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ANNIHILATORS OF SUBTRACTION ALGEBRAS

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Abstract. A characterization of prime ideals is discussed. A relation between prime ideals and ideals of the form $A_w^{\wedge}$ is given. The prime ideal theorem is established. The notion of annihilators is introduced, and basic properties are investigated.

1. Introduction

B. M. Schein [6] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [7] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. H. Kim and H. S. Kim [5] showed that a subtraction algebra is equivalent to an implicative BCK-algebra, and a subtraction semigroup is a special case of a BCI-semigroup (or, IS-algebra as a new name) which is a generalization of a ring. Y. B. Jun et al. [3] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [2], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Y. B. Jun and K. H. Kim [4] introduced the notion of prime and

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irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also provided a condition for an ideal to be a prime/irreducible ideal. In this paper, we give other characterization of prime ideals. We discuss a relation between prime ideals and ideals of the form $A_w^{\wedge}$. We construct prime ideal theorem. We also introduce the notion of annihilators, and investigate basic properties.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;

(S2) $x - (x - y) = y - (y - x)$;

(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [3, 4]):

(a1) $(x - y) - y = x - y$.

(a2) $x - 0 = x$ and $0 - x = 0$.

(a3) $(x - y) - x = 0$.

(a4) $x - (x - y) \leq y$.

(a5) $(x - y) - (y - x) = x - y$.

(a6) $x - (x - (x - y)) = x - y$.

(a7) $(x - y) - (z - y) \leq x - z$.

(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.

(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.

(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.

(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

3. Prime ideals and Annihilators

In what follows let X denote a subtraction algebra unless otherwise specified.

Definition 3.1. (Jun et al. [3]) A nonempty subset A of X is called an *ideal* of X if it satisfies

(I1) $0 \in A$

(I2) $y \in A$ and $x - y \in A$ imply $x \in A$ for all $x, y \in X$.

Note that an ideal A of X has the following property: If $x \leq y$ and $y \in A$, then $x \in A$ (see [4, Lemma 3.2]). For any subset A of X the minimal ideal of X containing A is called the *ideal generated* by A, and denoted by $\langle A \rangle$. Then by [2, Theorem 3.2], we have

$$\langle A \rangle = \{x \in X \mid (\cdots((x - a_1) - a_2) - \cdots) - a_n = 0 \text{ for some } a_1, a_2, \cdots, a_n \in A\}.$$

For any nonnegative integer n we define $x - n(y)$ recursively as follows: $x - 0(y) = x$, $x - 1(y) = x - y$, and $x - (n+1)(y) = (x - n(y)) - y$ for all $x, y \in X$.

Theorem 3.2. *Let A be an ideal of X and $w \in X$. Then*

$$\langle A \cup \{w\} \rangle = \{x \in X \mid x - n(w) \in A \text{ for some nonnegative integer } n\}.$$

Proof. Denote

$$\Omega := \{x \in X \mid x - n(w) \in A \text{ for some nonnegative integer } n\}.$$

Let $x \in \langle A \cup \{w\} \rangle$. Then

$$((\cdots((x - a_1) - a_2) - \cdots) - a_n) - k(w) = 0$$

for some $a_1, a_2, \cdots, a_n \in A$ and some nonnegative integer k. Using (S3) repeatedly, we have

$$(\cdots((x - k(w)) - a_1) - \cdots) - a_n = 0 \in A,$$

and so $x - k(w) \in A$ by (I2). This shows that $x \in \Omega$ so that $\langle A \cup \{w\} \rangle \subseteq \Omega$. Conversely if $x \in \Omega$, then there exists a nonnegative integer n such that $x - n(w) \in A$. Since $A \subseteq \langle A \cup \{w\} \rangle$, it follows that

$$(x - (n-1)(w)) - w = x - n(w) \in \langle A \cup \{w\} \rangle$$

so that $x - (n-1)(w) \in \langle A \cup \{w\} \rangle$. Repeating this process we conclude that $x = x - 0(w) \in \langle A \cup \{w\} \rangle$. Thus $\Omega \subseteq \langle A \cup \{w\} \rangle$. This completes the proof. □

Theorem 3.3. (Jun and Kim [4, Theorem 3.4]) *Let A be an ideal of X. For any $w \in X$, the set*

$$A_w^{\wedge} := \{x \in X \mid w \wedge x \in A\}$$

is an ideal of X containing A.

Definition 3.4. (Jun and Kim [4]) A *prime ideal* of X is defined to be an ideal P of X such that if $x \wedge y \in P$ then $x \in P$ or $y \in P$.

Proposition 3.5. *Let P and A be ideals of X such that $A \subseteq P$. If P is prime, then $A_w^{\wedge} \subseteq P$ for all $w \in X \setminus P$.*

Proof. Let $x \in A_w^{\wedge}$ for all $w \in X \setminus P$. Then $w \wedge x \in A \subseteq P$. Since P is prime, it follows that $x \in P$ because $w \notin P$. Hence $A_w^{\wedge} \subseteq P$ for all $w \in X \setminus P$. □

Proposition 3.6. *If P is a prime ideal of X, then $X \setminus P$ is $\wedge$-closed, that is, $x \wedge y \in X \setminus P$ whenever $x \in X \setminus P$ and $y \in X \setminus P$.*

Proof. The proof is straightforward. □

Theorem 3.7. *An ideal A of X is prime if and only if $A_w^\wedge = A$ for all $w \in X \setminus A$.*

Proof. Suppose A is a prime ideal of X and let $w \in X \setminus A$. If $x \in A_w^\wedge$, then $w \wedge x \in A$, and so $x \in A$. Hence $A_w^\wedge = A$. Conversely assume that $A_w^\wedge = A$ for all $w \in X \setminus A$. Let $x, y \in X$ be such that $x \wedge y \in A$ and $x \notin A$. Then $y \in A_x^\wedge = A$, hence A is prime. □

Proposition 3.8. *If A is an ideal of X, then $A = A_w^\wedge \cap \langle A \cup \{w\} \rangle$ for all $w \in X \setminus A$.*

Proof. Obviously, $A \subseteq A_w^\wedge \cap \langle A \cup \{w\} \rangle$. Let $x \in A_w^\wedge \cap \langle A \cup \{w\} \rangle$. Then $w \wedge x \in A$ and $x \in \langle A \cup \{w\} \rangle$. It follows from Theorem 3.2 that $x - n(w) \in A$ for some nonnegative integer n. Now

$$\begin{aligned} x - n(w) &= (x - (n-1)(w)) - w \\ &= (x - (n-1)(w)) - ((x - (n-1)(w)) \wedge w). \end{aligned} \tag{3.1}$$

Since $x - (n-1)(w) \leq x$, therefore

$$(x - (n-1)(w)) \wedge w \leq x \wedge w = w \wedge x$$

and so $(x-(n-1)(w)) \wedge w \in A$. Applying (3.1) and (I2), we have $x-(n-1)(w) \in A$. Continuing this process, we get $x \in A$ and, consequently, $A_w^\wedge \cap \langle A \cup \{w\} \rangle \subseteq A$. This completes the proof. □

Theorem 3.9. (prime ideal theorem) *Let A be an ideal of X and G a $\wedge$-closed subset of X for which A and G are disjoint. Then there exists a prime ideal P of X such that $A \subseteq P$ and $G \cap P = \emptyset$.*

Proof. Using an application of Zorn's lemma we know that there is an ideal P being the maximal element of the family of all ideals that contain A and have empty intersection with G. We now prove that P

is prime. Suppose that P is not prime. Then, by Theorem 3.7, there exists an element $w \in X \setminus P$ such that $P_w^\wedge \neq P$. Note that P is properly contained in both $P_w^\wedge$ and $\langle P \cup \{w\}\rangle$; therefore the maximality of P implies that $P_w^\wedge \cap G \neq \emptyset$ and $\langle P \cup \{w\}\rangle \cap G \neq \emptyset$. Let $x \in P_w^\wedge \cap G$ and $y \in \langle P \cup \{w\}\rangle \cap G$. Then $x \wedge y \in P_w^\wedge \cap \langle P \cup \{w\}\rangle = P$ by Proposition 3.8, and $x \wedge y \in G$ because G is $\wedge$-closed. Consequently, $x \wedge y \in P \cap G$ and so $P \cap G \neq \emptyset$, a contradiction. This completes the proof. □

Definition 3.10. Let S be a nonempty subset of X. The *annihilator* of S is the subset S^a of X given by

$$S^a = \{x \in X \mid x \wedge y = 0 \text{ for all } y \in S\}.$$

If $S = \{x\}$, we write x^a instead of $\{x\}^a$. It is obvious that $X^a = \{0\}$ and $0^a = X$.

Proposition 3.11. *Let S and T be nonempty subsets of X. Then*

(i) $x \in S^a$ *if and only if* $x - y = x$ *for all* $y \in S$ *if and only if* $y - x = y$ *for all* $y \in S$.
(ii) S^a *is an ideal of* X.
(iii) $S \subseteq T$ *implies* $T^a \subseteq S^a$.
(iv) $S \subseteq (S^a)^a$.
(v) $S^a = ((S^a)^a)^a$.
(vi) $(S \cup T)^a = S^a \cap T^a$.
(vii) $S^a = \bigcap_{x \in S} x^a$.
(viii) $S \cap S^a = \{0\}$.

Proof. (i) If $x \in S^a$, then $x - (x - y) = x \wedge y = 0$ for all $y \in S$ and so $x \leq x - y$. Since $x - y \leq x$ for all $x, y \in X$, it follows that $x - y = x$ for all $y \in S$. Using (S2), we conclude that $y - x = y$ for all $y \in S$. Conversely, if $x - y = x$ (or $y - x = y$) for all $y \in S$, then clearly $x \in S^a$.

(ii) Obviously $0 \in S^a$. Let $x, y \in X$ be such that $y \in S^a$ and $x - y \in S^a$. Using (i), we have $w - y = w$ and $w - (x - y) = w$ for all $w \in S$. It follows from (a7) that

$$w = w - (x - y) = (w - y) - (x - y) \leq w - x.$$

But since $w - x \leq w$, we conclude that $w - x = w$ for all $w \in S$, and so $x \in S^a$ by (i). Therefore S^a is an ideal of X.

(iii) Assume that $S \subseteq T$ and let $x \in T^a$. Then $x \wedge w = 0$ for all $w \in T$. Since $S \subseteq T$, it follows that $x \wedge w = 0$ for all $w \in S$, that is, $x \in S^a$. Hence $T^a \subseteq S^a$.

(iv) Let $y \in S$. Then $y \wedge x = x \wedge y = 0$ for all $x \in S^a$, which implies that $y \in (S^a)^a$. Thus $S \subseteq (S^a)^a$.

(v) Using (iii) and (iv), we get $\left((S^a)^a\right)^a \subseteq S^a$. Replacing S by S^a in (iv), we obtain $S^a \subseteq \left((S^a)^a\right)^a$. Therefore (v) is valid.

(vi) Using (iii), we have $(S \cup T)^a \subseteq S^a$ and $(S \cup T)^a \subseteq T^a$. Hence $(S \cup T)^a \subseteq S^a \cap T^a$. Now let $x \in S^a \cap T^a$. Then $x \in S^a$ and $x \in T^a$. Let $y \in S \cup T$. If $y \in S$, then $x \wedge y = 0$ since $x \in S^a$; and if $y \in T$ then $x \wedge y = 0$ since $x \in T^a$. It follows that $x \wedge y = 0$ for all $y \in S \cup T$ so that $x \in (S \cup T)^a$. Thus (vi) is valid.

(vii) Since $S = \bigcup_{x \in S} \{x\}$, we have $S^a = \bigcap_{x \in S} x^a$.

(viii) Let $x \in S \cap S^a$. Then $x = x \wedge x = 0$, and thus $S \cap S^a = \{0\}$. This completes the proof. □

Proposition 3.12. *Let S and T be ideals of X. Then*

(i) $S \cap T = \{0\}$ *if and only if* $S \subseteq T^a$.

(ii) $S \cap (S \cap T)^a \subseteq T^a$.

Proof. (i) Assume that $S \cap T = \{0\}$ and let $x \in S$. For any $y \in T$, if $y = 0$, then clearly $x \wedge y = 0$. If $y \neq 0$, then $y \notin S$ by assumption. But $x \wedge y \leq x$ implies $x \wedge y \in S$ because S is an ideal of X and $x \in S$; and $x \wedge y \leq y$ implies $x \wedge y \in T$ because T is an ideal of X and $y \in T$. Hence $x \wedge y \in S \cap T = \{0\}$, and so $x \wedge y = 0$. Therefore $x \in T^a$ and consequently $S \subseteq T^a$. Conversely, if $S \subseteq T^a$ then $S \cap T \subseteq T^a \cap T = \{0\}$. Since $\{0\} \subseteq S \cap T$, it follows that $S \cap T = \{0\}$.

(ii) Note from Proposition 3.11(viii) that

$$\{0\} = (S \cap T) \cap (S \cap T)^a = (S \cap (S \cap T)^a) \cap T,$$

which is equivalent to $S \cap (S \cap T)^a \subseteq T^a$ by (i). □

Theorem 3.13. *If S is an ideal of X, then $(S^a)^a = S$.*

Proof. Let $x \in X \setminus S$ and $y \in S$. Since $(x - y) - y = x - y$, it follows from Proposition 3.11(i) that $x - y \in S^a$. Note that $S^a \cap (S^a)^a = \{0\}$. Hence $x - y \notin (S^a)^a$, because if $x - y \in (S^a)^a$ then $x - y = 0$ and thus $x \in S$ by (I1) and (I2), a contradiction. But $y \in S \subseteq (S^a)^a$ and $(S^a)^a$ being an ideal imply that $x \notin (S^a)^a$. Consequently, $(S^a)^a \subseteq S$. This completes the proof. □

Corollary 3.14. *If S and T are ideals of X, then $S \cap T^a = \{0\}$ if and only if $S \subseteq T$.*

Theorem 3.15. *If S and T are ideals of X, then $(S \cap T)^a = \langle S^a \cup T^a \rangle$.*

Proof. Since $S \cap T \subseteq S, T$, it follows from Proposition 3.11(iii) that $S^a \subseteq (S \cap T)^a$ and $T^a \subseteq (S \cap T)^a$ so that $\langle S^a \cup T^a \rangle \subseteq (S \cap T)^a$. Note that $S^a, T^a \subseteq S^a \cup T^a \subseteq \langle S^a \cup T^a \rangle$. Hence, by Proposition 3.11(iii) and Theorem 3.13, we have $\langle S^a \cup T^a \rangle^a \subseteq (S^a)^a = S$ and $\langle S^a \cup T^a \rangle^a \subseteq (T^a)^a = T$, and so $\langle S^a \cup T^a \rangle^a \subseteq S \cap T$. Using Proposition 3.11(iii) and Theorem 3.13 again, we conclude that

$$(S \cap T)^a \subseteq \left(\langle S^a \cup T^a \rangle^a\right)^a = \langle S^a \cup T^a \rangle.$$

Consequently, $(S \cap T)^a = \langle S^a \cup T^a \rangle$. □

Theorem 3.16. *If S is a subset of X, then $\langle S \rangle = (S^a)^a$.*

Proof. From (ii) and (iv) of Proposition 3.11 it follows that $\langle S \rangle \subseteq (S^a)^a$. Since $S \subseteq \langle S \rangle$, by Proposition 3.11(iii) we get $\langle S \rangle^a \subseteq S^a$, and so $(S^a)^a \subseteq \left(\langle S \rangle^a\right)^a = \langle S \rangle$ by Proposition 3.11(iii) and Theorem 3.13. Hence $\langle S \rangle = (S^a)^a$. □

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References

[1] J. C. Abbott, *Sets, Lattices and Boolean Algebras,* Allyn and Bacon, Boston 1969.

[2] Y. B. Jun and H. S. Kim, *On ideals in subtraction algebras,* Sci. Math. Japon. (submitted)

[3] Y. B. Jun, H. S. Kim and E. H. Roh, *Ideal theory of subtraction algebras,* Sci. Math. Japon. Online **e-2004** (2004), 397–402.

[4] Y. B. Jun and K. H. Kim, *Prime and irreducible ideals in subtraction algebras,* Internat. J. Math. & Math. Sci. (submitted)

[5] Y. H. Kim and H. S. Kim, *Subtraction algebras and BCK-algebras,* Math. Bohemica **128** (2003), 21–24.

[6] B. M. Schein, *Difference Semigroups,* Comm. in Algebra **20** (1992), 2153–2169.

[7] B. Zelinka, *Subtraction Semigroups,* Math. Bohemica, **120** (1995), 445–447.

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