Honam Mathematical J. **27**(2005), No. 2, pp. 301–315

A FUNCTIONAL CENTRAL LIMIT THEOREM FOR MULTIVARIATE LINEAR PROCESS WITH POSITIVELY DEPENDENT RANDOM VECTORS

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Abstract. Let $\{A_u, u = 0, 1, 2, \cdots\}$ be a sequence of coefficient matrices such that $\sum_{u=0}^{\infty} \|A_u\| < \infty$ and $\sum_{u=0}^{\infty} A_u \neq O_{m\times m}$, where for any $m \times m(\ m \geq 1)$, matrix $A = (a_{ij})$, $\|A\| = \sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|$ and $O_{m\times m}$ denotes the $m \times m$ zero matrix. In this paper, a functional central limit theorem is derived for a stationary $m-$dimensional linear process $\mathbb{X}_t$ of the form $\mathbb{X}_t = \sum_{u=0}^{\infty} A_u \mathbb{Z}_{t-u}$, where $\{\mathbb{Z}_t,\ t = 0, \pm 1, \pm 2, \cdots\}$ is a stationary sequence of linearly positive quadrant dependent $m-$dimensional random vectors with $E(\mathbb{Z}_t) = \mathbb{O}$ and $E\|\mathbb{Z}_t\|^2 < \infty$.

1. Introduction

Lehmann(1966) introduced a simple and natural definition of positive dependence: A sequence $\{Y_j, j \geq 1\}$ of random variables is said to be pairwise positive quadrant dependent(pairwise PQD) if for any real y_i, y_j and $i \neq j$,

$$P\{Y_i > y_i,\ Y_j > y_j\} \geq P\{Y_i > y_i\}P\{Y_j > y_j\}.$$

Newman's original central limit theorem for associated random variables[7] requires only that positive linear combinations of the random

Received January 17, 2005. Revised May 25, 2005.

2000 Mathematics Subject Classification : 60F05, 60G10.

Key words and phrases : linearly positive quadrant dependence, functional central limit theorem, linear process, random vector.

variables are PQD. Primarily motivated by this, Newman(1984) introduced the following positive dependence: A sequence $\{Y_j, j \geq 1\}$ of random variables is said to be linearly positive quadrant dependent(LPQD) if for any disjoint subsets A, B and positive r_j's,

$$\sum_{i \in A} r_i Y_i \text{ and } \sum_{j \in B} r_j Y_j \text{ are PQD.}$$

We extend these positive dependence to random vectors as follows : Let $\{\mathbb{Z}_j, j \geq 1\}$ be a sequence of random vectors. $\{\mathbb{Z}_j, j \geq 1\}$ are said to be pairwise PQD if for any real vectors z_i, z_j and $i \neq j$,

$$(1) \qquad P\{\mathbb{Z}_i > z_i,\ \mathbb{Z}_j > z_j\} \geq P\{\mathbb{Z}_i > z_i\}P\{\mathbb{Z}_j > z_j\},$$

and they are said to be LPQD if for any disjoint subsets A, B and for any positive r_i's,

$$(2) \qquad \sum_{i \in A} r_i \mathbb{Z}_i \text{ and } \sum_{i \in B} r_j \mathbb{Z}_j \text{ are PQD.}$$

Throughout we assume that $\{A_u, u = 0, 1, 2, \cdots\}$ is a sequence of coefficient matrices such that

$$(3) \qquad \sum_{u=o}^{\infty} \|A_u\| < \infty \text{ and } \sum_{u=o}^{\infty} A_u \neq O_{m \times m},$$

where, for any $m \times m$, $m \geq 1$, matrix $A = (a_{ij})$, $\|A\| = \sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|$ and $O_{m \times m}$ denotes the $m \times m$ zero matrix.

Let $\mathbb{X}_t$, $t = 0, \pm 1, \cdots$, be an $m-$dimensional linear process of the form

$$(4) \qquad \mathbb{X}_t = \sum_{u=0}^{\infty} A_u \mathbb{Z}_{t-u},$$

defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\{\mathbb{Z}_t, t = 0, \pm 1, \pm 2, \cdots\}$ is a sequence of stationary LPQD $m-$dimensional random vectors with $E(\mathbb{Z}_t) = \mathbb{O}$, $E\|\mathbb{Z}_t\|^2 < \infty$ and positive definite covariance matrix $\Gamma_{m \times m}$

and let

$$(5) \qquad T = \Big(\sum_{j=o}^{\infty} A_j\Big)\Gamma\Big(\sum_{j=o}^{\infty} A_j\Big)',$$

where the prime denotes transpose and $\Gamma = [\sigma_{kj}]$ is the covariance matrix of $\mathbb{Z}_1 = (Z_1^{(1)}, \cdots, Z_1^{(m)})$ and $\mathbb{Z}_t = (Z_t^{(1)}, \cdots, Z_t^{(m)})$ with

$$(6) \qquad \sigma_{kj} = E(Z_1^{(k)} Z_1^{(j)}) + \sum_{t=2}^{\infty}\Big(E(Z_1^{(k)} Z_t^{(j)}) + E(Z_1^{(j)} Z_t^{(k)})\Big).$$

We also let W^m denote a Wiener measure on $\mathcal{D}^m[0,1]$, the space of all functions f defined on $[0,1]$ into $\mathbb{R}^m$, which have left hand limits and are continuous from the right. Put $\mathbb{S}_n = \sum_{t=1}^{n} \mathbb{X}_t$, $(n \geq 0)$ $(\mathbb{S}_0 = \mathbb{O})$, and define, for $n \geq 1$, the stochastic process ξ_n by

$$(7) \qquad \xi_n(u) = n^{-\frac{1}{2}} T^{-\frac{1}{2}} \mathbb{S}_{[nu]} \; 0 \leq u \leq 1.$$

For $m = 1$, under linearly positive quadrant dependence condition, Kim and Baek(2001) showed that the process ξ_n defined in (7) converges weakly to a Wiener measure W on $D[0,1]$, the set of all functions f defined on $[0,1]$ into $\mathbb{R}$, which have left hand limits and are continuous from the right (see Theorem A below).

Theorem A([5]). *Let $\{X_t,\ t \geq 1\}$ be a stationary linear process of the form $X_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$, where $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty} |a_j| < \infty$ and $\{\varepsilon_t\}$ is a strictly stationary sequence of LPQD random variables with $E\varepsilon_t = 0$ and $0 < E\varepsilon_t^2 < \infty$. Let $S_n = \sum_{t=1}^{n} X_t$, $\tau^2 = \sigma^2\Big(\sum_{j=0}^{\infty} a_j\Big)^2$, where*

$$(8) \qquad 0 < \sigma^2 = E(\varepsilon_1^2) + 2\sum_{t=2}^{\infty} E(\varepsilon_1 \varepsilon_t) < \infty.$$

Define for $n \geq 1$, the stochastic process

$$\eta_n(u) = n^{-1/2} S_{[nu]},\ u \in [0,1].$$

Assume

$$\sum_{t=n+1}^{\infty} E(\varepsilon_1\varepsilon_t) = O(n^{-\rho}) \text{ for some } \rho > 0, \tag{9}$$

$$E|\varepsilon_t|^s < \infty \text{ for some } s > 2. \tag{10}$$

Then the process η_n *converges weakly to a Wiener measure* W *with variance* τ^2 *on the space of all functions on* $[0,1]$, *which have left limits and are continuous from the right.*

In this paper we derive the functional central limit for LPQD m-dimensional random vectors and apply this result to extend Theorem A on invariance principle to the case of $m-$dimensional linear process.

2. Preliminaries

Lemma 2.1. Let $\{\epsilon_i,\ i \geq 1\}$ be a sequence of stationary LPQD random variables with $E(\epsilon_i) = 0$. If (9) and (10) hold, then we have, for all $n \geq 1$

$$E\Big(\max_{1\leq k\leq n} |S_k|^r\Big) \leq Cn^{r/2} \text{ for } r > 2 \text{ and some } C > 0, \tag{11}$$

where $S_n = \sum_{i=1}^n \epsilon_i$.

Proof. According to Lemma 3 of Birkel(1993), we have $E|S_n|^r = O(n^{\frac{r}{2}})$, for $r > 2$. Hence, Theorem 3.7.5 of Stout(1974) yields $E(\max_{1\leq k\leq n} |S_k|)^r = O(n^{\frac{r}{2}})$. Thus, the desired result(11) follows.

Lemma 2.2. Let $\{\mathbb{Z}_i, 1 \leq i \leq n\}$ be a sequence of stationary LPQD random vectors in $\mathbb{R}^m$ with $E(\mathbb{Z}_i) = \mathbb{O}$ and $E\|\mathbb{Z}_i\|^r < \infty$ for $r > 2$. If

$$\sum_{t=n+1}^{\infty} E\|\mathbb{Z}_1\mathbb{Z}_t\| = O(n^{-\rho}) \text{ for some } \rho > 0, \tag{12}$$

$$E\|\mathbb{Z}_t\|^s < \infty \text{ for some } s > 2, \tag{13}$$

then there is a constant $0 < B < \infty$ such that

$$E \max_{1\le k\le n} \|\sum_{i=1}^{k} \mathbb{Z}_i\|^r \le Bm^r n^{r/2}. \tag{14}$$

Proof. Note that

$$\max_{1\le k\le n} \|\sum_{i=1}^{k} \mathbb{Z}_i\| \le \sum_{j=1}^{m} \max_{1\le k\le n} |\sum_{i=1}^{k} Z_i^{(j)}| \tag{15}$$

and by Lemma 2.1,

$$E \max_{1\le k\le n} |\sum_{i=1}^{k} Z_i^{(j)}|^r \le Cn^{r/2}, \tag{16}$$

where $Z_i^{(j)}$ is the $j-$th component of $\mathbb{Z}_i$. Combining (15) and (16), the desired result (14) follows.

From the Newman's central limit theorem for LPQD random variables (Theorem 12 of [8]) we obtain the following result by means of the simple device due to Cramer Wold (see [1], [3]).

Lemma 2.3. Let $\{\mathbb{Z}_t\}$ be a sequence of stationary LPQD $m-$dimensional random vectors with $E(\mathbb{Z}_t) = \mathbb{O}$ and $E\|\mathbb{Z}_t\|^2 < \infty$. If

$$E\|\mathbb{Z}_1\|^2 + 2\sum_{t=2}^{\infty}\sum_{i=1}^{m}(Z_1^{(i)} Z_t^{(i)}) = \sigma^2 < \infty \tag{17}$$

holds, then we have

$$n^{-1/2}\sum_{t=1}^{n} \mathbb{Z}_t \longrightarrow^{\mathcal{D}} N(\mathbb{O}, \Gamma), \tag{18}$$

where $\Gamma = [\sigma_{kj}]$ is defined in (6), $N(\mathbb{O},\Gamma)$ denotes an $m-$dimensional normal random vector and the symbol $\longrightarrow^{\mathcal{D}}$ indicates convergence in distribution.

Lemma 2.4. (Burton et al. 1986) Let $\mathbb{Y}_1, \mathbb{Y}_2, \cdots, \mathbb{Y}_k$ and $\mathbb{Y}'_1, \mathbb{Y}'_2, \cdots, \mathbb{Y}'_k$ be two sets of $\mathbb{R}^m-$valued random vectors having finite moment generating functions. Suppose that, for all vectors $a_1, a_2, \cdots, a_n \ge 0$ the joint distributions of $(< a_1, \mathbb{Y}_1 >, < a_2, \mathbb{Y}_2 >, \cdots, < a_k, \mathbb{Y}_k >)$ and $(< a_1, \mathbb{Y}'_1 >, \cdots, < a_k, \mathbb{Y}'_k >)$ coincide. Then $(\mathbb{Y}_1, \mathbb{Y}_2, \cdots, \mathbb{Y}_k)$ and $(\mathbb{Y}'_1, \mathbb{Y}'_2, \cdots, \mathbb{Y}'_k)$ have the same joint distribution.

Theorem 2.5. *Let $\{\mathbb{Z}_i \ ; \ i \geq 1\}$ be a stationary sequence of $m-$dimensional LPQD random vectors with $E(\mathbb{Z}_t) = \mathbb{O}$ and $E\|\mathbb{Z}_1\|^2 < \infty$. Define for $u \in [0,1],\ n \geq 1$,*

$$W_n(u) = n^{-1/2} \sum_{j=1}^{[nu]} \mathbb{Z}_j, \tag{19}$$

where the covariance matrix $\Gamma = [\sigma_{kj}]$ is defined in (6).
If (12), (13) and (17) hold, then W_n converges weakly to Wiener measure W^m with covariance matrix $\Gamma = [\sigma_{kj}]$.

Proof. First for every $\varepsilon > 0$ we will prove

$$\lim_{n \geq 1} \sup P\{w(W_n, \delta) > \varepsilon\} \to 0 \text{ as } \delta \to 0, \tag{20}$$

where $w(W_n, \delta) = \sup_{|u-v|<\delta} \|W_n(u) - W_n(v)\|$. Let $\varepsilon > 0$ be given. Then

$$\begin{aligned} &P\{w(W_n, \delta) > \varepsilon\} \\ &\leq \sum_{i=0}^{1/\delta} P\{n^{-1/2} \max_{[ni\delta]<k\leq[n(i+1)\delta]} \|\mathbb{S}_k - \mathbb{S}_{[ni\delta]}\| > \varepsilon/3\} \\ &= \sum_{i=0}^{[1/\delta]} P\{n^{-1/2} \max_{0\leq k\leq[n\delta]} \|S_k\| > \varepsilon/3\} , \end{aligned} \tag{21}$$

where $\mathbb{S}_n = \sum_{i=1}^n \mathbb{Z}_i$. Using Markov inequality, Lemma 2.2 and (21), we obtain, for $r > 2$,

$$
\begin{aligned}
&P\{w(W_n,\delta) > \varepsilon\} \\
&\le (3/\varepsilon)^{-r} n^{-r/2} \sum_{i=0}^{[1/\delta]} E\Big(\max_{[ni\delta]<k\le[n(i+1)\delta]} \|\mathbb{S}_k - \mathbb{S}_{[ni\delta]}\| \Big)^r \\
(22)\qquad &\le c(\varepsilon) n^{-r/2} m^r \sum_{i=0}^{[1/\delta]} \Big([n(i+1)\delta] - [ni\delta] \Big)^{r/2} \\
&\le c(\varepsilon) m^r n^{-r/2} \Big([1/\delta] + 1 \Big) (n\delta + 1)^{r/2} \\
&= c(\varepsilon) m^r (\delta + \frac{1}{n})^{r/2} \Big([1/\delta] + 1 \Big),
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\limsup_{n\ge1} P\{w(W_n,\delta) > \varepsilon\} \\
&\le c(\varepsilon) m^r \delta^{r/2} \Big([1/\delta] + 1 \Big) \longrightarrow 0 \text{ as } \delta \to 0.
\end{aligned}
$$

This proves (20) and yields the tightness of the sequence $\{W_n,\ n \ge 1\}$ together with Theorem 15.5 of Billingsley(1968). It remains to show that the only possible limit point is Wiener measure W^m in $\mathbb{R}^m$ with covariance structure $\Gamma = [\sigma_{kj}]$. To prove it we will use the idea in the proof of Theorem 2 of [4]. Let $Z(\cdot)$ be a limit point of $(W_n(\cdot))$. We have to prove

(i) $Z(t+h) - Z(t)$ has normal distribution $h \cdot \Gamma$,

(ii) Z has independent increments.

Let $a \in \mathbb{R}^m$ be a non-negative vector. Then $< a, (Z(t+h) - Z(t)) >$ has a normal distribution with variance $ah\Gamma a'$ by Lemma 2.3. Now we apply Lemma 2.4 with $k = 1$ which yields (i). Let $0 \le t_1 < t_2 < \cdots < t_k \le 1$ be given.

$$
\Big(W_n(t_1), W_n(t_2) - W_n(t_1), \cdots, W_n(t_k) - W_n(t_{k-1}) \Big),
$$

converges in distribution to

$$\Big(Z(t_1), Z(t_2) - Z(t_1), \cdots, Z(t_k) - Z(t_{k-1})\Big).$$

Let $a_1, \cdots, a_k \in \mathbb{R}^m$ be non-negative vectors. Then

$$\Big(< a_1, W_n(t_1) >, < a_2, (W_n(t_2) - W_n(t_1)) >, \cdots$$
$$\cdots, < a_k, (W_n(t_k) - W_n(t_{k-1})) > \Big)$$

converges in distribution to

$$\Big(< a_1, Z(t_1) >, < a_2, (Z(t_2) - Z(t_1)), \cdots, < a_k, (Z(t_k) - Z(t_{k-1})) > \Big).$$

The coordinates of the last vector are hence LPQD and by a simple computation involving (17) also uncorrelated, which together imply independent(see[8]). Now we can again apply Lemma 2.4 to obtain the independence of the increments of the Z process.

3. Main result

Lemma 3.1. Let $\{\mathbb{Z}_t\}$ be a stationary LPQD sequence of $m-$dimensional random vectors with $E(\mathbb{Z}_t) = \mathbb{O}$, $E\|\mathbb{Z}_t\|^2 < \infty$. Let $\widetilde{\mathbb{X}}_t = (\sum_{j=0}^{\infty} A_j)\mathbb{Z}_t$ and $\widetilde{\mathbb{S}}_k = \sum_{t=1}^{k} \widetilde{\mathbb{X}}_t$. If (12), (13) and (17) hold, then

$$n^{-1/2} \max_{1 \le k \le n} \|\widetilde{\mathbb{S}}_k - \mathbb{S}_k\| = o_p(1). \tag{23}$$

Proof. Appendix

Theorem 3.2. *Let $\{\mathbb{Z}_t\ ;\ t \ge 1\}$ be a stationary LPQD sequence of $m-$dimen- sional random vectors with $E(\mathbb{Z}_1) = \mathbb{O}$, $E\|\mathbb{Z}_1\|^2 < \infty$ and let $\{\mathbb{X}_t\}$ be an $m-$dimensional process defined in (4). Set $\mathbb{S}_n = \sum_{t=1}^{n} \mathbb{X}_t (\mathbb{S}_0 = \mathbb{O})$, and define for $u \in [0,1]$, $n \ge 1$, the stochastic process ξ_n by*

$$\xi_n(u) = n^{-1/2}\mathbb{S}_{[nu]}. \tag{24}$$

If (12), (13) and (17) holds, then

$$\xi_n \longrightarrow^w W^m, \tag{25}$$

where $\longrightarrow^w$ *indicates weak convergence and* W^m *is an* $m-$*dimensional Wiener process with covariance matrix* T, $T = (\sum_{j=1}^{\infty} A_j)\Gamma(\sum_{j=1}^{\infty} A_j)'$ *and* $\Gamma = [\sigma_{kj}]$ *as in (6).*

Proof. Let $\widetilde{\mathbb{X}_t} = \sum_{j=0}^{\infty} A_j \mathbb{Z}_t$ and $\widetilde{\mathbb{S}_k} = \sum_{t=1}^{k} \widetilde{\mathbb{X}_t}$. Let $\widetilde{\xi}_n$ be the same as ξ_n defined in (24) with $\widetilde{\mathbb{S}}_{[nu]}$ in place of $\mathbb{S}_{[nu]}$, (i.e., $\widetilde{\mathbb{X}}_t$ in place of $\mathbb{X}_t$)

$$\widetilde{\xi}_n(u) = \Big(\sum_{j=0}^{\infty} A_j\Big) n^{-1/2} \Big(\sum_{t=1}^{[nu]} \mathbb{Z}_t\Big). \tag{26}$$

It is clear that $\widetilde{\mathbb{X}_t}'$s are LPQD. From (3), (12), (13) and (17), we have

$$\begin{aligned} & E\|\widetilde{\mathbb{X}_1}\|^2 + 2\sum_{t=2}^{\infty}\sum_{j=1}^{m} E(\widetilde{X_1^{(j)}}\widetilde{X_t^{(j)}}) \\ &= \|\sum_{j=0}^{\infty} A_j\|^2 \{E\|\mathbb{Z}_1\|^2 + 2\sum_{t=2}^{\infty}\sum_{j=1}^{m} E(Z_1^{(j)} Z_t^{(j)})\} < \infty, \end{aligned} \tag{27}$$

$$\sum_{t=n+1}^{\infty} E\|\widetilde{\mathbb{X}_1}\widetilde{\mathbb{X}_t}\| = \|\sum_{j=0}^{\infty} A_j\|^2 \sum_{t=n+1}^{\infty} E\|\mathbb{Z}_1\mathbb{Z}_t\|^2 = O(n^{-\rho})$$

for some $\rho > 0$,

$$E\|\widetilde{\mathbb{X}_t}\|^s = \|\sum_{j=0}^{\infty} A_j\|^s E\|\mathbb{Z}_t\|^s < \infty \text{ for some } s > 2. \tag{28}$$

Thus it follows from (27), (28) and (29) that $\{\widetilde{\mathbb{X}_t}\}$ satisfies the conditions of Theorem 2.5. This implies

$$\widetilde{\xi}_n \longrightarrow^w W^m. \tag{29}$$

By Lemma 3.1,

$$\widetilde{\xi}_n(u) - \xi(u) \longrightarrow^P 0 \text{ for all } 0 \le u \le 1. \tag{30}$$

By (30), (31) and Theorem 4.1 of Billingsley(1968), we obtain $\xi_n \longrightarrow^w W^m$.
So Theorem 3.2 is proved.

4. Appendix

Proof of Lemma 3.1. We prove Lemma 3.1 by using the ideas in the proof of Lemma 1 of [5]. First observe that

$$\begin{aligned}\tilde{\mathbb{S}}_k &= \sum_{t=1}^{k}\Big(\sum_{j=0}^{k-t} A_j\Big)\mathbb{Z}_t + \sum_{t=1}^{k}\Big(\sum_{j=k-t+1}^{\infty} A_j\Big)\mathbb{Z}_t \\ &= \sum_{t=1}^{k}\Big(\sum_{j=0}^{t-1} A_j\mathbb{Z}_{t-j}\Big) + \sum_{t=1}^{k}\Big(\sum_{j=k-t+1}^{\infty} A_j\Big)\mathbb{Z}_t\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathbb{S}}_k - \mathbb{S}_k &= -\sum_{t=1}^{k}\Big\{\sum_{j=t}^{\infty} A_j\mathbb{Z}_{t-j}\Big\} + \sum_{t=1}^{k}\Big(\sum_{j=k-t+1}^{\infty} A_j\Big)\mathbb{Z}_t \\ &= I_1 \ + \ I_2 \ (say).\end{aligned}$$

To prove

$$(A.1) \qquad n^{-\frac{1}{2}} \max_{1\le k\le n} \|I_1\| \ = \ o_p(1),$$

we observe that for $r > 2$

$$\begin{aligned} & n^{-\frac{r}{2}} E \max_{1\le k\le n} \left\| \sum_{t=1}^{k} \sum_{j=t}^{\infty} A_j \ \mathbb{Z}_{t-j} \right\|^r \\ &= n^{-\frac{r}{2}} \ E \ \max_{1\le k\le n} \left\| \sum_{j=1}^{\infty} \sum_{t=1}^{j\wedge k} A_j \ \mathbb{Z}_{t-j} \right\|^r \\ &\le n^{-\frac{r}{2}} \left(\sum_{j=1}^{\infty} \|A_j\| \left\{ E \max_{1\le k\le n} \left\| \sum_{t=1}^{j\wedge k} \mathbb{Z}_{t-j} \right\|^r \right\}^{\frac{1}{r}} \right)^r \end{aligned}$$

$$\leq B\, m^r \left[\sum_{j=1}^{\infty} \|A_j\| \left(\frac{j \wedge k}{n} \right)^{\frac{1}{2}} \right]^r \quad \text{for some } B > 0,$$

where we have used Lemma 1 in Kim and Baek(2001) and Lemma 2.2. The first inequality above is obtained by Minkowski's inequality and by the dominated convergence theorem the last term above tends to zero as $n \longrightarrow \infty$ from which (A.1) follows. Thus (A.1) is proved by the Markov inequality. Next, we show that

$$(A.2) \qquad n^{-\frac{1}{2}} \max_{1 \leq k \leq n} \|I_2\| \;=\; o_p(1).$$

Write

$$I_2 = I_{21} + I_{22}, \text{ where}$$
$$I_{21} = A_1 \mathbb{Z}_k + A_2(\mathbb{Z}_k + \mathbb{Z}_{k-1}) + \cdots + A_k(\mathbb{Z}_k + \cdots + \mathbb{Z}_1)$$

and

$$I_{22} = (A_{k+1} + A_{k+2} + \cdots)\, (\mathbb{Z}_k + \cdots + \mathbb{Z}_1).$$

Let p_n be a sequence of positive integers such that

$$(A.3) \qquad p_n \to \infty \text{ and } p_n/n \to 0 \text{ as } n \to \infty.$$

Note that

$$\begin{aligned} n^{-\frac{1}{2}} \max_{1 \leq k \leq n} \|I_{22}\| \;&\leq\; \left(\sum_{i=0}^{\infty} \|A_i\| \right) n^{-\frac{1}{2}} \max_{1 \leq k \leq p_n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \\ &+ \left(\sum_{i > p_n} \|A_i\| \right) n^{-\frac{1}{2}} \max_{1 \leq k \leq n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \\ &= III + IV \text{ (say)}. \end{aligned}$$

Note that for $r > 2$ and constants B_1, B_2

$$\left(\sum_{i=0}^{\infty}\|A_i\|\right)^r n^{-\frac{r}{2}} E \max_{1\le k\le p_n} \|\mathbb{Z}_1+\cdots+\mathbb{Z}_k\|^r \le \left(\sum_{i=0}^{\infty}\|A_i\|\right)^r B_1\, m^r (p_n/n)^{\frac{r}{2}},$$

$$\left(\sum_{i>p_n}\|A_i\|\right)^r n^{-\frac{r}{2}} E \max_{1\le k\le n} \|\mathbb{Z}_1+\cdots+\mathbb{Z}_k\|^r \le \left(\sum_{i>p_n}\|A_i\|\right)^r B_2\, m^r$$

by Lemma 2.2. Thus, by (3) and (A.3)

$$\begin{aligned} III+IV &= O_p\left(\frac{p_n}{n}\right) + O_p\left(\sum_{i>p_n}\|A_i\|\right) \\ &= o_p(1). \end{aligned}$$

It remains to prove that

$$Y_n \ := n^{-\frac{1}{2}} \max_{1\le k\le n} \|I_{21}\| \ = \ o_p(1).$$

To this end, define for each $l \ge 1$

$$I_{21,l} = B_1\mathbb{Z}_k + B_2(\mathbb{Z}_k+\mathbb{Z}_{k-1}) + \cdots + B_k(\mathbb{Z}_k+\cdots+\mathbb{Z}_1),$$

where

$$B_k = \begin{cases} A_k, & k \le l, \\ O_{m\times m}, & k > l. \end{cases}$$

Let $Y_{n,l} = n^{-\frac{1}{2}} \ \max_{1\le k\le n} \ \|I_{21,l}\|$. Clearly, for each $l \ge 1$,

$$Y_{n,l} = o_p(1). \tag{A.4}$$

On the other hand,

$$
\begin{aligned}
&|Y_{n,l} - Y_n| \\
&\leq n^{-\frac{1}{2}} \max_{1\leq k\leq n} \left\| \sum_{i=1}^{k} (A_i - B_i)\ (\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}) \right\| \\
&\leq n^{-\frac{1}{2}} \max_{l\leq k\leq n} \left(\sum_{i=l+1}^{k} \|A_i\| \cdot \|\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}\| \right) \\
&\leq n^{-\frac{1}{2}} \left(\sum_{i>l} \|A_i\| \right) \max_{l\leq k\leq n} \max_{l\leq i\leq k} \left(\|\mathbb{Z}_k + \cdots + \mathbb{Z}_{k-i+1}\| \right) \\
&\leq n^{-\frac{1}{2}} \left(\sum_{i>l} \|A_i\| \right) \max_{l\leq k\leq n} \max_{l\leq i\leq k} \left(\|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| + \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_{k-i}\| \right) \\
&\leq n^{-\frac{1}{2}} \left(\sum_{i>l} \|A_i\| \right) \Big(\max_{l\leq k\leq n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \\
&\qquad\qquad + \max_{l\leq k\leq n} \max_{1\leq i\leq n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_{k-i}\| \Big) \\
&\leq n^{-\frac{1}{2}} \left(\sum_{i>l} \|A_i\| \right) \left(\max_{l\leq j\leq n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_j\| + \max_{1\leq k\leq n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_k\| \right) \\
&\leq 2\, n^{-\frac{1}{2}} \left(\sum_{i>l} \|A_i\| \right) \max_{l\leq j\leq n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_j\|.
\end{aligned}
$$

From this result, (3) and Lemma 2.2, we have, for any $\delta > 0$ and $r > 2$,

$(A.5)$

$$
\begin{aligned}
&\lim_{l\to\infty} \limsup_{n\to\infty} P(|Y_{n,l} - Y_n| > \delta) \\
&\leq \lim_{l\to\infty} 2^r \delta^{-r} \left(\sum_{i>l} \|A_i\| \right)^r \limsup_{n\to\infty} n^{-\frac{r}{2}} E \max_{1\leq j\leq n} \|\mathbb{Z}_1 + \cdots + \mathbb{Z}_j\|^r \\
&\leq C \delta^{-r} 2^r m^r \lim_{l\to\infty} \left(\sum_{i>l} \|A_i\| \right)^r = 0.
\end{aligned}
$$

In view of (A.4) and (A.5), it follows from Theorem 4.2 of Billingsley (1968, p.25)(or Proposition 6.3.3 of Brockwell and Davis(1992, p.205)) that $Y_n = o_p(1)$. This completes the proof of Lemma 3.1.

5. Acknowledgements

The authors are exceptionally grateful to the referee for offering helpful remarks and comments that improved presentation of the paper and Tae-Sung Kim is the corresponding author.

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