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INVERTIBLE INTERPOLATION ON $AX = Y$ IN A TRIDIAGONAL ALGEBRA ALG$\mathcal{L}$

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Abstract. Given operators X and Y acting on a separable Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A such that $AX = Y$. We show the following : Let $\mathcal{L}$ be a subspace lattice acting on a separable complex Hilbert space $\mathcal{H}$ and let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators acting on $\mathcal{H}$. Then the following are equivalent:

(1) There exists an invertible operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $AX = Y$.

(2) There exist bounded sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathbb{C}$ such that

$$\alpha_{2k-1} \neq 0, \quad \beta_{2k-1} = \frac{1}{\alpha_{2k-1}}, \quad \beta_{2k} = -\frac{\alpha_{2k}}{\alpha_{2k-1}\alpha_{2k+1}} \quad \text{and}$$

$$y_{i1} = \alpha_1 x_{i1} + \alpha_2 x_{i2}$$

$$y_{i\ 2k} = \alpha_{4k-1} x_{i\ 2k}$$

$$y_{i\ 2k+1} = \alpha_{4k} x_{i\ 2k} + \alpha_{4k+1} x_{i\ 2k+1} + \alpha_{4k+2} x_{i\ 2k+2} \text{ for } k \in \mathbb{N}.$$

1. Introduction

Let $\mathcal{A}$ be a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all operators acting on a Hilbert space $\mathcal{H}$ and let x and y be vectors in $\mathcal{H}$. An *interpolation question* for $\mathcal{A}$ asks for which x and y is there a bounded operator

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$A \in \mathcal{A}$ such that $Ax = y$. An n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison[6]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance[7]: his result was extended by Hopenwasser[2] to the case that $\mathcal{U}$ is a CSL-algebra. Munch[8] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser[3] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation n-vectors, although necessity was not proved in that paper.

We establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. The symbol Alg$\mathcal{L}$ is the algebra of all bounded operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let x and y be two vectors in a Hilbert space $\mathcal{H}$. Then $\langle x, y \rangle$ means the inner product of the vectors x and y. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. Results

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots\}$. Let $x_1, x_2, \cdots, x_n$ be vectors in $\mathcal{H}$. Then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$. Let $\mathcal{L}$ be the subspace lattice generated by the subspaces $[e_{2k-1}]$, $[e_{2k-1}, e_{2k}, e_{2k+1}]$ $(k = 1, 2, \cdots)$. Then the algebra Alg$\mathcal{L}$ is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson[1]. These algebras have been found to be useful counterexample to a number of plausible conjectures.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, \cdots\}$, where all non-starred entries are zero. It is easy to see that Alg$\mathcal{L}$=$\mathcal{A}$.

We consider interpolation problems for the above tridiagonal algebra Alg$\mathcal{L}$.

Lemma 1. *Let $A = (a_{ij})$ be an operator in the tridiagonal algebra Alg$\mathcal{L}$. If $A = (a_{ij})$ is invertible, then A^{-1} is an operator in the tridiagonal algebra Alg$\mathcal{L}$.*

Proof. Suppose that A is invertible. Then there is an operator $B = (b_{ij})$ such that $AB = I = BA$. Hence b_{ij} is zero except for b_{kk}, $b_{2k-1\ 2k}$ and $b_{2k+1\ 2k}$ for all k in $\mathbb{N}$. So $A^{-1} = B$ is an operator in the tridiagonal algebra Alg$\mathcal{L}$.

Theorem 2. *Let $\mathcal{L}$ be a subspace lattice acting on a separable complex Hilbert space $\mathcal{H}$ and let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators acting on $\mathcal{H}$. Then the following are equivalent:*

(1) There exists an invertible operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $AX = Y$.

(2) There exist bounded sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathbb{C}$ such that

$$\alpha_{2k-1} \neq 0, \quad \beta_{2k-1} = \frac{1}{\alpha_{2k-1}}, \quad \beta_{2k} = -\frac{\alpha_{2k}}{\alpha_{2k-1}\alpha_{2k+1}} \quad \text{and}$$

$$y_{i1} = \alpha_1 x_{i1} + \alpha_2 x_{i2}$$

$$y_{i\ 2k} = \alpha_{4k-1} x_{i\ 2k}$$

$$y_{i\ 2k+1} = \alpha_{4k} x_{i\ 2k} + \alpha_{4k+1} x_{i\ 2k+1} + \alpha_{4k+2} x_{i\ 2k+2} \text{ for } k \in \mathbb{N}.$$

Proof. Suppose that A is an invertible operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $AX = Y$. By Lemma 1, $A^{-1} = B = (b_{ij})$ is an operator in the tridiagonal algebra Alg$\mathcal{L}$. Let $\alpha_n = a_{ij}$ and $\beta_n = b_{ij}$ for $i + j - 1 = n$. Since $AB = I = BA$,

$$\alpha_{2k-1} \neq 0, \quad \beta_{2k-1} = \frac{1}{\alpha_{2k-1}} \quad \text{and} \quad \beta_{2k} = -\frac{\alpha_{2k}}{\alpha_{2k-1}\alpha_{2k+1}}.$$

Since $AX = Y$,

$$y_{i1} = \alpha_1 x_{i1} + \alpha_2 x_{i2}$$

$$y_{i\ 2k} = \alpha_{4k-1} x_{i\ 2k}$$

$$y_{i\ 2k+1} = \alpha_{4k} x_{i\ 2k} + \alpha_{4k+1} x_{i\ 2k+1} + \alpha_{4k+2} x_{i\ 2k+2} \text{ for } k \in \mathbb{N}.$$

Conversely, there exist bounded sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathbb{C}$ such that

$$\alpha_{2k-1} \neq 0, \quad \beta_{2k-1} = \frac{1}{\alpha_{2k-1}}, \quad \beta_{2k} = -\frac{\alpha_{2k}}{\alpha_{2k-1}\alpha_{2k+1}} \quad \text{and}$$

$$y_{i1} = \alpha_1 x_{i1} + \alpha_2 x_{i2}$$

$$y_{i\ 2k} = \alpha_{4k-1} x_{i\ 2k}$$

$$y_{i\ 2k+1} = \alpha_{4k} x_{i\ 2k} + \alpha_{4k+1} x_{i\ 2k+1} + \alpha_{4k+2} x_{i\ 2k+2} \text{ for } k \in \mathbb{N}.$$

Let $A = (a_{ij})$ be a matrix such that $a_{ij} = \alpha_n$ and $B = (b_{ij})$ be a matrix such that $b_{ij} = \beta_n$ for $i + j - 1 = n$. Since $\{\alpha_n\}$ is bounded, A is a bounded operator. Since $AB = I = BA$, A is invertible. Since

$$y_{i1} = \alpha_1 x_{i1} + \alpha_2 x_{i2}$$

$$y_{i\ 2k} = \alpha_{4k-1} x_{i\ 2k}$$

$$y_{i\ 2k+1} = \alpha_{4k} x_{i\ 2k} + \alpha_{4k+1} x_{i\ 2k+1} + \alpha_{4k+2} x_{i\ 2k+2} \text{ for } k \in \mathbb{N},$$

$AX = Y$.

Theorem 3. *Let n be a fixed natural number ($n \geq 2$). Let $\mathcal{L}$ be a subspace lattice acting on a separable complex Hilbert space $\mathcal{H}$ and let $X_i = (x_{jk}^{(i)})$ and $Y_i = (y_{jk}^{(i)})$ be operators acting on $\mathcal{H}$ for all $i = 1, 2, \cdots, n$. Then the following are equivalent:*

(1) There exists an invertible operator $A = (a_{jk})$ in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for all $i = 1, 2, \cdots, n$.

(2) There exist bounded sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathbb{C}$ such that

$$\alpha_{2k-1} \neq 0, \quad \beta_{2k-1} = \frac{1}{\alpha_{2k-1}}, \quad \beta_{2k} = -\frac{\alpha_{2k}}{\alpha_{2k-1}\alpha_{2k+1}} \quad \text{and}$$

$$\begin{aligned} y_{j1}^{(i)} &= \alpha_1 x_{j1}^{(i)} + \alpha_2 x_{j2}^{(i)} \\ y_{j\,2k}^{(i)} &= \alpha_{4k-1} x_{j\,2k}^{(i)} \\ y_{j\,2k+1}^{(i)} &= \alpha_{4k} x_{j\,2k}^{(i)} + \alpha_{4k+1} x_{j\,2k+1}^{(i)} + \alpha_{4k+2} x_{j\,2k+2}^{(i)} \text{ for } k \in \mathbb{N}. \end{aligned}$$

Proof. Suppose that A is an invertible operator $A = (a_{jk})$ in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for all $i = 1, 2, \cdots, n$. By Lemma 1, $A^{-1} = B = (b_{jk})$ is an operator in the tridiagonal algebra Alg$\mathcal{L}$. Let $\alpha_n = a_{jk}$ and $\beta_n = b_{jk}$ for $j + k - 1 = n$. Since $AB = I = BA$,

$$\alpha_{2k-1} \neq 0, \quad \beta_{2k-1} = \frac{1}{\alpha_{2k-1}}, \quad \beta_{2k} = -\frac{\alpha_{2k}}{\alpha_{2k-1}\alpha_{2k+1}}.$$

Since $AX_i = Y_i$ for all $i = 1, 2, \cdots, n$,

$$\begin{aligned} y_{j1}^{(i)} &= \alpha_1 x_{j1}^{(i)} + \alpha_2 x_{j2}^{(i)} \\ y_{j\,2k}^{(i)} &= \alpha_{4k-1} x_{j\,2k}^{(i)} \\ y_{j\,2k+1}^{(i)} &= \alpha_{4k} x_{j\,2k}^{(i)} + \alpha_{4k+1} x_{j\,2k+1}^{(i)} + \alpha_{4k+2} x_{j\,2k+2}^{(i)} \text{ for } k \in \mathbb{N}. \end{aligned}$$

Conversely, there exist bounded sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathbb{C}$ such that

$$\alpha_{2k-1} \neq 0, \quad \beta_{2k-1} = \frac{1}{\alpha_{2k-1}}, \quad \beta_{2k} = -\frac{\alpha_{2k}}{\alpha_{2k-1}\alpha_{2k+1}} \quad \text{and}$$

$$
\begin{aligned}
y_{j1}^{(i)} &= \alpha_1 x_{j1}^{(i)} + \alpha_2 x_{j2}^{(i)} \\
y_{j\ 2k}^{(i)} &= \alpha_{4k-1} x_{j\ 2k}^{(i)} \\
y_{j\ 2k+1}^{(i)} &= \alpha_{4k} x_{j\ 2k}^{(i)} + \alpha_{4k+1} x_{j\ 2k+1}^{(i)} + \alpha_{4k+2} x_{j\ 2k+2}^{(i)} \text{ for } k \in \mathbb{N}.
\end{aligned}
$$

Let $A = (a_{jk})$ be a matrix such that $a_{jk} = \alpha_n$ and $B = (b_{jk})$ be a matrix such that $b_{jk} = \beta_n$ for $j + k - 1 = n$. Since $\{\alpha_n\}$ is bounded, A is a bounded operator. Since $AB = I = BA$, A is invertible. Since

$$
\begin{aligned}
y_{j1}^{(i)} &= \alpha_1 x_{j1}^{(i)} + \alpha_2 x_{j2}^{(i)} \\
y_{j\ 2k}^{(i)} &= \alpha_{4k-1} x_{j\ 2k}^{(i)} \\
y_{j\ 2k+1}^{(i)} &= \alpha_{4k} x_{j\ 2k}^{(i)} + \alpha_{4k+1} x_{j\ 2k+1}^{(i)} + \alpha_{4k+2} x_{j\ 2k+2}^{(i)} \text{ for } k \in \mathbb{N},
\end{aligned}
$$

$AX_i = Y_i$ for all $i = 1, 2, \cdots, n$.

By the similar way with the above, we have the following.

Theorem 4. *Let $\mathcal{L}$ be a subspace lattice acting on a separable complex Hilbert space $\mathcal{H}$ and let $X_i = (x_{jk}^{(i)})$ and $Y_i = (y_{jk}^{(i)})$ be operators acting on $\mathcal{H}$ for all $i = 1, 2, \cdots$. Then the following are equivalent:*

(1) There exists an invertible operator $A = (a_{jk})$ in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for all $i = 1, 2, \cdots$.

(2) There exist bounded sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $\mathbb{C}$ such that

$$
\alpha_{2k-1} \neq 0, \quad \beta_{2k-1} = \frac{1}{\alpha_{2k-1}}, \quad \beta_{2k} = -\frac{\alpha_{2k}}{\alpha_{2k-1}\alpha_{2k+1}} \quad \text{and}
$$

$$
\begin{aligned}
y_{j1}^{(i)} &= \alpha_1 x_{j1}^{(i)} + \alpha_2 x_{j2}^{(i)} \\
y_{j\ 2k}^{(i)} &= \alpha_{4k-1} x_{j\ 2k}^{(i)} \\
y_{j\ 2k+1}^{(i)} &= \alpha_{4k} x_{j\ 2k}^{(i)} + \alpha_{4k+1} x_{j\ 2k+1}^{(i)} + \alpha_{4k+2} x_{j\ 2k+2}^{(i)} \text{ for } i,\ k \in \mathbb{N}.
\end{aligned}
$$

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