition and $I \subseteq H$. Then the following statements are equivalent:
(i) I is a proper hyper$K$-ideal of H,
(ii) I is a proper positive implicative hyper$K$-ideal of type 6,
(iii) I is a proper positive implicative hyper$K$-ideal of type 8.

Proof. $(i) \Rightarrow (ii)$ Let $I = I_1$ be a hyper$K$-ideal of H . Then $b \circ a = \{b\}$ because in the other case we receive to a contradiction. So by Lemma 3.16, I_1 is a positive implicative hyper$K$-ideal of type 6. The proof of case I_2 is similar.

$(ii) \Rightarrow (i)$ Let $I = I_1$ be a positive implicative hyper$K$-ideal of type 6. Then by Lemmas 3.16 and 3.5, $b \circ a = \{b\}$ and $b \circ 0 = \{b\}$ and so I_1 is a hyper$K$-ideal of H .

$(ii) \Rightarrow (iii)$ By Theorem 2.4, hold.

$(iii) \Rightarrow (ii)$ Let $(x \circ y) \circ z < I_1$ and $y \circ z < I_1$, for $x, y, z \in H$. We must prove that $x \circ z < I_1$. If $y \circ z \subseteq I_1$ then by the hypothesis $x \circ z < I_1$. Now, let $y \circ z \not\subseteq I_1$. Hence $b \in y \circ z$. We consider three following cases for x.

Case 1. If $x = 0$ then $0 \in 0 \circ z = x \circ z$ and so $x \circ z < I_1$.

Case 2. If $x = a$ then we consider three cases for z. If $z = 0$ then by Lemma 3.5, $x \circ z = a \circ 0 = \{a\} < I_1$. If $z = a$ then $0 \in a \circ a = x \circ z$ and so $x \circ z < I_1$. If $z = b$, since $0 \notin a \circ b$ and by Lemma 3.5, $a \circ b \neq \{b\}$ then $a \circ b = \{a\}$ or $\{a, b\}$ and so $x \circ z = a \circ b < I_1$.

Case 3. If $x = b$ then we consider three following cases for z.
(3-1): Let $z = 0$. Since $b \in y \circ z = y \circ 0$ and by Lemma 3.5, $a \circ 0 = \{a\}$ then $y = b$ and so $\{b\} = b \circ 0 = y \circ z < I_1$, which is a contradiction.

(3-2): Let $z = a$. If $y = 0$ then by Lemma 3.5,

$$x \circ z = b \circ a = (b \circ 0) \circ a = (x \circ y) \circ z < I_1$$

If $y = a$ then $(b \circ a) \circ a = (x \circ y) \circ z < I_1$. We know that $b \circ a = \{b\}$ or $\{a, b\}$. If $b \circ a = \{b\}$ then $\{b\} = b \circ a = (b \circ a) \circ a = (x \circ y) \circ z < I_1$, which is impossible. Thus we let $b \circ a = \{a, b\}$ and so $x \circ z = b \circ a = \{a, b\} < I_1$. If $y = b$ then $x \circ z = b \circ a = y \circ z < I_1$.
(3-3): Let $z = b$. Since $0 \in b \circ b = x \circ z$ then $x \circ z < I_1$.

The proof of case I_2 is similar.

Note. The following example shows that the simple condition is necessary in Theorem 3.17.

Example 3.18. (i) Let $H = \{0, a, b\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	a	b
0	$\{0\}$	$\{0, b\}$	$\{0, a, b\}$
a	$\{a, b\}$	$\{0, a\}$	$\{0, a, b\}$
b	$\{b\}$	$\{a, b\}$	$\{0, b\}$

It is clear that H doesn't satisfy the simple condition and I_1 is a positive implicative hyper$K$-ideal of type 8, but it is not a positive implicative hyper$K$-ideal of type 6.

(ii) Let $H = \{0, a, b\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	a	b
0	$\{0, b\}$	$\{0, b\}$	$\{0, a, b\}$
a	$\{a, b\}$	$\{0, a, b\}$	$\{0, a\}$
b	$\{b\}$	$\{b\}$	$\{0, b\}$

It is clear that H doesn't satisfy the simple condition and I_2 is a positive implicative hyper$K$-ideal of type 6, but it is not a hyper$K$-ideal of H.

Theorem 3.19. Let 0 be a scalar element of H and $I \subseteq H$. Then I is a proper positive implicative hyper$K$-ideal of type 2 if and only I is a proper positive implicative hyper$K$-ideal of type 3.

Proof. ($\Longleftarrow$) The proof follows by Theorem 2.4.

($\Longrightarrow$) Let $I = I_1$ be a positive implicative hyper$K$-ideal of type 2. Since 0 is a scalar element of H then by Theorem 2.2,

$$0 \circ 0 = 0 \circ a = 0 \circ b = \{0\}, a \circ 0 = \{a\}, b \circ 0 = \{b\}$$

We know that, $0 \notin b \circ a$ or $0 \notin a \circ b$. Without loss of generality, let $0 \notin b \circ a$. First we proves that $b \notin a \circ b$ (and so $a \circ b \subseteq \{0, a\}$). Let $b \in a \circ b$, by the contrary. If $a \circ b = \{b\}$ then $0 \in 0 \circ b \subseteq (a \circ a) \circ b$ and $0 \notin b \circ a = (a \circ b) \circ a$. Therefore, $(a \circ a) \circ b \neq (a \circ b) \circ a$, which is a contradiction by (HK2). If $a \circ b \neq \{b\}$ then $a \circ b \cap \{0, a\} \neq \emptyset$. Since $b \in a \circ b$ hence

$$(a \circ 0) \circ b = a \circ b < I_1 \text{ and } 0 \circ b \subseteq I_1 \text{ but } a \circ b \not\subseteq I_1$$

which is a contradiction because I_1 is a positive implicative hyper$K$-ideal of type 2. Therefore, $b \notin a \circ b$.

Now, we prove that $b \circ a = \{b\}$. Let $b \circ a \neq \{b\}$, by the contrary. Since $0 \notin b \circ a$ then $a \in b \circ a$. But in this case,

$$(b \circ a) \circ 0 = b \circ a < I_1 \text{ and } a \circ 0 \subseteq I_1, \text{ but } b \circ 0 \not\subseteq I_1$$

which is a contradiction because I_1 is a positive implicative hyper$K$-ideal of type 2.

Therefore, by the above comments we have the following results:

$$0 \circ 0 = 0 \circ a = 0 \circ b = \{0\}, a \circ 0 = \{a\}, b \circ 0 = \{b\}, a \circ b \subseteq \{0, a\}, b \circ a = \{b\} \quad (1)$$

Now, we prove that I_1 is a positive implicative hyper$K$-ideal of type 3. Let $(x \circ y) \circ z < I_1$ and $y \circ z < I_1$, but $x \circ z \not\subseteq I_1$, by the contrary. By (1), the only cases that probably $x \circ z \not\subseteq I_1$ are $b \circ 0, b \circ a, b \circ b$ or $a \circ a$.

Case 1. Let $x = b$ and $z = 0$. Then we consider three cases for y. If $y = 0$ then $(x \circ y) \circ z = (b \circ 0) \circ 0 = \{b\} \not< I_1$, which is impossible.

If $y = a$ then $(x \circ y) \circ z = (b \circ a) \circ 0 = \{b\} \not< I_1$, which is impossible.
If $y = b$ then $y \circ z = b \circ 0 = \{b\} \not\subseteq I_1$, which is impossible.

Case 2. Let $x = b$ and $z = a$.
If $y = 0$, then $(x \circ y) \circ z = (b \circ 0) \circ a = b \circ a = \{b\} \not< I_1$, which is impossible.
If $y = a$ then $(x \circ y) \circ z = (b \circ a) \circ a = b \circ a = \{b\} \not< I_1$, which is impossible.
If $y = b$, then $y \circ z = b \circ a = \{b\} \not\subseteq I_1$, which is impossible.

Case 3. Let $x = b$ and $z = b$.
If $b \circ b \subseteq \{0, a\}$ then $x \circ z = b \circ b \subseteq I_1$, which is a contradiction by hypothesis. Now, let $b \circ b \not\subseteq \{0, a\}$. Then $b \in b \circ b$. But in this case

$$(b \circ 0) \circ b = b \circ b < I_1 \text{ and } 0 \circ b \subseteq I_1 \text{ but } b \circ b \not\subseteq I_1$$

which is a contradiction because I_1 is a positive implicative hyper$K$-ideal of type 2.

Case 4. Let $x = a$ and $z = a$.
If $a \circ a \subseteq \{0, a\}$ then $x \circ z = a \circ a \subseteq I_1$ which is a contradiction by hypothesis. Let $a \circ a \neq \{0, a\}$. Hence $b \in a \circ a$. But in this case,

$$(a \circ 0) \circ a = a \circ a < I_1 \text{ and } 0 \circ a \subseteq I_1 \text{ but } a \circ a \not\subseteq I_1$$

which is a contradiction because I_1 is a positive implicative hyper$K$-ideal of type 2.

Therefore, I_1 is a positive implicative hyper$K$-ideal of type 3. The proof of case $I = I_2$ is similar to the case $I = I_1$ by some modifications.

Note. In the following example we shows that if 0 does't satisfy the scalar element then the Theorem 3.19 is not correct in general.

Example 3.20. (i) Let $H = \{0, a, b\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	a	b
0	$\{0, a\}$	$\{0\}$	$\{0, b\}$
a	$\{a\}$	$\{0, a\}$	$\{a, b\}$
b	$\{b\}$	$\{b\}$	$\{0, a, b\}$

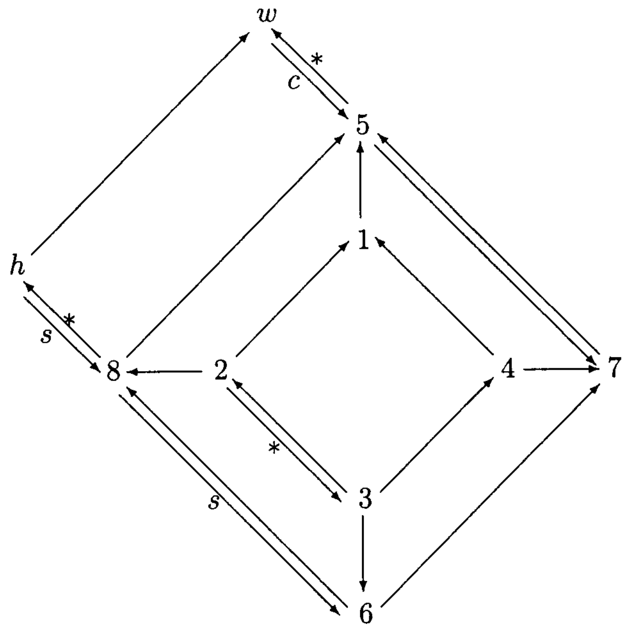
We can check that H satisfies the simple condition, but 0 is not a scalar element of H. Moreover, I_1 is a positive implicative hyper$K$-ideal of type 2 but it is not a positive implicative hyper$K$-ideal of type 3.

(ii) Let $H = \{0, a, b\}$. Then the following table shows a hyper$K$-algebra structure on H.

$\circ$	0	a	b
0	$\{0, a\}$	$\{0\}$	$\{0, b\}$
a	$\{a\}$	$\{0, a\}$	$\{0, a, b\}$
b	$\{b\}$	$\{b\}$	$\{0, a, b\}$

We can check that H satisfies the normal condition, but 0 is not a scalar element of H. Moreover, I_1 is a positive implicative hyper$K$-ideal of type 2 but it is not a positive implicative hyper$K$-ideal of type 3.

Corollary 3.21. Let H be a hyper$K$-algebra of order 3. Then, the following diagram shows the some relationships between the all of types of positive implicative hyper$K$-ideals and (weak) hyper$K$-ideals of H:



h= hyper$K$-ideal, w= weak hyper$K$-ideal, $*$: when 0 is a scalar element of H, s=with simple condition, c= with closed condition

Proof. The proof follows by Theorems 2.4, 3.1, 3.9, 3.17 and 3.19.

Example 3.22. Let $H = \{0, a, b\}$. Then the following tables shows hyper$K$-algebras structure on H.

$\circ_1$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a\}$

$\circ_2$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{a\}$
b	$\{b\}$	$\{a,b\}$	$\{0,a\}$

$\circ_3$	0	a	b
0	$\{0,a,b\}$	$\{0,a,b\}$	$\{0,a,b\}$
a	$\{a\}$	$\{0,a,b\}$	$\{0,a,b\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

In $(H, \circ_1)$, I_1 is a positive implicative hyper$K$-ideal of type 1, but it is not of type 4. In $(H, \circ_2)$, I_1 is a positive implicative hyper$K$-ideal of type 5 and 7, but it is not of types 6 and 8. In $(H, \circ_3)$, I_1 and I_2 are positive implicative hyper$K$-ideal of types 1 and 4, but they are not of types 2 and 3.

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