Honam Mathematical J. **27** (2005), No. 3, pp. 465–476

NON-COMPACT MINIMAL SURFACES BOUNDED BY CONVEX CURVES IN PARALLEL PLANES OF $\mathbf{R}^3$

SUN SOOK JIN

Abstract. In this article, we solve some kinds of non-compact Douglas-Plateau problem for two convex curves in parallel planes.

1. Introduction

The classical Douglas-Plateau problem for two *compact* contours is to find a minimal annulus bounded by two disjoint Jordan curves. In 1931, Douglas [3] proved that if A_1 and A_2 are the least area disks bounded by Jordan curves γ_1 and γ_2, respectively, satisfying

$$\inf\{\text{Area}(S)\} < \text{Area}(A_1) + \text{Area}(A_2)$$

where the infimum is over all surfaces' areas of annular type bounded by γ_1 and γ_2, then there is a minimal annulus with the boundary $\gamma_1 \cup \gamma_2$.

If γ_1 and γ_2 are coaxial unit circles in parallel planes, then it is well known that there is a constant $h > 0$ such that when the distance between the centers is smaller than h, there are exactly two catenoids bounded by $\gamma_1 \cup \gamma_2$; when the distance between the centers is larger then h, there are no catenoids bounded by $\gamma_1 \cup \gamma_2$. Furthermore, by Shiffman's third theorem [13], any minimal annuli bounded by $\gamma_1 \cup \gamma_2$ must be a rotation surface hence is a piece of a catenoid. Thus there are

Received May 3, 2005. Revised July 30, 2005.

2000 Mathematics Subject Classification : Primary 53A10.

Key words and phrases : minimal surfaces, Douglas-Plateau problem, Riemann minimal examples, Courant-Lebesgue lemma.

either two, one, or zero minimal annuli bounded by $\gamma_1 \cup \gamma_2$ depending on the distance between their centers.

Meeks and White [10] generalized the above observation to minimal annuli bounded by two smooth convex Jordan curves $\gamma_1 \cup \gamma_2$ in different parallel planes, i.e., there are either two, one, or zero minimal annuli bounded by $\gamma_1 \cup \gamma_2$. But unlike the coaxial circles case, there are no simple criteria to tell us when do we have two, one, or zero minimal annuli bounded by $\gamma_1 \cup \gamma_2$. However, there are some partial conditions in this case, for example, Hoffman and Meeks [6] gave a sufficient condition to ensure that there are two solutions, i.e., if there is a connected compact non-planar minimal surface (could be branched) whose boundary is contained in open planar disks bounded by $\gamma_1 \cup \gamma_2$, then there are two minimal annuli bounded by $\gamma_1 \cup \gamma_2$. Yi Fang and J-H. Hwang generalized the above results to the continuous boundary case as follows;

Proposition 1 ([5]). *Let D_α and D_β be two open disks lying on parallel planes, and their boundaries α and β are continuous convex Jordan curves, respectively. Suppose that there is a (maybe branched) minimal surface Σ with $\partial\Sigma \subset D_\alpha \cup D_\beta$, then there exist at least two embedded minimal annuli A and B bounded by $\alpha \cup \beta$ such that*

- **(1):** *A is stable, B is unstable. Recall a minimal surface is called stable if, with respect to any non-trivial normal variation that fixes the boundary, the second derivative of area functional is positive. If the second derivative of area is negative for some variation, then this surface is called unstable.*
- **(2):** *If $M \neq A$ is a compact (maybe branched) minimal surface with $\partial M \subset D_\alpha \cup D_\beta$, then M is contained in the compact region of $\mathbf{R}^3$ bounded by $A \cup D_\alpha \cup D_\beta$ such that $\mathrm{Int}(A) \cap \mathrm{Int}(M) = \emptyset$ and $\mathrm{Int}(B) \cap \mathrm{Int}(M) \neq \emptyset$.*
- **(3):** *Both A and B have the same symmetry group of the boundary.*

In this paper, we consider some kinds of *non-compact Douglas-Plateau problem*, i.e., for two embedded proper complete curves in parallel planes, at least one of which is non-compact, find a minimal annulus bounded by the given curves. There is a classical example bounded by two parallel straight lines, which is a piece of one of *Riemann's minimal examples*. Recall the one-parameter family of Riemann's minimal examples are the only complete minimal surfaces of $\mathbf{R}^3$ foliated by circles and straight lines in parallel planes except planes, catenoids, and helicoids, see [12]. Recently, Yi Fang and J-F. Hwang [5] proved the existence of two embedded minimal annuli bounded by continuously embedded, proper, complete, non-compact, *non-flat* convex curves in parallel planes. They used one of Jenkins and Serrin's minimal graphs of [9] as the barrier confining all of approximating surfaces.

In [7] we use one of Riemann's minimal examples instead of a Jenkin's and Serrin's graph as the barrier and obtained the existence result of a pair of embedded minimal annuli in a slab which are bounded by a convex Jordan curve and a straight line in parallel planes. Now, we generalize this technique to the flat or non-flat boundary case and get the following theorems;

Theorem 1. *Let $\gamma \subset P_1$ be a convex Jordan curve and let $\Gamma \subset P_0$ be a continuously embedded, proper, complete, non-compact convex curve, where $P_t := \{(x, y, z) \in \mathbf{R}^3 \mid z = t\}$ is a horizontal plane at the height $t \in \mathbf{R}$. Denote D_γ by the compact planar disk in P_1 with $\partial D_\gamma = \gamma$ and let P_0^+ be the half-plane of P_0 bounded by Γ. Suppose that there is a (maybe branched) compact minimal surface Σ with $\partial\Sigma \subset D_\gamma \cup \overline{P_0^+}$. Then there are two embedded minimal annuli $\mathcal{A}$ and $\mathcal{B}$ such that*

1. $\partial\mathcal{A} = \partial\mathcal{B} = \gamma \cup \Gamma$.
2. *For each $t \in (0, 1)$, $\mathcal{A} \cap P_t$ and $\mathcal{B} \cap P_t$ are strictly convex Jordan curves.*
3. $\mathrm{Int}(\mathcal{A}) \cap \mathrm{Int}(\mathcal{B}) = \emptyset$.

4. *If Σ' is a connected compact non-planar (maybe branched) minimal surface such that $\partial\Sigma' \subset D_\gamma \cup \overline{P_0^+}$, then*

$$\mathrm{Int}(\mathcal{A}) \cap \mathrm{Int}(\Sigma') = \emptyset, \quad \mathcal{B} \cap \Sigma' \neq \emptyset.$$

5. *$\mathcal{A}$ and $\mathcal{B}$ have the same symmetry groups as that of $\gamma \cup \Gamma$.*

Theorem 2. *Let $\Gamma^1 \subset P_{t_1}$ and $\Gamma^2 \subset P_{t_2}$, $t_1 < t_2$, be continuously embedded, proper, complete, non-compact convex curves. Suppose that for all $i = 1, 2$, there are parallel straight lines ℓ^i lying in P_{t_i} such that $\ell^i \cap \Gamma^i = \emptyset$, respectively. Observe Γ^i, $i = 1, 2$, separates the plane P_{t_i} by two parts, i.e., the right one $P_{t_i}^{\Gamma^i,+}$ and the left one $P_{t_i}^{\Gamma^i,-}$, respectively. If there is a (maybe branched) compact minimal surface Σ with $\partial\Sigma \subset \overline{P_{t_1}^{\Gamma^1,-}} \cup \overline{P_{t_2}^{\Gamma^2,+}}$, then there are two embedded minimal annuli $\mathcal{M}$ and $\mathcal{N}$ such that*

1. $\partial\mathcal{M} = \partial\mathcal{N} = \Gamma^1 \cup \Gamma^2$.
2. *For each $t \in (t_1, t_2)$, $\mathcal{M} \cap P_t$ and $\mathcal{N} \cap P_t$ are strictly convex Jordan curves.*
3. $\mathrm{Int}(\mathcal{M}) \cap \mathrm{Int}(\mathcal{N}) = \emptyset$
4. *Now let Σ' be a connected compact non-planar (maybe branched) minimal surface such that $\partial\Sigma' \subset \overline{P_{t_1}^{\Gamma^1,-}} \cup \overline{P_{t_2}^{\Gamma^2,+}}$, then*

$$\mathrm{Int}(\mathcal{M}) \cap \mathrm{Int}(\Sigma') = \emptyset, \quad \mathcal{N} \cap \Sigma' \neq \emptyset.$$

5. *$\mathcal{M}$ and $\mathcal{N}$ have the same symmetry groups as that of $\Gamma^1 \cup \Gamma^2$.*

The basic idea to prove the first theorem is to approximate a non-compact curve $\Gamma \subset P_0$ with covex Jordan curves $\Gamma_n \subset P_0$, $n = 1, 2, 3, \cdots$. Then, by the above Proposition 1, we can get embedded minimal annuli A_n and B_n bounded by $\gamma \cup \Gamma_n$. Take a straight line $\ell \subset P_0$ such that $\ell \cap P_0^+ = \emptyset$, and a Riemann's minimal example bounded by the straight line ℓ and a circle lying on P_1 containing γ in its interior. Then we can use it as the barrier confining all of A_n's and B_n's. Together with the similar method of the proof of Theorem 3.1 in [5], it leads us to prove that there are subsequences of $\{A_n\}$ and $\{B_n\}$ convergent to embedded minimal annuli $\mathcal{A}$ and $\mathcal{B}$, respectively, in the interior of the slab bounded by P_0 and P_1. Moreover, since the boundary curves γ and Γ_n are convex, M. Shiffman's first theorem in [13] shows that the intermediate curves $A_n \cap P_t$ and $B_n \cap P_t$, $0 < t < 1$, are all strictly convex Jordan curves. Therefore we can divide the approximating annulus A_n into two graphs over a vertical plane, each of which is simply connected. The same is true for B_n. Then we can use the Courant-Lebesgue lemma in [2] to prove that the convergence can be extended to their boundaries, respectively, and $\partial\mathcal{A} = \partial\mathcal{B} = \gamma \cup \Gamma$.

2. Proof of the theorem 1

Let $\ell \subset P_0$ denote the y-axis. We may assume without loss of generality that

$$\ell \cap P_0^+ = \emptyset.$$

Recall that P_0^+ denotes the right-half plane of P_0 bounded by Γ. Let $\{D_n\}$, $n = 1, 2, \cdots$, be a sequence of circular disks in P_0 such that

$$(D_n \cap \ell) \subset (D_{n+1} \cap \ell), \quad (D_n \cap P_0^+) \subset (D_{n+1} \cap P_0^+),$$
$$\lim_{n\to\infty} D_n = \ell, \quad \partial\Sigma \subset D_\gamma \cup (D_1 \cap P_0^+).$$

For example, if we take

$$r_n = a + n^2 + n + \frac{1}{n}, \quad a_n = a + n^2 + n$$

for a large $a \geq 0$ satisfying the given conditions, then the disk

$$D_n := \{(x, y, 0) \mid (x - a_n)^2 + y^2 \leq r_n^2 \}$$

satisfies all the above conditions.

On the other hand, let

$$\Gamma_n := (\partial D_n \cap P_0^+) \cup (\Gamma \cap D_n),$$

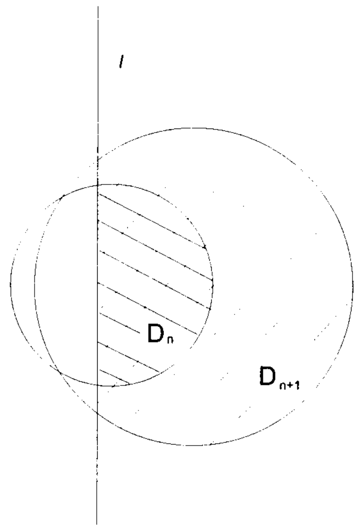

FIGURE 1.

and let D_{Γ_n} be the compact set in the half-plane P_0^+ with the boundary Γ_n. Then,

$$D_{\Gamma_1} \subset D_{\Gamma_2} \subset \cdots \longrightarrow P_0^+.$$

Observe that for any $r > 0$ there is an integer $n_0 > 0$ such that

$$\Gamma_n \cap C_r = \Gamma \cap C_r \quad \text{whenever } n > n_0$$

where $C_r = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 \leq r^2\}$ is the vertical solid cylinder of $\mathbf{R}^3$ with the radius $r > 0$. Since both γ and Γ_n are continuous convex curves in the parallel planes and $\partial\Sigma \subset D_\gamma \cup D_{\Gamma_n}$, by Proposition 1, there exist a stable embedded minimal annulus A_n and an unstable embedded minimal annulus B_n bounded by $\gamma \cup \Gamma_n$ for all $n = 1, 2, \cdots$, respectively. Now we will show that both sequences $\{A_n\}$ and $\{B_n\}$ converge to the desired minimal annuli.

Claim 1. Denote by $D \subset P_1$ a disk whose boundary ∂D is a circle containing γ in its interior. Then there is a piece of one of Riemann's minimal examples $\mathcal{R}$ bounded by $\partial D \cup \ell$ with

$$\text{(1)} \qquad \mathrm{Int}(\mathcal{R}) \cap \mathrm{Int}(A_n) = \emptyset, \quad \mathrm{Int}(\mathcal{R}) \cap \mathrm{Int}(B_n) = \emptyset$$

for all $n \in \mathbf{N}$.

proof. Recall $\partial\Sigma \subset D \cup D_n$ for all n. Together with Proposition 1 again, it leads us to take a *stable* embedded minimal annulus R_n such that

$$\partial R_n = \partial D \cup \partial D_n.$$

Then, using the same argument in [7], we can show that

- The sequence of Gaussian curvatures $\{K_{R_n}\}$ of R_n is uniformly bounded.
- Given $p \in \mathbf{R}^3$, there exists a positive number $r_0 = r_0(p)$ and a positive constant $c = c(p)$ such that for all n,

$$\mathrm{Area}(R_n \cap B_r) \leq cr^2$$

  if $r > r_0$, where $B_r := B(p,r)$ is the ball of radius r with center p.

These allow us to prove that there exists an embedded minimal annulus $\mathcal{R}$ such that

$$\lim_{n\to\infty} R_n = \mathcal{R}, \quad \partial\mathcal{R} = \partial D \cup \ell.$$

It is well-known that $\mathcal{R}$ must be a piece of one of Riemann's minimal examples which are foliated by circles in parallel planes.

On the other hand, if $m \geq n$ then the boundary curve $\partial A_n = \partial B_n$ is contained in $D \cup D_m$. Since R_m is stable and $\partial R_m = \partial D \cup \partial D_m$, by Proposition 1-(2), neither A_n nor B_n meets R_m in their interiors, the equation (1) follows.

Claim 2. Both sequences $\{A_n\}$ and $\{B_n\}$ converge to minimal annuli $\mathcal{A}$ and $\mathcal{B}$, respectively, with boundary $\gamma \cup \Gamma$.

proof In the proof of *Claim 1*, we have shown that $A_n, B_n \subset V_m$ for all $m \geq n$, where V_m is a solid in $\mathbf{R}^3$ bounded by $R_m \cup D_m \cup D$. Let V

be a solid such that $\partial V = \mathcal{R} \cup \overline{P_0^+} \cup D$. Then we can say that

$$A_n,\ B_n \subset V \quad \text{for all } n \in \mathbf{N}.$$

Notice that ∂V meets the intermediate plane along the circles, so for given $0 < t < 1$ we can choose a positive number $r(t)$ and a solid cylinder $C_{r(t)}$ such that

$$V \cap P_t \subset C_{r(t)}.$$

From now on, we have shown that for all n,

$$A_n \cap S(t,1) \subset C_{r(t)}, \quad B_n \cap S(t,1) \subset C_{r(t)}$$

where $0 < t < 1$, respectively. Now we can use the interior curvature estimate in [5] and then the compactness lemma in [1], to prove the smooth convergence of $\{A_n \cap S(t,1)\}$ and $\{B_n \cap S(t,1)\}$, for all $0 < t < 1$. Furthermore, being the limit curve of the sequence of strictly convex curves, all the intermediate curves of $\mathcal{A}$ and $\mathcal{B}$ are strictly convex. Hence $\mathcal{A}$ and $\mathcal{B}$ are minimal annuli.

Next, we extend the above convergence result to the boundary. Since every intermediate curve $A_n \cap P_t$, $0 < t < 1$, is a strictly convex Jordan curve, A_n consists of two subsets A_n^+ and A_n^- where $\mathrm{Int}(A_n^+)$ and $\mathrm{Int}(A_n^-)$ are minimal graphs over a domain Ω_Π^n in a vertical plane Π, where $\partial\Omega_\Pi^n$ is a piecewise smooth convex curve. If we choose a solid cylinder C_r with a sufficiently large radius $r > 0$ containing γ in its interior, then $A_n \setminus C_r$ becomes a graph over a domain in the horizontal plane P_0, and $A_n \cap \partial C_r$ is a simple curve. Therefore, both $A_n^+ \cap C_r$ and $A_n^- \cap C_r$ are simply connected minimal surfaces. Now we can take a closed unit disk Δ in the plane and a conformal embedding

$$X_n : \Delta \to \mathbf{R}^3$$

of $A_n^+ \cap C_r$ satisfying the three point condition. Observe all of the Dirichlet integrals $\int_\Delta |DX_n|^2$, $n \in \mathbf{N}$, are uniformly bounded, since each X_n is conformal and $\mathcal{A}^+ \cap C_r$ has a bounded area. Then by the well-known Courant-Lebesgue lemma, see [3], we can say that $A_n^+ \cap C_r =$

$X_n(\Delta)$ converges to $(\mathcal{A}^+ \cup \gamma \cup \Gamma) \cap C_r$ and is continuous up to the boundary. Similar argument for $\mathcal{A}^-$ also holds. Thus we see that

$$\partial(\mathcal{A} \cap C_r) \cap (P_0 \cup P_1) = (\gamma \cup \Gamma) \cap C_r$$

for all $r > 0$ large enough. Moreover, it is clear that $\partial A \subset P_0 \cup P_1$. Therefore $\mathcal{A}$ has the boundary $\gamma \cup \Gamma$. With the similar method to $\mathcal{B}$,

$$\partial \mathcal{A} = \partial \mathcal{B} = \gamma \cup \Gamma$$

and they are continuous up to boundary.

Now let Σ' be a connected non-planar compact (maybe branched) minimal surface such that

$$\partial \Sigma' \subset \overline{D}_\gamma \cup \overline{P}_0^+$$

Let W_n be the solid bounded by $A_n \cup D_\gamma \cup D_{\Gamma_n}$, and W be the solid bounded by $\mathcal{A} \cup \overline{D}_\gamma \cup \overline{P}_0^+$. Since each A_n is stable, by Proposition 1-(2),

$$\mathrm{Int}(A_n) \cap \mathrm{Int}(\Sigma') = \emptyset.$$

It follows that $\Sigma' \subset W_n$ for all n, and $\Sigma' \subset W$ for $W_n \to W$. By the comparison principle for minimal surfaces, either $\mathcal{A} = \Sigma'$ or $\mathrm{Int}(\mathcal{A}) \cap \mathrm{Int}(\Sigma') = \emptyset$. Since Σ' is compact and $\mathcal{A}$ is not compact, both cannot be equal. On the other hand all of B_n's are unstable, so using Proposition 1-(2) again we have $B_n \cap \Sigma' \neq \emptyset$ for all n, and so $\mathcal{B} \cap \Sigma' \neq \emptyset$. Now let W_n' be the solid bounded by $B_n \cup D_\gamma \cup D_{\Gamma_n}$ and W' be the solid bounded by $\mathcal{B} \cup \overline{D}_\gamma \cup \overline{P}_0^+$. Since $W_n' \subset W_n$ for all n and $\lim_{n\to\infty} W_n = W$, we have $\lim_{n\to\infty} W_n' = W' \subset W$. By the comparison principle for minimal surfaces again, it follows that

$$\mathrm{Int}\mathcal{A} \cap \mathrm{Int}\mathcal{B} = \emptyset.$$

Finally, since A_n and B_n have the same symmetry group as that of boundary for all n, the same holds for the limits $\mathcal{A}$ and $\mathcal{B}$.

3. Proof of the theorem 2

We may assume that there is a constant $a > 0$ such that

$$\begin{aligned}\ell^1 &= \{(a, y, t_1) \,|\, y \in \mathbf{R}\}\\ \ell^2 &= \{(-a, y, t_2) \,|\, y \in \mathbf{R}\}\end{aligned}$$

For $r_n = a + n^2 + n + \frac{1}{n}$ and $a_n = a + n^2 + n$, take the circular disks

$$\begin{aligned}D_n^1 &:= \{(x, y, t_1) \mid (x + a_n)^2 + y^2 \le r_n^2 \} \subset P_{t_1}\\ D_n^2 &:= \{(x, y, t_2) \mid (x - a_n)^2 + y^2 \le r_n^2 \} \subset P_{t_2}\end{aligned}$$

Let

$$\begin{aligned}\Gamma_n^1 &:= (\partial D_n^1 \cap P_{t_1}^{\Gamma^1,-}) \cup (\Gamma^1 \cap D_n^1),\\ \Gamma_n^2 &:= (\partial D_n^2 \cap P_{t_2}^{\Gamma^2,+}) \cup (\Gamma^2 \cap D_n^2).\end{aligned}$$

Note that Γ_n^i are continuous and con vex cureves lying in the plane P_{t_i}, $i = 1, 2$, respectively. Let $D_{\Gamma_n^1}^1 \subset P_{t_1}^{\Gamma^1,-}$, $D_{\Gamma_n^2}^2 \subset P_{t_2}^{\Gamma^2,+}$ be the compact sets bounded by Γ_n^1 and Γ_n^2, respectively. Then

$$D_{\Gamma_1^1}^1 \subset D_{\Gamma_2^1}^1 \subset \cdots \longrightarrow P_{t_1}^{\Gamma^1,-}, \quad D_{\Gamma_1^2}^2 \subset D_{\Gamma_2^2}^2 \subset \cdots \longrightarrow P_{t_2}^{\Gamma^2,+}.$$

Observe that for any $r > 0$ there is an integer $n_0 > 0$ such that

$$\Gamma_n^i \cap C_r = \Gamma^i \cap C_r \quad \text{whenever } n > n_0$$

where $i = 1, 2$. It is clear that $\partial\Sigma \subset D_{\Gamma_n^1}^1 \cup D_{\Gamma_n^2}^2 \subset D_n^1 \cup D_n^2$ for all $n = 1, 2, 3, \cdots$, by Proposition 1, there exist a stable embedded minimal annulus M_n and an unstable embedded minimal annulus N_n bounded by $\Gamma_n^1 \cup \Gamma_n^2$ for all $n = 1, 2, \cdots$, respectively.

On the other hand, with the same argument of the proof of Theorem 1, we can take a sequence $\{R_n'\}$ of pieces of stable Riemann's minimal examples in the slab $S(t_1, t_2)$ such that

$$\partial R_n' = \partial D_n^1 \cup \partial D_n^2$$

and show that this sequence converges to a piece of Riemann's minimal example $\mathcal{R}'$ bounded by $\ell^1 \cup \ell^2$. Using this Riemann's minimal example as the barrier, together with the same argument of the proof of Theorem 1, we can prove that

$$M_n \to \mathcal{M}, \quad N_n \to \mathcal{N} \quad \text{as } n \to \infty$$

where $\mathcal{M}$ and $\mathcal{N}$, having the boundary $\Gamma^1 \cup \Gamma^2$, are the desired minimal annuli.

References

[1] M. Anderson, *Curvature estimates for minimal surfaces in 3-manifolds*, Ann. Scient. Éc. Norm. Sup. **18** (1985), 89-105

[2] U. Dierkes, S. Hildebrandt, A. Küster, and O. Wohlrab, *Minimal Surfaces 1*, Springer-Verlag 1992

[3] J. Douglas, *The problem of Plateau for two contours*, J. Math. Phys. Masschusetts Inst. of Tech. **10** (1930/31), 315-359

[4] Yi Fang, On minimal annuli in a slab, Comm. Math. Helv. **69** (1994), 417-430

[5] Yi Fang and Jenn-Fang Hwang, *Curvature estimates for minimal annuli and non-compact Douglas-Plateau problem*, Comm. Anal. Geom. **8(4)** (2000), 871-904

[6] D. Hoffman and W. Meeks, *A variational approach to the existence of complete embedded minimal surfaces*, Duke. Math. **57** (1988), 877-893

[7] S.S. Jin, *Non-compact Douglas-Plateau problem*, Jour. K.M.S, to appear

[8] S.S. Jin, *Non-compact Douglas-Plateau problem bounded by a line and a Jordan curve*, Comm. K.M.S, to appear

[9] H. Jenkins and J. Serrin, *Variational problems of minimal surface type, 2, Boundary value problems for the minimal surface equation*, Arch. Rational Mech. Anal. **21** (1966), 321-342

[10] W. Meeks and B. White, *Minimal surfaces bounded by convex curves in parallel planes*, Comm. Math. Helv. **66** (1991), 263-278

[11] J. Perez and A. Ros, *Properly embedded minimal surfaces with finite total curvature, in THE GLOBAL THEORY OF MINIMAL SURFACES IN FLAT SPACES*, Lecture Notes in Mathematics, Springer-Verlag Vol.**1775** (2002), 15-66

[12] B. Riemann, *Über die Fläche vom kleinsten Inhalt bei gegebener Begrebzung*, Abh. Königl. d. Wiss. Göttingen, Mathem. Cl. **13** (1867), 3-52

[13] M. Shiffman, *On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes*, Annals of Math. **63** (1956), 77-90

[14] B. White, *Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals*, Invent. Math. **88** (1987), 243-256

Sun Sook Jin
Major in Mathematics and Applies Mathematics
College of Electronics and Information
Kyung Hee University
Yongin 449-701, Korea
Email : ssjin@khu.ac.kr