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ASYMPTOTICS FOR MULTIVARIATE MOVING AVERAGE PROCESS WITH NA RANDOM VECTORS

KWANG-HEE HAN

Abstract. The aim of this paper is to establish a functional central limit theorem for multivariate moving average process generated by negatively associated random vectors under the finite second moments.

1. Introduction

A finite family of random variables $\{\epsilon_i, 1 \le i \le n\}$ is said to be negatively associated if for every pair of disjoint subsets A and B of $\{1, \cdots, n\}$, $Cov(f(\epsilon_i, i \in A), g(\epsilon_j, j \in B)) \le 0$ whenever f and g are coordinatewise non-decreasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated. The concept of the negative association was introduced by Alam and Saxena(1981) and Joag-Dev and Proschan(1983). As pointed out and proved by Joag-Dev and Proschan(1983), a number of well-known multivariate distributions possess the negative association property. Negative association has found important and wide applications in multivariate statistical analysis and reliability theory. Many investigators also discuss applications of negative association to probability, stochastic processes and statistics. The notions of negatively associated random variables have received increasing attention recently. We refer to Joag-Dev and

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Proschan(1983) for fundamental properties, Newman(1984) for the central limit theorem and Matula(1992) for the three series theorem and Su et al.(1997) for a moment inequality and a weak invariance principle. More recently, Shao(2000) established the functional central limit theorem for negatively associated random variables with finite variances.

We can still extend the concept of negative association to the random vectors as follows: Let $\{\mathbb{Z}_i, 1 \leq i \leq n\}$ be $\mathbb{R}^m$-valued random vectors. They are said to be negatively associated(NA) for every pair of disjoint subsets A and B of $\{1, \cdots, n\}$

$$(1) \qquad Cov(f(\mathbb{Z}_i, i \in A), g(\mathbb{Z}_j, j \in B)) \leq 0$$

whenever f and g are coordinatewise increasing and the covariance exists. An infinite family is nagatively associated if every finite subfamily is negatively associated.

Define a multivariate moving average process by

$$(2) \qquad \mathbb{X}_t = \sum_{u=0}^{\infty} A_u \mathbb{Z}_{t-u}, \ t = 0, \pm 1, \pm 2, \cdots, \ u = 0, 1, 2, \cdots$$

where $\{\mathbb{Z}_t\}$ is a sequence of m-dimensional random vectors with $E(\mathbb{Z}_t) = \mathbb{O}$ and $\{A_u\}$ is a sequence of $m \times m$ matrices such that

$$(3) \qquad \sum_{u=0}^{\infty} \|A_u\| < \infty \ and \ \sum_{u=0}^{\infty} A_u \neq O_{m \times m},$$

where for any $m \times m, m \geq 1$, matrix $A = (a_{ij})$, $\|A\| := \sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|$ and $O_{m \times m}$ denotes the $m \times m$ zero matrix. In time-series analysis, this process is of great importance. Many important time-series models, such as the causal multivariate auto regressive moving average(MARMA) process(Brockwell and Davis, 1987), have type (2) satisfying (3).

In Section 2 we establish the functional central limit theorem(FCLT) for negatively associated random vectors with finite variances. Applying this result we also derive a functional central limit theorem for a multivariate linear process generated by negatively associated random vectors in Section 3.

2. A FCLT for negatively associated random vectors

Lemma 2.1(Su et al., 1997) Let $r \geq 2$ and let $\{\epsilon_i,\ 1 \leq i \leq n\}$ be a sequence of NA random variables with $E\epsilon_i = 0$ and $E|\epsilon_i|^r < \infty$. Then, there exists a constant $A_r > 0$ such that

$$(4) \qquad E \max_{1\leq k\leq n} |\sum_{i=1}^{k} \epsilon_i|^r \leq A_r\{(\sum_{i=1}^{n} E\epsilon_i^2)^{r/2} + \sum_{i=1}^{n} E|\epsilon_i|^r\}.$$

Lemma 2.2(Shao, 2000) Let $\{\epsilon_i, i \geq 1\}$ be a sequence of NA random variables with zero means and finite second moments. Let $T_k = \sum_{i=1}^{k} \epsilon_i$ and $B_k = \sum_{i=1}^{k} E\epsilon_i^2$. Then for all $x > 0$ and $a > 0$,

$$P(\max_{1\leq k\leq n} |T_k| \geq x)$$

$$(5) \qquad \leq 2P(\max_{1\leq k\leq n} |\epsilon_k| > a) + 4exp(-\frac{x^2}{8B_n}) + 4(\frac{B_n}{4(xa + B_n)})^{x/(12a)}$$

The proofs of Lemmas 2.1 and 2.2 can be found in Shao(2000).

Lemma 2.3 Let $r \geq 2$ and let $\{\mathbb{Z}_i, 1 \leq i \leq n\}$ be a sequence of negatively associated random vectors in $\mathbb{R}^m$ with $E\mathbb{Z}_i = \mathbb{O}$ and $E\|\mathbb{Z}_i\|^r < \infty$ where $\|\mathbb{Z}_i\| = (Z_{11}^2 + \cdots + Z_{1m}^2)^{\frac{1}{2}}$. Then there exist a constant $0 < A_r < \infty$ such that

$$(6) \quad E \max_{1\leq k\leq n} \|\sum_{i=1}^{k} \mathbb{Z}_i\|^r \leq A_r m^r\{(\sum_{i=1}^{n} E\|\mathbb{Z}_i\|^2)^{\frac{r}{2}} + \sum_{i=1}^{n} E\|\mathbb{Z}_i\|^r\}$$

Proof. Note that

$$(7) \qquad \max_{1\leq k\leq n} \|\sum_{i=1}^{k} \mathbb{Z}_i\| \leq \sum_{j=1}^{m} \max_{1\leq k\leq n} |\sum_{i=1}^{k} Z_i^{(j)}|$$

and by Lemma 2.1

$$E \max_{1\le k\le n} |\sum_{i=1}^{k} Z_i^{(j)}|^r \le A_r\{(\sum_{i=1}^{n} E(Z_i^{(j)})^2)^{\frac{r}{2}} + \sum_{i=1}^{n} E|Z_i^{(j)}|^r\}$$

$$\le A_r\{(\sum_{i=1}^{n} E\|Z_i\|^2)^{\frac{r}{2}} + \sum_{i=1}^{n} E\|Z_i\|^r\}. \tag{8}$$

Hence, from (7) and (8) equation (6) follows.

Lemma 2.4 Let $\{\mathbb{Z}_i, 1 \le i \le n\}$ be a sequence of negatively associated random vectors in $\mathbb{R}^m$ with $E(\mathbb{Z}_i) = \mathbb{O}$ and $E\|\mathbb{Z}_i\|^2 < \infty$. Then for all $x > 0$ and $a > 0$,

$$P(\max_{1\le k\le n} \|\sum_{i=1}^{k} \mathbb{Z}_i\| \ge mx)$$

$$\le 2mP(\max_{1\le k\le n} \|\mathbb{Z}_k\| > a) + 4mexp(-\frac{x^2}{8\sum_{i=1}^{n} E\|\mathbb{Z}_i\|^2})$$

$$+4m(\frac{\sum_{i=1}^{n} E\|\mathbb{Z}_i\|^2}{4(xa + \sum_{i=1}^{n} E\|\mathbb{Z}_i\|^2)})^{x/(12a)} \tag{9}$$

Proof. From (7) and Lemma 2.2, (9) follows easily.

Theorem 2.5 Let $\{\mathbb{Z}_i, i \ge 1\}$ be a strictly stationary sequence of negatively associated random vectors in $\mathbb{R}^m$ with $E(\mathbb{Z}_i) = \mathbb{O}$ and $E\|\mathbb{Z}_i\|^2 < \infty$. Define, for $t \in [0,1]$, $n \ge 1$

$$S_n(t) = n^{-\frac{1}{2}} \sum_{i=1}^{[nt]} \mathbb{Z}_i. \tag{10}$$

If

$$E\|\mathbb{Z}_1\|^2 + 2\sum_{i=1}^{\infty}\sum_{j=1}^{m} E(Z_1^{(j)} Z_i^{(j)}) = \sigma^2 < \infty, \tag{11}$$

then, as $n \to \infty$,

$$S_n \to^w W^m \tag{12}$$

where $\to^{w}$ indicates weak convergence, and W^m is an m-dimensional Wiener measure with covariance matrix $\Gamma = (\sigma_{kj})$,

$$\sigma_{kj} = E(Z_1^{(k)} Z_1^{(j)}) + \sum_{i=2}^{\infty}[E(Z_1^{(k)} Z_i^{(j)}) + E(Z_1^{(j)} Z_i^{(k)})]. \tag{13}$$

Proof. By means of the simple device due to Cramer Wold(see[2],[3]), from the Newman's central limit theorem for negatively associated random variables we obtain

$$n^{-\frac{1}{2}} \sum_{i=1}^{n} \mathbb{Z}_i \to^{D} N(\mathbb{O}, \Gamma), \tag{14}$$

where $N(\mathbb{O}, \Gamma)$ denotes an m-dimensional normal random vector and the symbol $\to^{D}$ indicates convergence in distribution. Hence, by the similar proof to that of Theorem 2 of Burton et al.(1986) on the functional central limit theorem for weakly associated random vectors, the limit point of $S_n(\cdot)$ is Wiener measure W^m in $\mathbb{R}^m$ with covariance matrix $\Gamma = (\sigma_{kj})$.

It remains to verify the tightness of $S_n(\cdot)$(see Theorem 15.1 of [2]). By Theorem 8.4 of [2] we only need to show that for any $\epsilon > 0$, there exist a positive number λ and an integer n such that for every $n \geq n_0$

$$P(\max_{1 \leq k \leq n} \| \sum_{i=1}^{k} \mathbb{Z}_i \| > \lambda n^{\frac{1}{2}}) \leq m^3 \epsilon \lambda^{-2} \tag{15}$$

Applying (9) with $\lambda = m\lambda'$, $x = \lambda' n^{\frac{1}{2}}$ and $a = \lambda' n^{\frac{1}{2}}/48$

$$\begin{aligned}
& P(\max_{1\le k\le n} \|\sum_{i=1}^{k} \mathbb{Z}_i\| > \lambda n^{\frac{1}{2}}) \\
= \quad & P(\max_{1\le k\le n} \|\sum_{i=1}^{k} \mathbb{Z}_i\| > m\lambda' n^{\frac{1}{2}}) \\
\le \quad & 2mP(\max_{1\le k\le n} \|\mathbb{Z}_k\| > \lambda' n^{\frac{1}{2}}/48) \\
& +4mexp(-\frac{\lambda'^2 n}{8nE\|\mathbb{Z}_1\|^2}) + 4m(\frac{nE\|\mathbb{Z}_1\|^2}{4(nE\|\mathbb{Z}_1\|^2 + \lambda'^2 n/48)})^4 \\
\le \quad & 2m(48)^2 \lambda'^{-2} E\|\mathbb{Z}_1\|^2 I\{\|\mathbb{Z}_1\| > \lambda' n^{\frac{1}{2}}/48\} \\
& +4mexp(-\frac{\lambda'^2}{8E\|\mathbb{Z}_1\|^2}) + 4m(\frac{12E\|\mathbb{Z}_1\|^2}{\lambda'^2})^4 \\
\le \quad & m\epsilon\lambda'^{-2} = m^3\epsilon\lambda^{-2}
\end{aligned}$$

provided that λ is sufficiently large. This proves (15), and hence the proof of Theorem 2.5 is complete.

3. A FCLT for a linear process generated by NA random vectors

Theorem 3.1 Lex $\mathbb{X}_t$ satisfy model (2) and let $\{\mathbb{Z}_t, t \ge 1\}$ be a stationary sequence of negatively associated random vectors in $\mathbb{R}^m$ with $E(\mathbb{Z}_t) = \mathbb{O}$ and $E\|\mathbb{Z}_t\|^2 < \infty$. Define, for $s \in [0,1]$, $n \ge 1$, the stochastic process ξ_n by

$$\xi_n(s) = n^{-\frac{1}{2}} \sum_{t=1}^{[ns]} \mathbb{X}_t.$$

If (11) holds then

$$\xi_n \to^w W^m \tag{16}$$

where $\to^w$ indicates weak convergence and W^m is an m-dimensional Wiener measure with covariance matrix $T = (\sum_{j=1}^{\infty} A_j)\Gamma(\sum_{j=1}^{\infty} A_j)'$ and Γ is defined in (13).

Proof. For every fixed $l \geq 1$, put

$$\begin{aligned} \mathbb{X}_t &= \sum_{u=0}^{\infty} A_u \mathbb{Z}_{t-u} \\ &= \sum_{u=0}^{l} A_u \mathbb{Z}_{t-u} + \sum_{u=l+1}^{\infty} A_u \mathbb{Z}_{t-u} \\ &= \mathbb{X}_{1t}^{(l)} + \mathbb{X}_{2t}^{(l)}. \end{aligned}$$

From the idea in Fuller(1996, p.320) we obtain that for any $k \geq 1$,

$$\begin{aligned} \sum_{t=1}^{k} \mathbb{X}_{1t}^{(l)} &= \sum_{t=1}^{k}\sum_{u=0}^{l} A_u \mathbb{Z}_{t-u} \\ &= \sum_{u=0}^{l} A_u \sum_{t=1}^{k} \mathbb{Z}_t + \sum_{i=1}^{l} \mathbb{Z}_{1-i} \sum_{t=i}^{l} A_t + \sum_{i=0}^{l-1} \mathbb{Z}_{k-i} \sum_{t=i+1}^{l} A_t \\ &= \sum_{u=0}^{l} A_u \sum_{t=1}^{k} \mathbb{Z}_t + R(k,l)\ (say). \end{aligned}$$

Therefore, it follows that for every fixed $l \geq 1$,

$$\begin{aligned} n^{-\frac{1}{2}} \sum_{t=1}^{[ns]} \mathbb{X}_t &= n^{-\frac{1}{2}} \sum_{i=1}^{[ns]} \mathbb{X}_{1t}^{(l)} + n^{-\frac{1}{2}} \sum_{i=1}^{[ns]} \mathbb{X}_{2t}^{(l)} \\ &= (\sum_{u=0}^{l} A_u) n^{-\frac{1}{2}} \sum_{t=1}^{[ns]} \mathbb{Z}_t + n^{-\frac{1}{2}} R([ns], l) \\ &\quad + n^{-\frac{1}{2}} \sum_{t=1}^{[ns]} \mathbb{X}_{2t}^{(l)}. \end{aligned} \tag{17}$$

By (17), Theorem 4.1 given in Billingsley(1968, p.25) and noting that $\sum_{u=0}^{l} \|A_u\| < \infty$ as $l \to \infty$, to prove (16), it suffices to show that for any $\delta > 0$,

$$(18) \qquad \limsup_{n\to\infty} P\{\sup_{0\le s\le 1} \|R([ns],l)\| \ge \delta n^{\frac{1}{2}}\} = 0,$$

for every fixed $l \ge 1$ and

$$(19) \qquad \lim_{l\to\infty} \limsup_{n\to\infty} P\{\sup_{0\le s\le 1} \|\sum_{t=1}^{[ns]} \mathbb{X}_{2t}^{(l)}\| \ge \delta n^{\frac{1}{2}}\} = 0.$$

By $\sum_{u=0}^{l} \|A_u\| < \infty$ and $E\|Z_t\|^2 < \infty$ (18) holds, and hence as $n \to \infty$,

$$\begin{aligned} & n^{-\frac{1}{2}} \sup_{0\le s\le 1} \|R([ns],l)\| \\ & \le n^{-\frac{1}{2}} \max_{-l\le t\le n} \|\mathbb{Z}_t\| \sum_{i=0}^{l} (\sum_{u=i}^{l} \|A_u\| + \sum_{u=i+1}^{l} \|A_u\|) \to^P 0. \end{aligned}$$

We next prove (19). Noting that

$$\sum_{t=1}^{k} \mathbb{X}_{2t}^{(l)} = \sum_{u=l+1}^{\infty} A_u \sum_{t=1}^{k} \mathbb{Z}_{t-u} \; for\ any\ k \ge 1,$$

by applying Hölder inequality and Lemma 2.3, we have

$$\begin{aligned} E \sup_{1\le s\le 1} \|\sum_{t=1}^{[ns]} \mathbb{X}_{2t}^{(l)}\|^2 &\le (\sum_{u=l+1}^{\infty} \|A_u\|)^2 E \max_{1\le k\le n} \|\sum_{u=1}^{k} \mathbb{Z}_{t-u}\|^2 \\ &\le Cm^2 nE\|Z_1\|^2 (\sum_{u=l+1}^{\infty} \|A_u\|)^2. \end{aligned}$$

Now, (19) follows immediately from the Markov inequality and $\sum_{u=l+1}^{\infty} \|A_u\| \to 0 \; as \; l \to \infty$. The proof of Theorem 3.1 is complete.

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Kwang-Hee Han
Department of Computer Science
Howon University
Jeonbuk, 573-718, Korea
Email : khhan@sunny.howon.ac.kr