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GENERALIZED FUZZY VECTOR QUASIVARIATIONAL-LIKE INEQUALITIES†

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Abstract. In this paper, we introduce a generalized vector quasivariational like inequality for fuzzy mappings and show the existence of solutions under compact assumption.

1. Introduction

By Giannessi [14], a vector variational inequality problem was introduced in a finite dimensional Euclidean space with its applications. Later many authors [1, 5, 9, 11, 15, 17-18, 21-24, 28-29, 34-35] have extensively studied the problem in infinite dimensional spaces under different assumptions. Especially, vector variational-like inequalities and vector quasivariational inequalities were considered on topological vector spaces were considered in [1, 5, 23, 35] and [17-18, 28, 36], respectively.

On the other hand, since Chang and Zhu [6] introduced a variational inequality problem for fuzzy mappings, it has been generalized to many kinds of variational inequality problems [2-4, 19-20, 22, 25-27, 31, 33], even though its history is very short. Recently some authors [3, 4, 20, 22,

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25-26] considered vector variational inequality problems for fuzzy mappings. In particular, Chang et al. [3-4] studied vector quasivariational inequality problems for fuzzy mappings and Lee et al. [20] obtained a fuzzy extension of Siddiqi et al.'s results for vector variational-like inequalities. Also, in [2] vector variational-like inequalities for fuzzy mappings were dealt with, respectively.

In this paper, packaging "quasi"and "like", we introduce a generalized vector qua-sivariational-like inequality problems for fuzzy mappings and show the existence of solutions to our inequality problems, which are general outcomes of Lee et al.'s recent results [26].

2. Preliminaries

Let X and Y be topological spaces, and $T : X \to 2^Y$ a multivalued mapping. The following are easily known facts.

Definition 2.1. T is upper semicontinuous (in short, u.s.c.) at $x \in X$ if every open set V in Y containing $T(x)$, there is an open set U containing x such that $T(u) \subseteq V$ for all $u \in U$; T is u.s.c. on X if T is u.s.c. at every point of X. T is lower semicontinuous (in short, l.s.c.) at $x \in X$ if for every open set V in Y with $T(x) \cap V \neq \emptyset$, there is an open set U containing x such that $T(u) \cap V \neq \emptyset$ for all $u \in U$; T is l.s.c. on X if T is l.s.c. at every point of X. T is continuous at x if T is both u.s.c. and l.s.c. at x. T is compact if $T(X)$ is contained in some compact subset of Y, and T is closed if the graph of T, $G_rT = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$. Let $T^- : Y \to 2^X$ be a multivalued mapping defined by

$$x \in T^-(y) \quad \text{if and only if} \quad y \in T(x).$$

T is said to have open lower sections if for any $y \in Y$, $T^-(y)$ is open in X.

Lemma 2.1. *T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$ and for any net $\{x_\alpha\}$ in X converging to x, there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ for each α, and y_α converges to y.*

Let X, Y be sets, $F : X \to \Im(Y)$ be a fuzzy mapping and denote a fuzzy set $F(x)$ by F_x in Y for $x \in X$, where $\Im(Y)$ is the collection of all fuzzy sets in Y.

Definition 2.2. [2-4, 6] F is said to be convex on a set X if Y is a convex subset of a topological vector space and for any $x \in X$, y, $z \in Y$ and $\lambda \in [0,1]$,

$$F_x(\lambda y + (1-\lambda)z) \geq \min\{F_x(y), F_x(z)\}.$$

F is closed fuzzy set-valued if for each $y \in Y$, $F_x(y)$ is u.s.c. on $X \times Y$ as a real ordinary function. F is topologically open fuzzy set-valued if for each $x_0 \in X$ and for each open subset V of Y with $F_{x_0}(y) \geq \gamma$ for some $y \in V$ ($\gamma \in (0,1]$), there is a neighborhood U of x_0 in X such that if $x \in U$, then $F_x(y) \geq \gamma$ for some $y \in V$. F is weakly open fuzzy set-valued if for each $y \in Y$, $F_x(y)$ is l.s.c. on $X \times Y$ as a real ordinary function.

Lemma 2.2. [2] *Let K be a nonempty closed convex subset of a real Hausdorff topological vector space X, D a nonempty closed convex subset of a real Hausdorff topological vector space Y and $\beta : K \to (0,1]$ a l.s.c. function. Let $F : K \to \Im(D)$ be a fuzzy mapping with the cut set $(F_x)_{\beta(x)} := \{d \in D | F_x(d) \geq \beta(x)\} \neq \emptyset$ for any $x \in K$. Let $\bar{F} : K \to 2^D$ be a multivalued mapping defined by $\bar{F}(x) = (F_x)_{\beta(x)}$. If F is a convex fuzzy mapping with closed fuzzy set-values, then $\bar{F}$ is a closed mapping with nonempty convex-values.*

Lemma 2.3. [3] *Let X and Y be topological spaces, and $F : X \to \Im(Y)$ be a fuzzy mapping such that for any $x \in X$, the cut set $(F_x)_\gamma := \{y \in Y : F_x(y) \geq \gamma\}$ is nonempty for $\gamma \in (0,1]$. Let $\bar{F} : X \to 2^Y$ be a multivalued mapping defined by $\bar{F}(x) = (F_x)_\gamma$. If F is convex and* *topologically open fuzzy set-valued, then $\bar{F}$ is a l.s.c. mapping with nonempty convex-values.*

Lemma 2.4. [4] *Let K be a nonempty closed convex subset of a real Hausdorff topological vector space X, D a nonempty closed convex subset of a Hausdorff topological vector space Y and $\beta : X \to (0,1]$ an u.s.c. function. Let $F : K \to \Im(D)$ be a fuzzy mapping such that for any $x \in K$, the strong cut set $[F_x]_{\beta(x)} := \{d \in D : F_x(d) > \beta(x)\}$ is nonempty. Let $\bar{F} : K \to 2^D$ be a multivalued mapping defined by $\bar{F}(x) = [F_x]_{\beta(x)}$.*
(1) If F is convex, then $\bar{F}$ has nonempty convex-values.
(2) If F is weakly open fuzzy set-valued, then $\bar{F}$ has open lower sections.

Definition 2.3. In an ordered Hausdorff topological vector space Z, usually a closed convex pointed solid proper cone P in Z defines partial orders $<$ and $\leq$ as

$$x <_P y \quad \text{iff} \quad x - y \in -int\, P$$
$$x \leq_P y \quad \text{iff} \quad x - y \in -P$$

for x, $y \in Z$. To an arbitrary subset C of Z, the orders can be extended by setting

$$C <_P 0 \quad \text{iff} \quad C \subseteq -int\, P$$
$$C \leq_P 0 \quad \text{iff} \quad C \subseteq -P.$$

A point z_0 in a nonempty subset C of Z is called a vector maximal point of C [37] iff the set $\{z \in C : z_0 \leq_P z, z \neq z_0\} = \emptyset$, which is equivalent to

$$C \cap (z_0 + P) = \{z_0\}.$$

Lemma 2.5. [30] *Let C be a nonempty compact subset of an ordered Banach space Z, then $\max C \neq \emptyset$, where $\max C$ denotes the set of all vector maximal points of C.*

3. Main results

The following definitions on P-convexity and linearity of two variable functions are primitive concepts to obtain our results.

Definition 3.1. Let X, Z be two vector spaces, $K \subset X$ a nonempty convex set, and $P \subset Z$ a pointed, closed convex cone with apex at the origin and nonempty interior $int\, P$.

A multivalued mapping $H : K \times K \to 2^Z$ is said to be P-convex with respect to the first variable if for x_1, x_2, $y \in K$, $u_1 \in H(x_1, y)$, $u_2 \in H(x_2, y)$ and $\lambda \in [0, 1]$, there exists $u \in H(\lambda x_1 + (1-\lambda)x_2, y)$ such that

$$\lambda u_1 + (1-\lambda)u_2 \in u + P.$$

Definition 3.2. Let X, Z be vector spaces. We say that $\eta : X \times X \to Z$ is linear if

$$\eta(\lambda(x_1, y_1) + (x_2, y_2)) = \lambda\eta(x_1, y_1) + \eta(x_2, y_2),$$

$$\text{for } (x_1, x_2),\ (y_1, y_2) \in X \times X \text{ and } \lambda \in \mathbb{R}.$$

Throughout this section X denotes a Hausdorff topological vector space, Y a topological vector space and Z an ordered topological vector space. Let K be a nonempty convex subset of X, D a nonempty subset of Y and $\{C(x)|x \in K\}$ a family of solid convex cones in Z, that is, for any $x \in K$, $int\, C(x)$ is nonempty and $C(x) \neq Z$. $L(X, Z)$ denotes the space of all linear continuous operators from X to Z. Let $F : K \to \Im(D)$, $G : K \to \Im(K)$ be two fuzzy mappings and $M : K \times D \to 2^{L(X,Z)}$, $H : K \times K \to 2^Z$ two multivalued mappings, $\eta : X \times X \to X$ a mapping, $\beta : X \to (0, 1]$ a function and γ a constant in $(0, 1]$.

We consider the following generalized vector quasivariational-like inequality for fuzzy mappings;

(F-VQVLI) find $\bar{x} \in K$ such that for any $x \in K$ there exists $\bar{s} \in (F_{\bar{x}})_{\beta(\bar{x})}$ satisfying the following inequality;

$$\max\langle M(\bar{x},\bar{s}),\eta(x,z)\rangle + u \notin -int\, C(\bar{x})$$

for any $z \in (G_{\bar{x}})_\gamma$ and $u \in H(x,\bar{x})$, where

$$\max\langle M(\bar{x},\bar{s}),\eta(x,z)\rangle = \max_{s\in M(\bar{x},\bar{s})}\langle s,\eta(x,z)\rangle,$$

and $\langle s,\eta(x,z)\rangle$ denotes the evaluation of continuous linear operator s from X into Z at $\eta(x,z)$.

Replacing $\Im(K)$ and $\Im(D)$ with 2^K and 2^D, respectively in **(F-VQVLI)**, we obtain the following generalized vector quasivariational-like inequality for multivalued mappings;

(VQVLI) find $\bar{x} \in K$ such that for any $x \in K$ that exists $\bar{s} \in F(\bar{x})$ satisfying the following inequality;

$$\max\langle M(\bar{x},\bar{s}),\eta(x,z)\rangle + u \notin -int\, C(\bar{x})$$

for any $z \in G(\bar{x})$ and $u \in H(x,\bar{x})$.

Deleting a topological vector space Y and a fuzzy mapping F first, and then replacing Z with an ordered topological vector space Y, $H : K \times K \to 2^Z$ with $H : K \times K \to Y$, and $M : K \times D \to 2^{L(X,Z)}$ with $S : K \to 2^{L(X,Y)}$ in **(F-VQVLI)** we obtain the following vector variation-like inequality for fuzzy mappings;

(F-VVLI) find $\bar{x} \in K$ satisfying the following inequality;

$$\max\langle S(\bar{x}),\eta(x,y)\rangle + H(x,\bar{x}) \notin -int\, C(\bar{x})$$

for any $x \in K$ and any $y \in (G_{\bar{x}})_{\beta(\bar{x})}$, where $\{C(x) : x \in K\}$ is a family of closed convex cones in Y.

Replacing a fuzzy mapping $G : K \to \Im(K)$ with a multivalued mapping $G : K \to 2^K$ defined by $G(x) = K$ for $x \in K$ and putting $H \equiv 0$ in **(F-VVLI)**, we obtain the following vector variational-like inequalities for multivalued mappings,

(VVLI) find $\bar{x} \in K$ such that

$$\max\langle S(\bar{x}), \eta(x, y)\rangle \notin -int\, C(\bar{x})$$

for any x, $y \in K$,
which is a generalized form of the following vector variational-like inequalities for multivalued mappings, introduced and studied by Chang, Thompson and Yuan [5];

(VVLI)′ Find $\bar{x} \in K$ satisfying the following inequality;

$$\max\langle S(\bar{x}), \eta(x, \bar{x})\rangle \notin -int\, C(\bar{x}) \quad \text{for} \quad x \in K.$$

Putting $Z = Y$, $\eta(x, z) = x - z$ and $H = \bar{0}$, and replacing $M : K \times D \to 2^{L(X,Z)}$ with $S : K \to L(X, Y)$ in **(VQVLI)**, we have the following variational inequality;

(VVI) find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - z\rangle \notin -int\, C(\bar{x}) \quad \text{for any } x \in K \text{ and } z \in G(\bar{x}).$$

Putting $C(x) \equiv C$ for $x \in K$ and $\eta(x, y) = x - y$ in **(VVLI)′**, we obtain the following vector-valued variational inequality considered by Lee et al. [27]; find $\bar{x} \in K$ such that for each $x \in K$, there exists $\bar{s} \in S(\bar{x})$ such that

$$\langle \bar{s}, x - \bar{x}\rangle \not\geq_{-int\, C} 0,$$

where $x \not\geq_P y$ means $x - y \notin P$.

Putting $Z = \mathbb{R}$, $L(X, Z) = X^*$, the dual of X and $C(x) \equiv \mathbb{R}^+$, the positive orchant for $x \in K$ in **(VVLI)′**, we obtain the following scalar-valued variational inequality considered by Cottle and Yao [12], Isac [16], and Noor [32]; find $\bar{x} \in K$ such that

$$\sup_{u \in S(\bar{x})} \langle u, \eta(x, \bar{x})\rangle \geq 0, \quad \text{for } x \in K.$$

Replacing $S : K \to 2^{L(X,Z)}$ with $S : X \to L(X,Z)$ and putting $\eta(x,z) = x - g(z)$, where $g : K \to K$ is a mapping, then **(VVLI)**$'$ reduces to the following vector variational inequality **(VVI)** considered by Siddiqi et al. [35];

(VVI)$'$ find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - g(\bar{x})\rangle \not\geq_{-int\, C(\bar{x})} 0, \quad \text{for } x \in K.$$

Putting $G(x) = \{x\}$ for $x \in K$ in **(VVI)** or $g(x) = x$ for $x \in K$ in **(VVI)**$'$, we obtain the following vector-valued variational inequality considered by Chen [7]; find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - \bar{x}\rangle \not\geq_{-int\, C(\bar{x})} 0, \quad \text{for } x \in K.$$

Putting $C(x) \equiv C$ and $g(x) = x$ for $x \in K$ in **(VVI)**$'$, we obtain the following vector-valued variational inequality considered by Chen et al. [7-8, 10]; find $\bar{x} \in K$ such that

$$\langle S(\bar{x}), x - \bar{x}\rangle \not\geq_{-int\, C} 0, \quad \text{for } x \in K.$$

The following theorem called Ky Fan's Section Theorem is a very important tool in nonlinear analysis.

Theorem 3.1. [**13**] *Let K be a nonempty compact convex subset of a Hausdorff topological vector space. Let A be a subset of $K \times K$ having the following properties*

(i) $(x,x) \in A$ for all $x \in K$;

(ii) for any $x \in K$, the set $A_x := \{y \in K : (x,y) \in A\}$ is closed in K;

(iii) for any $y \in K$, the set $A^y := \{x \in K : (x,y) \notin A\}$ is convex in K. Then there exists $\bar{y} \in K$ such that $K \times \{\bar{y}\} \subset A$.

We deal with **(F-VQVLI)** for the compact set case using Ky Fan's Section Theorem.

Theorem 3.2. *Let K be a nonempty compact convex subset of X and D a nonemtpy closed convex subset of Y. Let $F : K \to \Im(D)$ be a convex fuzzy mapping with closed fuzzy set-values, $G : K \to \Im(K)$*

a convex fuzzy mapping with topologically open fuzzy set-values, $M : K \times D \to 2^{L(X,Z)}$ *a multivalued mapping, and a multivalued mapping* $W : K \to 2^Z$ *defined by* $W(x) = Z \backslash \{-int\, C(x)\}$, $x \in K$, *closed. Let* $\eta : X \times X \to X$ *be linear,* $y \mapsto \eta(\cdot, y)$ *continuous, and* $H : K \times K \to 2^Z$ *P-convex with respect to the first variable and l.s.c. with respect to the second, where* $P := \bigcap_{x \in K} C(x)$.

Suppose further that

(1) there exist a l.s.c. function $\beta : X \to (0,1]$ *and a constant* $\beta \in (0,1]$ *such that for any* $x \in K$, *the cut sets* $(F_x)_{\beta(x)}$ *and* $(G_x)_\gamma$ *are nonempty;*

(2) $\bigcup_{x \in K} (F_x)_{\beta(x)}$ *is contained in some compact subset of* D*;*

(3) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha)\rangle$ *converges to* $\max\langle M(y,s), \eta(x,z)\rangle$ *provided that* $y_\alpha \to y$, $s_\alpha \to s$ *and* $z_\alpha \to z$*; and*

(4) $\langle M(x,\cdot), \eta(x,\cdot)\rangle = 0$ *and* $H(x,x) = \{0\}$ *for all* $x \in K$.

Then **(F-VQVLI)** *is solvable.*

Proof. Define multivalued mappings $\bar{F} : K \to 2^D$ and $\bar{G} : K \to 2^K$ by $\bar{F}(x) = (F_x)_{\beta(x)}$ and $\bar{G}(x) = (G_x)_\gamma$, respectively. Let $A = \{(x,y) \in K \times K$: there exists $s \in \bar{F}(y)$ such that $\max\langle M(y,s), \eta(x,z)\rangle + u \notin -int\, C(y)$ for any $z \in \bar{G}(y)$ and $u \in H(x,y)\}$. By condition (4), it is easily shown that $(x,x) \in A$, for all $x \in K$. Moreover $A_x = \{y \in K : (x,y) \in A\}$, $x \in K$ is closed. In fact, since G is topologically open fuzzy set-valued, by Lemma 2.3 $\bar{G}$ is l.s.c.. Let $\{y_\alpha\}$ be a net in A_x such that $y_\alpha \to y$. Then by Lemma 2.1, for any $z \in \bar{G}(y)$ there exists a net $\{z_\alpha\}$ converging to z such that $z_\alpha \in \bar{G}(y_\alpha)$ for each α. Also by the lower semicontinuity of H with respect to the second variable, for any $u \in H(x,y)$ there exists a net $\{u_\alpha\}$ converging to u such that $u_\alpha \in H(x, y_\alpha)$ for each α. Since $y_\alpha \in A_x$ we can choose $s_\alpha \in \bar{F}(y_\alpha)$ such that

$$\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha)\rangle + u_\alpha \in W(y_\alpha)$$

for $z_\alpha \in \bar{G}(y_\alpha)$ and $u_\alpha \in H(x, y_\alpha)$. By condition (2) and the closedness of $\bar{F}$ owing to Lemma 2.2, we can assure the existence of limit s of $\{s_\alpha\}$

such that $s \in \bar{F}(y)$. Hence by condition (3) and the closedness of W, we have

$$\max\langle M(y,s), \eta(x,z)\rangle + u \in W(y)$$

for any $z \in \bar{G}(y)$ and $u \in H(x,y)$. Thus $A^y = \{x \in K : (x,y) \notin A\}$, $y \in K$ is convex. Indeed, let x_1, $x_2 \in A^y$ and $\lambda \in [0,1]$. Then from the fact that $(x_1,y) \notin A$ for any $s \in \bar{F}(y)$, there exist $z_1 \in \bar{G}(y)$ and $u_1 \in H(x_1,y)$ such that

$$\max\langle M(y,s), \eta(x_1,z_1)\rangle + u_1 \in -int\, C(y)$$

and from the fact that $(x_2,y) \notin A$ for any $s \in \bar{F}(y)$, there exist $z_2 \in \bar{G}y)$ and $u_2 \in H(x_2,y)$ such that

$$\max\langle M(y,s), \eta(x_2,z_2)\rangle + u_2 \in -int\, C(y).$$

On the other hand, since G is convex, by Lemma 2.4(1) $\bar{G}$ is convex-valued.

Hence for any $s \in \bar{F}(y)$, there exist $u \in H(\lambda x_1 + (1-\lambda)x_2, y)$ and $z := \lambda z_1 + (1-\lambda)z_2 \in \bar{G}(y)$ for $\lambda \in [0,1]$ such that

$$\begin{aligned}
&\max\langle M(y,s), \eta(\lambda x_1 + (1-\lambda)x_2, z)\rangle + u \\
&= \max\langle M(y,s), \eta(\lambda x_1 + (1-\lambda)x_2, \lambda z_1 + (1-\lambda)z_2)\rangle + u \\
&= \max\langle M(y,s), (\lambda\eta(x_1,z_1) + (1-\lambda)\eta(x_2,z_2))\rangle + u \\
&\leq \lambda \max\langle M(y,s), \eta(x_1,z_1)\rangle + (1-\lambda)\max\langle M(y,s), \eta(x_2,z_2)\rangle + u \\
&\in \lambda \max\langle M(y,s), \eta(x_1,z_1)\rangle + (1-\lambda)\max\langle M(y,s), \eta(x_2,z_2)\rangle \\
&\qquad + \lambda u_1 + (1-\lambda)u_2 - P \\
&= \lambda(\max\langle M(y,s), \eta(x_1,z_1)\rangle + u_1) \\
&\qquad + (1-\lambda)(\max\langle M(y,s), \eta(x_2,z_2)\rangle + u_2) - P \\
&\subseteq -\, int\, C(y) - int\, C(y) - C(y) \\
&= -\, int\, C(y).
\end{aligned}$$

Thus $\lambda x_1 + (1-\lambda)x_2 \in A^y$, which shows that A^y is convex. Hence by Ky Fan's Section Theorem there exists $\bar{x} \in K$ such that

$$K \times \{\bar{x}\} \subset A,$$

which implies that for any $x \in K$, there exists $\bar{s} \in (F_{\bar{x}})_{\beta(\bar{x})}$ such that

$$\max\langle M(\bar{x},\bar{s}), \eta(x,z)\rangle + u \notin -int\, C(\bar{x})$$

for $z \in (G_{\bar{x}})_\gamma$ and $u \in H(x,\bar{x})$.

The following theorem is a direct corollary of Theorem 3.2.

Theorem 3.3. [26] *Let K be a nonempty compact convex subset of X. Let $F : K \to \Im(L(X,Y))$ be a fuzzy mapping with closed fuzzy set-values, $G : K \to \Im(K)$ a convex fuzzy mapping with topologically open fuzzy set-values, and a multivalued mapping $W : K \to 2^Y$ defined by $W(x) = Y\backslash[-int\, C(x)]$, $x \in K$ closed, where $\{C(x) : x \in K\}$ is a family of solid convex cones in Y. Let $P = \bigcap_{x\in K} C(x)$ and $h : K \to Y$ be a continuous P-convex function. Suppose further that*
(1) there exist a l.s.c. function $\beta : X \to (0,1]$ and a constant $\gamma \in (0,1]$ such that for any $x \in K$ the cut sets $(F_x)_{\beta(x)}$ and $(G_x)_\gamma$ are nonempty;
(2) $\bigcup_{x\in K} (F_x)_{\beta(x)}$ is contained in some compact subset of $L(X,Y)$; and
(3) for any $x \in K$, there exists $s \in (F_x)_{\beta(x)}$ such that $\langle s, x-z\rangle \notin -int\, C(x)$ for any $z \in (G_x)_\gamma$.
Then the following variational inequality;
(F-VVI) *find $\bar{x} \in K$ such that for any $x \in K$, there exists $\bar{s} \in (F_{\bar{x}})_{\beta(\bar{x})}$ such that*

$$\langle \bar{s}, x-z\rangle + h(x) - h(\bar{x}) \notin -int\, C(\bar{x}) \quad \text{for any} \quad z \in (G_{\bar{x}})_\gamma,$$

is solvable.

The following theorem for the existence of solutions to **(VQVLI)** is a direct consequence of Theorem 3.2.

Corollary 3.4. Let K be a nonempty compact convex subset of X and D a nonempty subset of Y. Let $F : K \to 2^D$ be closed, $G : K \to 2^K$ be l.s.c. and nonempty convex-valued, $M : K \times D \to 2^{L(X,Z)}$ be nonempty compact-valued, and a multivalued mapping $W : K \to 2^Z$ defined by $W(x) = Z \backslash \{-int\, C(x)\}$, $x \in K$, closed. Let $\eta : X \times X \to X$ be linear, and $H : K \times K \to 2^Z$ be P-convex with respect to the first variable and l.s.c. with respect to the second, where $P := \bigcap_{x \in K} C(x)$.

Suppose further that

(1) $\langle M(x,\cdot), \eta(x,\cdot)\rangle = 0$ and $H(x,x) = \{0\}$ for all $x \in K$;

(2) F is compact; and

(3) $\max\langle M(y_\alpha, s_\alpha), \eta(x, z_\alpha)\rangle$ converges to $\max\langle M(y,s), \eta(x,z)\rangle$ provided that $y_\alpha \to y$, $s_\alpha \to s$ and $z_\alpha \to z$.

Then **(VQVLI)** is solvable.

Remark 3.1. Corollary 3.4 is a generalized result of many outcomes in [1-7, 15, 17, 20-22, 26-28] and therein.

References

[1] Q. H. Ansari, *A note on generalized vector variational-like inequalities*, Optimization **41** (1997), 197-205.

[2] S. S Chang, *Coincidence theorems and variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **61** (1994), 359-368.

[3] S. S. Chang, G. M. Lee and B. S. Lee, *Vector quasivariational inequalities for fuzzy mappings (I)*, Fuzzy Sets and Systems **87** (1997), 307-315.

[4] S. S. Chang, G. M. Lee and B. S. Lee, *Vector quasivariational inequalities for fuzzy mappings (II)*, Fuzzy Sets and Systems **102** (1999), 333-344.

[5] S. S. Chang, H. B. Thompson and G. X. Z. Yuan, *The existence theorems of solutions for generalized vector-valued variational-like inequalities*, Comput. Math. Appl. **37** (1999), 1–9.

[6] Shin-sen Chang and Yuan-guo Zhu, *On variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **32** (1989), 359-367.

[7] G. Y. Chen, *Existence of solutions for a vector variational inequality: An extension of the Hartmann-Stampacchia theorem*, J. Optim. Th. Appl. **74(3)** (1992), 445-456.

[8] G. Y. Chen and G. M. Cheng, *Vector Variational Inequalities and Vector Optimization*, Lecture Notes in Economics and Math. Systems **285**, Springer Verlag, Berlin, 1987.

[9] G. Y. Chen and B. D. Craven, *Approximate dual and approximate vector variational inequality for multiobjective optimization*, J. Austral. Math. Soc. (Series A) **47** (1989), 418-423.

[10] G. Y. Chen and B. D. Craven, *A vector variational inequality and optimization over an efficient set*, Zeitscrift für Operations Research **3** (1990), 1-12.

[11] G. Y. Chen and X. Q. Yang, *The vector complementarity problem and its equivalences with the weak minimal element in ordered spaces*, J. Math. Anal. Appl. **153** (1990), 136-158.

[12] R. W. Cottle and J. C. Yao, *Pseudo-monotone complementarity problems in Hilbert spaces*, J. Optim. Th. Appl. **75(2)** (1992), 281-295.

[13] Ky Fan, *A generalization of Tychonoff's fixed-point theorem*, Mathematische Annalen **142** (1961), 305-310.

[14] F. Giannessi, *Theorems of alternative, quadratic programs, and complementarity problems*, Variational Inequalities and Complementarity Problems (Cottle, Giannessi and Lions, eds.), John Wiley and Sons, New York, 1980.

[15] ______, *Vector Variational Inequality and Vector Equilibria*, Mathematical Theories, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.

[16] G. Isac, *A special variational inequality and the implicit complementarity problem*, J. of the Faculty of Sciences, Univ. of Tokyo, Section IA, Mathematics **37** (1990), 107–127.

[17] M. K. Kang and B. S. Lee, *Generalized vector quasivariational-like inequalities*, Honam Math. J. **26** (4) (2004).

[18] W. K. Kim and K. K. Tan, *On generalized vector quasi-variational inequalities*, Optimization **46** (1999), 185-198.

[19] S. Kum, *Sharpened forms of coincidence theorems and variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **79** (1996), 341-346.

[20] B. S. Lee and D. Y. Jung, *A fuzzy extension of Siddiqi et al.'s results for vector variational-like inequalities*, Indian J. pure appl. Math. **34** (10) (2003), 1495-1502.

[21] B. S. Lee and G. M. Lee, *A vector version of Minty's lemma and application*, Appl. Math. Lett. **12** (1999), 43-50.

[22] B. S. Lee, G. M. Lee and D. S. Kim, *Generalized vector-valued variational inequalities and fuzzy extensions*, J. Korean Math. Soc. **33** (1996), 609-624.

[23] B. S. Lee, G. M. Lee and D. S. Kim, *Generalized vector variational-like inequalities on locally convex Hausdorff topological vector spaces*, Indian J. pure appl. Math. **28** (1997), 33-41.

[24] G. M. Lee, D. S. Kim and B. S. Lee, *Generalized vector variational inequality*, Appl. Math. Lett. **9** (1) (1996), 39-42.

[25] G. M. Lee, D. S. Kim and B. S. Lee, *Strongly quasivariational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **78** (1996), 381-386.

[26] G. M. Lee, D. S. Kim and B. S. Lee, *Vector variational inequalities for fuzzy mappings*, Nonlinear Analysis Forum **4** (1999), 119–129.

[27] G. M. Lee, D. S. Kim, B. S. Lee and S. J. Cho, *Generalized vector variational inequality and fuzzy extension*, Appl. Math. Lett. **6** (1993), 47–51.

[28] G. M. Lee, B. S. Lee and S. S. Chang, *On vector quasivariational inequalities*, J. Math. Anal. Appl. **203** (1996), 626-639.

[29] K. L. Lin, D. P. Yang and J. C. Yao, *Generalized vector variational inequalities*, J. Optim. Th. Appl. **92** (1997), 117-125.

[30] D. C. Luc, *Theory of Vector Optimization*, Lectures Notes in Econ. and Mathem. System, vol. 319, Springer-Verlag, Berlin, 1989.

[31] M. A. Noor, *Variational inequalities for fuzzy mappings (I)*, Fuzzy Sets and Systems **55** (1993), 309-312.

[32] M. A. Noor, *General variational inequalities*, Appl. Math. Lett. **1** (1988), 119-121.

[33] S. Park, B. S. Lee and G. M. Lee, *A general vector-variational inequality and its fuzzy extension*, Internat. J. Math. & Math. Sci. **21** (1998), 637-642.

[34] A. H. Siddiqi, Q. H. Ansari and R. Ahmad, *On vector variational-like inequalities*, Indian J. pure appl. Math. **28(8)** (1997), 1009-1016.

[35] A. H. Siddiqi, Q. H. Ansari and A. Khaliq, *On vector variational inequalities*, J. Optim. Th. Appl. **84** (1995), 171-180.

[36] N. X. Tan, *Quasi-variational inequalities in topological linear locally convex Hausdorff spaces*, Mathematische Nachrichten, **122** (1985), 231-245.

[37] P. L. Yu, *Cone convexity, cone extreme points and nondominated solutions in decision problems with multiobjectives*, J. Optim. Th. Appl. **14** (1974), 319-377.

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