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ON THE STABILITY OF FUNCTIONAL EQUATIONS CONCERNED A MULTIPLICATIVE DERIVATION

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Abstract. In this paper we study the Hyers-Ulam stability and the superstability of functional equations related to a multiplicative derivation.

1. introduction

In 1940, S. M. Ulam [24] posed the following problem concerning the stability of homomorphism. We are given a group G_1 and a metric group (G_2, d). Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) \leq \delta$ for all $x,\ y \in G_1$, then a homomorphism $g : G_1 \to G_2$ exists with $d(f(x), g(x)) \leq \varepsilon$ for all $x \in G_1$?

As an answer to the problem of the Ulam problem, D. H. Hyers [3] in 1941 has proved the stability of linear functional equation, which states that if $\delta > 0$ and $f : X \to Y$ is a mapping with X, Y Banach spaces, such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \tag{1.1}$$

for all $x,\ y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \delta$$

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for all $x,\ y \in X$.

In such a case, the additive functional equation $f(x+y) = f(x) + f(y)$ is said to have the Hyers-Ulam stability property on (X, Y). This terminology is applied to all kinds of functional equations which have been studied by many authors (see, for example, [4]-[6], [15]-[20]).

In 1978, Th. M. Rassias [14] succeeded in generalizing the Hyers' result by weakening the condition for the bound of the left side of the inequality (1.1). Due to this fact, the additive functional equation $f(x+y) = f(x) + f(y)$ is said to have the Hyers-Ulam-Rassias stability property on (X, Y). A number of Rassias type results concerning the stability of different functional equations can be found in [2, 4], [7]-[10], [18].

If each solution $f : X \to Y$ of the inequality (1.1) is a solution of the additive functional equation $f(x+y) = f(x)+f(y)$, then we say that the additive functional equation has the superstability property on (X, Y). This property is also applied to the case of other functional equations (Refs, [1, 9, 17]).

We now consider a functional equation which defines multiplicative derivations in algebras:

$$f(xy) = xf(y) + yf(x). \tag{1.2}$$

It is immediate to observe that the real-valued function $f(x) = xlnx$ is a solution of the functional equation (1.2) on the interval $(0, \infty)$.

During the 34-*th International Symposium on Functional Equations*, Gy. Maksa [12] posed the problem concerning the Hyers-Ulam stability on the interval $(0, 1]$ of the functional equation (1.2), and J. Tabor gave an answer to the question of Maksa in [23]. On the other hand, Zs. Páles [13] remarked that the functional equation (1.2) on the interval $[1, \infty)$ for real-valued functions is superstable.

Here we introduce the following functional equations motivated by the functional equation (1.2):

$$(1.3) \qquad f(\alpha x+\alpha y+xy+\alpha\beta)=\alpha f(x)+\alpha f(y)+xf(y)+yf(x),$$

where $\alpha, \beta \in \mathbb{R}$ with $\alpha-\beta=1$ and

$$(1.4) \qquad f(xyz)=xf(yz)+xzf(y)+zf(xy).$$

In this paper, we will solve the functional equation (1.3) and then, by following the ideas of J. Tabor [23] and Zs. Páles [13], the Hyers-Ulam stability on the interval and the superstability on the interval of the functional equation (1.3) will be investigated, respectively. In addition, we will consider this problem for the functional equation (1.4).

Throughout this paper, we will denote by $\mathbb{R}$ and $\mathbb{N}$ the sets of real numbers and of positive integers, respectively.

2. Preliminary Examinations

It is easy to see that the real-valued function $f(x)=(x+\alpha)ln(x+\alpha)$ is a solution of the functional equation (1.3) on the interval $(-\alpha,\infty)$, where $\alpha \geq 0$. In the following theorem, we will find out the general solution of the functional equation (1.3) on the interval $(-\alpha,\infty)$, where $\alpha \geq 0$.

Theorem 2.1. *Let X be a real (or complex) vector space. A function $f:(-\alpha,\infty)\to X$ satisfies the functional equation (1.3) for all $x,y \in (-\alpha,\infty)$ if and only if there exists a solution $D:(0,\infty)\to X$ of the functional equation (1.2) such that*

$$f(x)=D(x+\alpha)$$

for all $x \in (-\alpha,\infty)$.

PROOF. Assume that a function $f : (-\alpha, \infty) \to X$ satisfies the functional equation (1.3), for all $x, y \in (-\alpha, \infty)$. Then we can define the mapping $D : (0, \infty) \to X$ by $D(x) := f(x - \alpha)$. So we get

$$\begin{aligned} D(xy) &= f(xy - \alpha) \\ &= f(\alpha(x-\alpha) + \alpha(y-\alpha) + (x-\alpha)(y-\alpha) + \alpha\beta) \\ &= \alpha f(x-\alpha) + \alpha f(y-\alpha) + (x-\alpha)f(y-\alpha) + (y-\alpha)f(x-\alpha) \\ &= xD(y) + yD(x). \end{aligned}$$

for all $x,\ y \in (0, \infty)$. Therefore D is a solution of the functional equation (1.2), as desired, and $f(x) = D(x + \alpha)$ for all $x \in (-\alpha, \infty)$.

The converse is obvious. □

In particular, the previous theorem holds for $\alpha < 0$ and is a generalization of [11, Theorem 2.1]. In the following theorem we consider the general solution of the functional equation (1.4).

Theorem 2.2. *Let X be a real (or complex) vector space. A function $f : (0, \infty) \to X$ satisfies the functional equation (1.4) for all $x, y, z \in (0, \infty)$ if and only if there exists a solution $D : (0, \infty) \to X$ of the functional equation (1.2) such that $f(x) = D(x)$ for all $x \in (0, \infty)$.*

PROOF. Suppose that a function $f : (0, \infty) \to X$ satisfies the functional equation (1.4) for all $x, y, z \in (0, \infty)$. By letting $x = y = z = 1$ in (1.4), we get $f(1) = 0$. Now let us define the mapping $D : (0, \infty) \to X$ by $D(x) := f(x)$. We claim that D is a solution of the functional equation (1.4).

Indeed, for all $x,\ y \in (0, \infty)$, we have

$$\begin{aligned} D(xy) &= f(x \cdot 1 \cdot y) \\ &= xf(y) + xyf(1) + yf(x) \\ &= xD(y) + yD(x). \end{aligned}$$

Therefore D is a solution of the functional equation (1.4), as claimed, and $f(x) = D(x)$ for all $x \in (0, \infty)$.

The reverse assertion is trivial. □

First, we demonstrate a theorem of F. Skof [21] concerning the stability of the additive functional equation $f(x+y) = f(x) + f(y)$ on a restricted domain:

Theorem 2.3. *Let X be a real (or complex) Banach space. Given $c > 0$, let a mapping $f : [0, c) \to X$ satisfy the inequality*

$$||f(x+y) - f(x) - f(y)|| \leq \delta$$

for some $\delta \geq 0$ and for all $x,\ y \in [0, c)$ with $x + y \in [0, c)$. Then there exists an additive mapping $A : \mathbb{R} \to X$ such that

$$||f(x) - A(x)|| \leq 3\delta$$

for all $x \in [0, c)$.

For the purpose, we will introduce the next result [22] which is essential to prove the theorem 3.3:

Theorem 2.4. *Let X be a real (or complex) Banach space, and let $c > 0$ be a given constant. Suppose that a mapping $f : \mathbb{R} \to X$ satisfies the inequality*

$$||f(x+y) - f(x) - f(y)|| \leq \delta$$

for some $\delta \geq 0$ and for all $x,\ y \in \mathbb{R}$ with $|x| + |y| > c$. Then there exists a unique additive mapping $A : \mathbb{R} \to X$ such that

$$||f(x) - A(x)|| \leq 9\delta$$

for all $x \in \mathbb{R}$.

3. Hyers-Ulam Stability and Superstability

Our main theorem in this section is the Hyers-Ulam stability on the interval $(-\alpha, 0]$ of the functional equation (1.3) and the proof is similar to the one given in [23]. In the rest of this paper $\alpha \in \mathbb{R}$ with $\alpha > 0,\ \alpha - \beta = 1$.

Theorem 3.1. *Let X be a real (or complex) Banach space, and let $f : (-\alpha, 0] \to X$ be a mapping satisfying the inequality*

$$(3.1)\qquad ||f(\alpha x + \alpha y + xy + \alpha\beta) - \alpha f(x) - \alpha f(y) - xf(y) - yf(x)|| \leq \delta$$

for some $\delta > 0$ and for all $x,\ y \in (-\alpha, 0]$. Then there exists a solution $H : (-\alpha, 0] \to X$ of the functional equation (1.3) such that

$$(3.2)\qquad ||f(x) - H(x)|| \leq (4e)\delta$$

for all $x \in (-\alpha, 0]$.

Proof. Let $g : (-\alpha, 0] \to X$ be a mapping defined by

$$g(x) = \frac{f(x)}{x + \alpha}$$

for all $x \in (-\alpha, 0]$. Then, by (3.1), we see that g satisfy the inequality

$$||g(\alpha x + \alpha y + xy + \alpha\beta) - g(x) - g(y)|| \leq \frac{\delta}{(x + \alpha)(y + \alpha)}$$

for all $x,\ y \in (-\alpha, 0]$. If we define the mapping $F : [0, \infty) \to X$ by

$$F(-ln(x + \alpha)) = g(x)$$

for all $x \in (-\alpha, 0]$, then, by setting $u = -ln(x + \alpha)$ and $v = -ln(y + \alpha)$, we obtain

$$(3.3)\qquad ||F(u + v) - F(u) - F(v)|| \leq \delta e^{u+v}$$

for all $u,\ v \in [0, \infty)$. This means that

$$||F(u + v) - F(u) - F(v)|| \leq \delta e^{c}$$

for all $u, v \in [0, c)$ with $u + v < c$, where $c > 1$ is an arbitrary given constant.

According to Theorem 2.3, there exists an additive mapping $A : \mathbb{R} \to X$ such that $||F(u) - A(u)|| \leq 3\delta e^c$ for all $u \in [0, c)$. If we let $c \to 1$ in the last inequality, we then get

(3.4)
$$||F(u) - A(u)|| \leq 3e\delta$$

for all $u \in [0, 1]$. Moreover, it follows from (3.3) that

$$\begin{aligned} || \quad & F(u+1) - F(u) - F(1)|| \leq \delta e^{u+1} \\ || \quad & F(u+2) - F(u+1) - F(1)|| \leq \delta e^{u+2} \\ & \vdots \\ ||&F(u+k) - F(u+k-1) - F(1)|| \leq \delta e^{u+k} \end{aligned}$$

for all $u \in [0, 1]$ and $k \in \mathbb{N}$. Summing up these inequalities we obtain

(3.5)
$$||F(u+k) - F(u) - kF(1)|| \leq \delta e \cdot e^{u+k}(1 + e^{-1} + \cdots + e^{-k+1}) \leq \delta e \cdot e^{u+k}$$

for all $u \in [0, 1]$ and $k \in \mathbb{N}$. We assert that

(3.6)
$$||F(v) - A(v)|| \leq 4\delta e \cdot e^v$$

for all $v \in [0, \infty)$.

In fact, let $v \geq 0$ and let $k \in \mathbb{N} \cup \{0\}$ be given with $v - k \in [0, 1]$. Then, by (3.4) and (3.5), we have

$$\begin{aligned} ||F(v) - A(v)|| &\leq ||F(v) - F(v-k) - kF(1)|| \\ &\quad + ||F(v-k) - A(v-k)|| + ||A(k) - kF(1)|| \\ &\leq \delta e \cdot e^v + 3\delta e + ||A(k) - kF(1)|| \\ &\leq \delta e \cdot e^v + 3\delta e + k||A(1) - F(1)|| \\ &\leq \delta e \cdot e^v + 3\delta e + 3\delta ev \\ &\leq \delta e(e^v + 3(1+v)) \\ &\leq 4\delta e \cdot e^v. \end{aligned}$$

Hence, from (3.6) and the definition of F, it follows that

$$||g(x) - A(-ln(x+\alpha))|| \leq 4\delta e \cdot e^{-ln(x+\alpha)} = \frac{4\delta e}{x+\alpha}$$

for all $x \in (-\alpha, 0]$, i.e.,

$$(3.7) \qquad \left|\left|\frac{f(x)}{x+\alpha} - A(-ln(x+\alpha))\right|\right| \leq \frac{4\delta e}{x+\alpha}$$

for all $x \in (-\alpha, 0]$. If we put $H(x) = (x+\alpha)A(-ln(x+\alpha))$ for all $x \in (-\alpha, 0]$, we can easily check that H is a solution of the functional equation (1.3) by Theorem 2.1. This and (3.7) yield that

$$||f(x) - H(x)|| \leq (4e)\delta$$

for all $x \in (-\alpha, 0]$ which proves (3.7). The proof of the theorem is complete. □

It is interesting that in Theorem 3.1 the $\alpha = 1$ case is included (cf. [11, Theorem 3.2]).

Theorem 3.2. *Let X be a real (or complex) Banach space, and let $f : (0, 1] \to X$ be a mapping satisfying the inequality*

$$(3.8) \qquad ||f(xyz) - xf(yz) - xzf(y) - zf(xy)|| \leq \delta$$

for some $\delta > 0$ *and for all* $x,\ y,\ z \in (0,1]$. *Then there exists a solution* $H : (0,1] \to X$ *of the functional equation* (1.2) *such that*

$$(3.9) \qquad ||f(x) - H(x)|| \le (6e)\delta$$

for all $x \in (0,1]$.

PROOF. By putting $x = y = z = 1$ in (3.8), we know that $\|f(1)\| \le \frac{\delta}{2}$. In (3.8), we set $y = 1$, then replace z by y, and get

$$||f(xy) - xf(y) - yf(x)|| \le \frac{3}{2}\delta$$

for some $\delta > 0$ and for all $x,\ y \in (0,1]$. From the result in [23], we obtain the inequality (3.9), as desired. □

Now let us prove the main theorem of the section which is the superstability of the functional equation (1.3) on the interval $[0,\infty)$.

Theorem 3.3. *Let* X *be a real* (*or complex*) *Banach space, and let* $f : [0,\infty) \to X$ *be a mapping satisfying the inequality*

$$(3.10) \quad ||f(\alpha x + \alpha y + xy + \alpha\beta) - \alpha f(x) - \alpha f(y) - xf(y) - yf(x)|| \le \delta$$

for some $\delta > 0$ *and for all* $x,\ y \in [0,\infty)$. *Then* f *satisfies the functional equation* (1.3) *for all* $x,\ y \in [0,\infty)$.

PROOF. Defining the mapping $g : [0,\infty) \to X$ by

$$g(x) = \frac{f(x)}{x + \alpha}$$

for all $x \in [0,\infty)$ as in the proof of Theorem 3.1, and defining the mapping $F : [0,\infty) \to X$ by

$$F(ln(x + \alpha)) = g(x)$$

for all $x \in [0,\infty)$, we see, by putting $u = ln(x + \alpha)$ and $v = ln(y + \alpha)$, that

$$(3.11) \qquad ||F(u + v) - F(u) - F(v)|| \le \delta e^{-(u+v)}$$

for all $u,\ v \in [0,\infty)$. We claim that F is additive.

From (3.11) with $\delta_n = \delta e^{-n}$ $(n \in \mathbb{N})$, we obtain

$$||F(u+v) - F(u) - F(v)|| \leq \delta_n$$

for all $u,\ v \in [0, \infty)$ with $u + v > n$.

We now define a mapping $T : \mathbb{R} \to X$ by

$$T(u) = \begin{cases} F(u) \text{ for } u \geq 0, \\ -F(-u) \text{ for } u < 0. \end{cases}$$

It is not difficult to see that

$$||T(u+v) - T(u) - T(v)|| \leq \delta_n$$

for all $u,\ v \in \mathbb{R}$ with $|u| + |v| > n$. Therefore, by Theorem 2.4, there exists a unique additive mapping $A_n : \mathbb{R} \to X$ satisfying

$$||T(u) - A_n(u)|| \leq 9\delta_n \tag{3.12}$$

for all $u \in \mathbb{R}$. Let $m,\ n \in \mathbb{N}$ with $n > m$. Then the additive mapping $A_n : \mathbb{R} \to X$ satisfies $||T(u) - A_n(u)|| \leq 9\delta_m$ for all $u \in \mathbb{R}$. The uniqueness argument now implies $A_n = A_m$ for all $n \in \mathbb{N}$ with $n > m > 0$, and thus $A_1 = A_2 = \cdots = A_n = \cdots$. Taking the limit in (3.12) as $n \to \infty$, we obtain $T = A_1$ and we deduce that F is additive.

Now, according to the definitions of F and g, we have

$$\frac{f(x)}{x+\alpha} = F(ln(x+\alpha))$$

for all $x \in [0, \infty)$, i.e.,

$$f(x) = (x+\alpha)F(ln(x+\alpha))$$

for all $x \in [0, \infty)$, and hence we see that f satisfies the functional equation (1.3) for all $x,\ y \in [0, \infty)$ by Theorem 2.1 since F is additive and $D(x) = xF(ln(x))$ $(x \in [1, \infty))$ is a solution of the functional equation (1.2). This completes the proof of the theorem. □

The theorem 4.2 introduced in [11] is a special case of the preceding theorem.

Theorem 3.4. *Let X be a real (or complex) Banach space, and let $f : [1,\infty) \to X$ be a mapping satisfying the inequality*

$$||f(xyz) - xf(yz) - xzf(y) - zf(xy)|| \leq \delta$$

for some $\delta > 0$ and for all $x,\ y,\ z \in [1,\infty)$, where $f(1) = 0$. Then f satisfies the functional equation (1.2) for all $x,\ y \in [1,\infty)$.

PROOF. If we use the same method as the proof of Theorem 3.2, then we have

$$||f(xy) - xf(y) - yf(x)|| \leq \delta$$

for some $\delta > 0$ and for all $x,\ y \in [1,\infty)$. Therefore, by the result of in [13], f satisfies the equation (1.2). The proof of Theorem is complete. □

We remarked that the two theorems mentioned above hold for $\alpha < 0$.

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