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ON THE UNIQUENESS OF SOLUTION OF A FIRST ORDER NON-LINEAR COMPLEX ELLIPTIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS IN SOBOLEV SPACE

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Abstract. In this paper we discuss on the uniquenss of the solution of partial differential equations:

(1) $$\frac{\partial w}{\partial \bar{z}} = F(z, w, \frac{\partial w}{\partial z}) + G(z, w.\bar{w})$$

in the sobolev space $W_{1,p}(D)$.

1. introduction

Suppose that D is a domain with finite area in the complex plane and $F = F(z, w, \frac{\partial w}{\partial z}) \in L_p(D)$, $G = G(z, w, \bar{w}) \in L_p(D)$, $1 < p < \infty$ and with define the weakly singular and strongly singular operators T_D and $\prod_D$:

$$T_D f(z) = -\frac{1}{\pi} \int\int_D \frac{1}{\xi - z} f(\xi) d\zeta d\eta$$

$$\prod_D f(z) = -\frac{1}{\pi} \int\int_D \frac{1}{(\xi - z)^2} f(\xi) d\zeta d\eta$$

that $\xi = \zeta + i\eta$ and $z = x + iy$.
with the following assumptions:

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I. As a function of the variables $z \in D, w, \bar{w}; F(z, w, \frac{\partial w}{\partial z}) + G(z, w, \bar{w})$ is a continuous function of its variables.

II. The functions $F(z, w, \frac{\partial w}{\partial z})$ and $G(z, w, \bar{w})$ satisfy a Lipschitz condition of the form:

$$|F(z, w, h) - F(z, \tilde{w}, \tilde{h})| \leq L_1|w - \tilde{w}| + L_2|h - \tilde{h}|$$

$$|G(z, w, \bar{w}) - G(z, w, \bar{\tilde{w}})| \leq L_3|w - \tilde{w}|$$

almost everywhere in D; whereas the constant L_2 is strictly less than 1, L_1 and L_3 are arbitrary positive numbers.

III. There exist $w, h \in L_p(D),\ 1 < p < \infty$, such that $F(z, w, h)$, $G(z, w, \bar{w}) \in L_p(D)$. If we denote by $\Im_p(D)$ the set of pairs (w, h) for which $w, h \in L_p(D),\ 1 < p < \infty$, and define the norm by the relation

$$||(w, h)|| = ||(w, h)||_{p,\lambda} = max(\lambda||w||_p, ||h||_p)\lambda > 0$$

the set $\Im_p(D)$ is then a Banach Space.
For a pair $(w, h) \in \Im_p(D)$ we define an operator L as follows:

$$L(w, h) = (W, H)$$

$$(1) \qquad \begin{cases} W(z) = \phi(z) + T_D[F(z, w, h) + G(z, w.\bar{w})] \\ H(z) = \phi'(z) + \prod_D[F(z, w, h) + G(z, w, \bar{w})] \end{cases}$$

where ϕ is a fixed holomorphic function in D and it belongs to $W_{1,p}(D)$, $1 < p < \infty$.
If $(W, H),\ (\tilde{W}, \tilde{H})$ are the images of two arbitrrarily chosen elements $(w, h),\ (\tilde{w}, \tilde{h}) \in \Im_p(D)$ respectively:

$$\begin{cases} W(z) = \phi(z) + T_D[F(z, w, h) + G(z, w, \bar{w})] \\ H(z) = \phi'(z) + \prod_D[F(z, w, h) + G(z, w, \bar{w})] \end{cases}$$

$$\begin{cases} \tilde{W}(z) = \phi(z) + T_D[F(z, \tilde{w}, \tilde{h}) + G(z, \tilde{w}, \bar{\tilde{w}})] \\ \tilde{H}(z) = \phi'(z) + \prod_D[F(z, \tilde{w}, \tilde{h}) + G(z, \tilde{w}, \bar{\tilde{w}})] \end{cases}$$

It then follows that

$$\lambda||W-\tilde{W}||_p \leq B(D)[(L_1+L_3)+\lambda L_2]||(w,h)-(\tilde{w},\tilde{h})||_{p,\lambda}$$

$$||H-\tilde{H}||_p \leq A(D)[\frac{1}{\lambda}(L_1+L_3)+L_2]||(w,h)-(\tilde{w},\tilde{h})||_{p,\lambda}.$$

This means that

$$\begin{aligned} ||(W,H)-(\tilde{W},\tilde{H})|| &\leq [(L_1+L_3)+\lambda L_2]max(B(D) \\ &+\frac{1}{\lambda}A(D))||(w,h)-(\tilde{w},\tilde{h})||. \end{aligned}$$

And if

$$[(L_1+L_3)+\lambda L_2]max(B(D)+\frac{1}{\lambda}A(D))<1 \tag{2}$$

then the operator L is contractive in $\Im_p(D)$ and , as such, by fixed point theorem , there exists the fixed element (w,h) of the operetor L so that

$$L(w,h)=(w,h)$$

that w is the solution of the partial differential equation (1).

2. uniqueness of solution

Suppose that

$$\begin{cases} L(w,h)=(w,h) \\ L(\tilde{w},\tilde{h})=(\tilde{w},\tilde{h}) \end{cases} \tag{3}$$

then we will have

$$\begin{aligned} &||(w,h)-(\tilde{w},\tilde{h})|| \\ &\quad =||L(w,h)-L(\tilde{w},\tilde{h})|| \\ &\quad \leq [(L_1+L_3)+\lambda L_2]max(B(D)+\frac{1}{\lambda}A(D))||(w,h)-(\tilde{w},\tilde{h})|| \end{aligned}$$

so that

$$k=[(L_1+L_3)+\lambda L_2]max(B(D)+\frac{1}{\lambda}A(D))<1 \tag{4}$$

and

$$\|(w,h)-(\tilde{w},\tilde{h})\| \leq k\|(w,h)-(\tilde{w},\tilde{h})\| \tag{5}$$

then we should have

$$\|(w,h)-(\tilde{w},\tilde{h})\| = 0$$

then

$$(w,h) = (\tilde{w},\tilde{h})$$

Consequently there exists a unique fixed element (w, h) of the operator L, which w is also a solution of partial differential equation (1).
If we have

$$\|(w,h)-(\tilde{w},\tilde{h})\| \neq 0$$

then

$$k \geq 1$$

that is opposite of the (4).

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