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NOTE ON REAL HYPERSURFACES OF NONFLAT COMPLEX SPACE FORMS IN TERMS OF THE STRUCTURE JACOBI OPERATOR AND RICCI TENSOR

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Abstract. Let M be a real hypersurface with almost contact metric structure (ϕ, ξ, η, g) in a nonflat complex space form $M_n(c)$. We denote by A and S be the shape operator and the Ricci tensor of M respectively. In the present paper we investigate real hypersurfaces with $g(SA\xi, A\xi) =$ const. of $M_n(c)$ whose structure Jacobi operator R_ξ commute with both ϕ and S. We give a characterization of Hopf hypersurfaces of $M_n(c)$.

0. Introduction

An n-dimensional complex space form $M_n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature c. As is well known, complete and simply connected complex space forms are isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}_n$ or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0, c = 0$ or $c < 0$.

Let M be a real hypersurface of $M_n(c)$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure J and the Kaehlerian metric of $M_n(c)$. This structure plays an important role in the study of the geometry of a real hypersurface. The structure

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vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. A real hypersurface is said to be a Hopf hypersurface if the structure vector field ξ of M is principal.

In a complex projective space $P_n\mathbb{C}$, Hopf hypersurfaces with constant principal curvatures are just the homogeneous real hypersurfaces ([7]). Further, Hopf hypersurfaces with constant principal curvatures in a nonflat complex space forms were completely classified as follows:

Theorem T ([9]). *Let M be a homogeneous real hypersurface of $P_n\mathbb{C}$. Then M is a tube of radius r over one of the following Kaehlerian submanifolds:*

(A_1) *a hyperplane $P_{n-1}\mathbb{C}$, where $0 < r < \frac{\pi}{2}$,*

(A_2) *a totally geodesic $P_k\mathbb{C}$ $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$,*

(B) *a complex quadric Q_{n-1}, where $0 < r < \frac{\pi}{4}$,*

(C) *$P_1\mathbb{C} \times P_{(n-1)/2}\mathbb{C}$, where $0 < r < \frac{\pi}{4}$ and $n(\geq 5)$ is odd,*

(D) *a complex Grassmann $G_{2,5}\mathbb{C}$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,*

(E) *a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.*

Theorem B ([1]). *Let M be a real hypersurface of $H_n\mathbb{C}$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following:*

(A_0) *a self-tube, that is, a horosphere,*

(A_1) *a geodesic hypersphere or a tube over a hyperplane $H_{n-1}\mathbb{C}$,*

(A_2) *a tube over a totally geodesic $H_k\mathbb{C}(1 \leq k \leq k-2)$,*

(B) *a tube over a totally real hyperbolic space $H_n\mathbb{R}$.*

We denote by S and R_ξ be the Ricci tensor and the structure Jacobi operator with respect to the structure vector field ξ of M respectively. Then it is a very important problem to investigate real hypersurfaces satisfying $R_\xi S = SR_\xi$ in $M_n(c)$. From this point of view, Kim, Lee and one of the present authors ([4]) was recently proved the following:

Theorem KKL ([4]). *Let M be a real hypersurface in a nonflat complex space form $M_n(c)$. If it satisfies $R_\xi \phi = \phi R_\xi, R_\xi S = S R_\xi$ and $g(S\xi, \xi) = const.$, then M is a Hopf hypersurface. Further, M is locally congruent to one of* (A_1), (A_2) *type if $c > 0$, or* (A_0), (A_1), (A_2) *type if $c < 0$ provided that $\eta(A\xi) \neq 0$.*

Further, Nagai, Takagi and one of the present authors ([5]) have been also proved the following:

Theorem KNT ([5]). *Let M be a real hypersurface with $R_\xi \phi = \phi R_\xi$ and at the same time $R_\xi S = S R_\xi$ in $M_n(c), c \neq 0$. If $\theta = 3\{(\rho - \lambda)^2 - \frac{c}{4}\} \neq 0$, then M is a Hopf hypersurface* (for the definitions of ρ and λ see section 2).

The main purpose of this paper is to establish the following theorem:

Theorem 3.3. *Let M be a real hypersurface in a nonflat complex space form $M_n(c)$ which satisfies $R_\xi \phi = \phi R_\xi$ and at the same time $R_\xi S = S R_\xi$. If $g(SA\xi, A\xi)$ is constant, then M is a Hopf hypersurface. Further, M is locally congruent to one of* (A_1), (A_2) *type if $c > 0$, or* (A_0), (A_1), (A_2) *type if $c < 0$ provided that $\eta(A\xi) \neq 0$.*

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces supposed to be orientable.

1. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\tilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric $\tilde{g}$ of

$M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_Y X = \nabla_Y X + g(AY, X)N, \quad \tilde{\nabla}_X N = -AX,$$

for any vector fields X and Y on M, where g denoted the Riemannian metric of M induced from $\tilde{g}$ and A is the shape operator of M in $M_n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

Then we may see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M, that is, we have

$$\phi^2 X = -X + \eta(X)\xi, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$\eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi)$$

for any vector fields X and Y on M.

Since J is parallel, we find from the Gauss and Weingarten formulas the following:

$$(1.1) \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX.$$

The ambient space being of constant holomorphic sectional curvature c, we obtain the following Gauss and Codazzi equations respectively:
(1.2)

$$\begin{aligned} R(X,Y)Z = \tfrac{c}{4}\{&g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &-2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(1.3) \quad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}$$

for any vector fields X, Y and Z on M, where R denotes Riemann-Christoffel curvature tensor of M.

Notation. In the sequel, we denote by $\alpha = \eta(A\xi), \beta = \eta(A^2\xi), \gamma = \eta(A^3\xi)$ and $h = \operatorname{Tr} A$, and for a function f we denote by ∇f the gradient vector field of f.

Putting $U = \nabla_\xi \xi$, we see that U is orthogonal to ξ. Thus we have

$$\phi U = -A\xi + \alpha\xi, \tag{1.4}$$

which leads to $g(U, U) = \beta - \alpha^2$.

From (1.2) the Ricci tensor S of type (1,1) on M is given by

$$S = \frac{c}{4}\{(2n+1)I - 3\eta \otimes \xi\} + hA - A^2, \tag{1.5}$$

where I is the identity tensor, which shows that

$$S\xi = \frac{c}{2}(n-1)\xi + hA\xi - A^2\xi. \tag{1.6}$$

If we put

$$A\xi = \alpha\xi + \mu W, \tag{1.7}$$

where W is a unit vector field orthogonal to ξ. Then we have $U = \mu\phi W$. So we verify that W is also orthogonal to U. Further we have

$$\mu^2 = \beta - \alpha^2. \tag{1.8}$$

Therefore, we easily see that ξ is a principal curvature vector, that is, $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$ or $\mu = 0$.

From the definition of U, and (1.1) and (1.7), we verify that

$$g(\nabla_X \xi, U) = \mu g(AW, X). \tag{1.9}$$

Differentiating (1.4) covariantly along M and making use of (1.1), we find

$$\begin{aligned} &\eta(X) g(AU + \nabla\alpha, Y) + g(\phi X, \nabla_Y U) \\ &= g((\nabla_Y A) X, \xi) - g(A\phi AX, Y) + \alpha g(A\phi X, Y), \end{aligned} \tag{1.10}$$

which enables us to obtain

$$(\nabla_\xi A)\xi = 2AU + \nabla\alpha \tag{1.11}$$

because of (1.3) and (1.9). Since W is orthogonal to U, we verify, using (1.1), that

$$\mu g(\nabla_X W, \xi) = g(AU, X). \tag{1.12}$$

Because of (1.1), (1.9) and (1.10), it is seen that

$$\nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha. \tag{1.13}$$

2. The structure Jacobi operator

Let M be a real hypersurface of a complex space form $M_n(c), c \neq 0$. Then the structure Jacobi operator R_ξ with respect to ξ is given by

$$R_\xi X = R(X,\xi)\xi = \frac{c}{4}(X - \eta(X)\xi) + \alpha AX - \eta(AX)A\xi \tag{2.1}$$

for any vector X on M, where we have used (1.2).

Now, suppose that $R_\xi\phi = \phi R_\xi$. Then above equation implies that

$$\alpha(\phi AX - A\phi X) = g(A\xi, X)U + g(U,X)A\xi. \tag{2.2}$$

We set Ω be a set of points such that $\mu(p) \neq 0$ at $p \in M$ and suppose that $\Omega \neq \emptyset$. In what follows we discuss our arguments on the open subset Ω of M unless otherwise stated. Then, it is, using (2.2), clear that $\alpha \neq 0$ on M. So a function λ given by $\beta = \alpha\lambda$ is defined. Therefore, replacing X by U in (2.1) and taking account of (1.4), we find

$$\phi AU = \lambda A\xi - A^2\xi. \tag{2.3}$$

Further, we assume that $R_\xi S = SR_\xi$. Then we see from (1.6) and (2.1) that

$$\begin{aligned} &g(A^3\xi, Y)g(A\xi, X) - g(A^3\xi, X)g(A\xi, Y) \\ &= g(A^2\xi, Y)g(hA\xi - \tfrac{c}{4}\xi, X) - g(A^2\xi, X)g(hA\xi - \tfrac{c}{4}\xi, Y) \\ &\quad + \tfrac{c}{4}h\{g(A\xi, Y)\eta(X) - g(A\xi, X)\eta(Y)\}, \end{aligned}$$

which shows that

$$\alpha A^3\xi = \left(\alpha h - \frac{c}{4}\right)A^2\xi + \left(\gamma - \beta h + \frac{c}{4}\right)A\xi + \frac{c}{4}(\beta - h\alpha)\xi. \tag{2.4}$$

Combining above two equations and using (1.7), we obtain

$$\mu\left\{g\left(A^2\xi,Y\right)w(X)-g\left(A^2\xi,X\right)w(Y)\right\}$$
$$=\beta\left\{\eta(Y)g(A\xi,X)-\eta(X)g(A\xi,Y)\right\}$$

where a 1-form w is defined by $w(X)=g(W,X)$. Putting $Y=A\xi$ in this, we find

$$A^2\xi=\rho A\xi+(\beta-\rho\alpha)\xi, \tag{2.5}$$

where we have put $\mu^2\rho=\gamma-\beta\alpha$ and $\mu^2(\beta-\rho\alpha)=(\beta^2-\alpha\gamma)$ on Ω, which implies

$$A^3\xi=(\rho^2-\beta-\rho\alpha)A\xi+\rho(\beta-\rho\alpha)\xi.$$

Comparing this with (2.4), we verify that

$$\mu\,(h-\rho)\left(\beta-\rho\alpha-\frac{c}{4}\right)=0. \tag{2.6}$$

Remark 2.1. $h-\rho=0$ on Ω.

In fact, if not, then we see from (2.6) that $\beta=\rho\alpha+\frac{c}{4}$ on Ω. Hence, (2.5) turns out to be $A^2\xi=\rho A\xi+\frac{c}{4}\xi$, which connected to (2.1) implies that $R_\xi A=AR_\xi$. Thus, by Corollary 4.2 of [4], it is seen that $\Omega=\emptyset$. Hence $h=\rho$ on Ω is proved. In what follows $h=\rho$ is satisfied everywhere.

Since we have $\beta=\alpha\lambda$, (2.5) becomes

$$A^2\xi=\rho A\xi+\alpha(\lambda-\rho)\xi. \tag{2.7}$$

Thus, (2.3) implies that

$$AU=(\rho-\lambda)\,U. \tag{2.8}$$

We also have by (1.7) and (2.7)

$$AW=\mu\xi+(\rho-\alpha)\,W \tag{2.9}$$

because of $\mu\neq 0$.

Differentiating (2.7) covariantly along Ω and making use of (1.1), we find

(2.10)
$$g\left((\nabla_X A)\,A\xi,Y\right)+g\left(A\,(\nabla_X A)\,\xi,Y\right)+g\left(A^2\phi AX,Y\right)-\rho g(A\phi AX,Y)$$
$$=(X\rho)\,g(A\xi,Y)+\rho g\left((\nabla_X A)\,\xi,Y\right)$$
$$+X\left(\alpha\lambda-\alpha\rho\right)\eta(Y)+\alpha\left(\lambda-\rho\right)g(\phi AX,Y)$$

for any vectors X and Y on M, which together with (1.3) and (1.11) yields

$$(\nabla_\xi A)\,A\xi=\rho AU-\frac{c}{4}U+\frac{1}{2}\nabla\beta.$$

Putting $X=\xi$ in (2.10) and taking account of (1.11), (2.8) and above equation, we obtain

$$\begin{aligned}\tfrac{1}{2}\nabla\beta=&-A\nabla\alpha+\rho\nabla\alpha+(\xi\rho)\,A\xi+\xi\left(\alpha\lambda-\alpha\rho\right)\xi\\&-\left\{(\rho-\alpha)\left(\rho+\alpha-3\lambda\right)-\tfrac{c}{4}\right\}U,\end{aligned}\tag{2.11}$$

which connected to $\beta=\alpha\lambda$ implies that

$$\alpha\xi\lambda=(2\alpha-\lambda)\xi\alpha+2\mu W\alpha.\tag{2.12}$$

Because of (2.9) and (2.11), we also have

$$\alpha W\lambda=(2\alpha-\lambda)W\alpha+2\mu\left(\xi\rho-\xi\alpha\right).\tag{2.13}$$

If we take account of (2.7) and (2.8), then (2.11) implies that

(2.14)
$$\begin{aligned}\tfrac{1}{2}\left(A\nabla\beta-\rho\nabla\beta\right)=&-A^2\nabla\alpha+2\rho A\nabla\alpha-\rho^2\nabla\alpha+(\xi\sigma)\,A\xi\\&+\left(\sigma\xi\rho-\rho\xi\sigma\right)\xi+\lambda\{(\rho-\lambda)\left(\rho+\alpha-3\lambda\right)-\tfrac{c}{4}\}U.\end{aligned}$$

Now, differentiating (2.9) covariantly along Ω, we find

$$(\nabla_X A)\,W+A\nabla_X W=(X\mu)\xi+\mu\nabla_X\xi+X(\rho-\alpha)W+(\rho-\alpha)\nabla_X W,$$

which together with (1.3), (1.12) and (2.8) yields

$$\begin{aligned}\mu\,(\nabla_W A)\,\xi&=\{(\rho-\lambda)(\rho-2\alpha)-\frac{c}{2}\}U+\frac{1}{2}\nabla\beta-\alpha\nabla\alpha,\\(\nabla_W A)\,W&=-2(\rho-\lambda)U+X\rho-X\alpha.\end{aligned}$$

If we replace X by $A\xi$ in (2.10) and make use of (1.3), (1.7), (1.11), (2.7), (2.8) and the last two equations, we obtain

$$\tfrac{1}{2}(A\nabla\beta - \rho\nabla\beta) + \alpha^2\nabla\lambda + \mu^2\nabla\rho = g(A\xi, \nabla\rho)A\xi + g(A\xi, \nabla\sigma)\xi$$
$$+\{(\rho-\lambda)(2\rho\lambda - 3\alpha\rho + 2\alpha\lambda) + \tfrac{c}{4}(3\alpha - 2\lambda)\}U.$$

Substituting (2.14) into this, we find

$$\begin{aligned}&\alpha^2\nabla\lambda + \mu^2\nabla\rho - A^2\nabla\alpha + 2\rho A\nabla\alpha - \rho^2\nabla\alpha\\ &= \{g(A\xi,\nabla\rho) - \xi\sigma\}\, A\xi + \{g(A\xi, \nabla\sigma) + \rho\xi\sigma - (\beta - \rho\alpha)\xi\rho\}\xi\\ &\quad+\{(\rho-\lambda)(\rho\lambda - 3\alpha\rho + \alpha\lambda + 3\lambda^2) + \tfrac{c}{4}(3\alpha - \lambda)\}U.\end{aligned} \tag{2.15}$$

On the other hand, from (1.10), (2.1) and (2.8) we have

$$\nabla_U U = \phi\,(\nabla_U A)\,\xi + (\rho - \lambda)(2\alpha - \rho)U. \tag{2.16}$$

If we differentiate (2.8) covariantly, we find

$$(\nabla_X A)\,U + A\nabla_X U = X(\rho - \lambda)U + (\rho-\lambda)\nabla_X U, \tag{2.17}$$

which together with (1.3), (1.13) and (2.1) implies that

$$\begin{aligned}\phi\,(\nabla_U A)\,\xi \;&=\; \{3\,(\lambda-\rho)\,(\lambda-\alpha) - \frac{c}{4} - \frac{1}{\alpha}U\alpha\}U + \mu(\xi\rho - \xi\lambda)W\\ &\quad+(\rho-\lambda)\,(\nabla\alpha - (\xi\alpha)\,\xi) - A\nabla\alpha + \frac{1}{\alpha}g(A\xi, \nabla\alpha)A\xi.\end{aligned}$$

Combining this to (2.16), we obtain (for detail, see [4])

$$\begin{aligned}A\,(\nabla_U U) - (\rho-\lambda)\nabla_U U &= A^2\nabla\alpha - 2(\rho-\lambda)A\nabla\alpha + (\rho-\lambda)^2\nabla\alpha\\ &\quad+\{(\rho-\lambda)\xi\alpha - g(A\xi, \nabla\alpha)\}\{A\xi - (\rho-\lambda)\xi\}\\ &\quad-\mu(\xi\rho - \xi\lambda + \tfrac{1}{\alpha}g(A\xi, \nabla\alpha))\{AW - (\rho-\lambda)W\},\end{aligned}$$

which connected to (1.3), (1.4), (2.8) and (2.17) implies that

$$\begin{aligned}&A^2\nabla\alpha - 2\,(\rho-\lambda)\,A\nabla\alpha + (\rho-\lambda)^2\,\nabla\alpha\\ &= \{g(A\xi, \nabla\alpha) - (\rho-\lambda)\,\xi\alpha\}\,\{A\xi - (\rho-\lambda)\,\xi\}\\ &\quad+\mu\,\{\xi\rho - \xi\lambda + \tfrac{1}{\alpha}g\,(A\xi, \nabla\alpha)\}\,\{AW - (\rho-\lambda)\,W\}\\ &\quad+\mu^2\,(\nabla\lambda - \nabla\rho) + U(\rho-\lambda)U.\end{aligned} \tag{2.18}$$

Substituting (2.15) into this, we have (for detail, see [4])

$$\alpha\,(\nabla\rho - \nabla\lambda) = \theta U \tag{2.19}$$

on Ω, where θ is given by

$$\theta = 3(\rho-\lambda)^2 - \frac{3}{4}c. \tag{2.20}$$

From this we obtain $\mu^2(\nabla\lambda - \nabla\rho) + U(\rho-\lambda)U = 0$ and $\xi\lambda = \xi\rho$. Thus, (2.18) is reduced to

$$\begin{aligned} &A^2\nabla\alpha - 2(\rho-\lambda)A\nabla\alpha + (\rho-\lambda)^2\nabla\alpha \\ &= \{g(A\xi,\nabla\alpha) - (\rho-\lambda)\,\xi\alpha\}\,\{A\xi - (\rho-\lambda)\,\xi\} \\ &\quad + \tfrac{\mu}{\alpha} g\,(A\xi,\nabla\alpha)\,\{AW - (\rho-\lambda)\,W\}. \end{aligned} \tag{2.21}$$

Now, we define a 1-form u by $u(X) = g(U,X)$ for any vector X. Then the exterior derivative du of 1-form u is given by

$$du(X,Y) = \frac{1}{2}\{Yu(X) - Xu(Y) - u([X,Y])\}.$$

Therefore, we see, using (1.9), (1.13) and (2.8), that

$$du(\xi,X) = (3\lambda - 2\rho)\mu w(X) + g(\phi\nabla\alpha, X) \tag{2.22}$$

for any vector X.

We prepare the following without proof in order to prove our Theorem 3.3 (See Lemma 3.5 of [4]).

Remark 2.2. Let M be a real hypersurface in $M_n(c), c \neq 0$ such that $R_\xi\phi = \phi R_\xi$ and $R_\xi S = SR_\xi$. If $du = 0$, then Ω is void.

3. Proof of theorems

We will continue our arguments under the same hypotheses $R_\xi\phi = \phi R_\xi$ and at the same time $R_\xi S = SR_\xi$ as in section 2. Because of Theorem KNT and Remark 2.1, we may only consider the case where $\theta = 0$ and hence

$$(h-\lambda)^2 = \frac{c}{4} \tag{3.1}$$

by virtue of (2.20). From (1.6), (2.7) and Remark 2.1, it follows that

$$g(S\xi,\xi)=\frac{c}{2}(n-1)+(h-\lambda)\alpha,$$

which together with (3.1) implies that $g(S\xi,\xi)=$ const. if α is constant.

According to Theorem KKL, we have

Lemma 3.1. *Let M be a real hypersurface with* (3.1) *satisfying $R_\xi\phi=\phi R_\xi$, and $R_\xi S=SR_\xi$ in $M_n(c), c\neq 0$. If α is constant, then $\Omega=\emptyset$.*

Remark 3.2. We have $(x-y\lambda)\,\xi\alpha=0$ on Ω if

$$x\nabla\alpha=y\alpha\nabla\lambda \tag{3.2}$$

for certain scalars x and y.

In fact, from (2.12) and (3.2) we have

$$2\mu yW\alpha=\{x-(2\alpha-\lambda)y\}\,\xi\alpha. \tag{3.3}$$

We also have by (2.13) and (3.2)

$$xW\alpha=y\,\{(2\alpha-\lambda)\,W\alpha+2\mu\,(\xi\lambda-\xi\alpha)\},$$

which together with $\mu^2=\alpha(\lambda-\alpha)$ gives

$$\mu\,\{x-(2\alpha-\lambda)y\}\,W\alpha=2\,(\lambda-\alpha)\,y\alpha\xi\lambda-2\alpha(\lambda-\alpha)y\xi\alpha,$$

or, using (2.12) and (3.3), it follows that $(x-\lambda y)\,\xi\alpha=0$. Hence we arrive at the conclusion.

Now, suppose that $g\,(SA\xi,A\xi)=$ const. $=: a'$. They by (1.5) and (2.7), we have

$$\alpha\left\{\frac{c}{4}\,(2n+1)\,\lambda-\frac{3}{4}c\alpha+\alpha\lambda(h-\lambda)\right\}=a'.$$

This, together with (3.1), yields

$$\alpha\,\{(2n+1)\,(h-\lambda)\lambda+\alpha\,(4\lambda-3h)\}=a \tag{3.4}$$

because of $h-\lambda\neq 0$, where $a\,(h-\lambda)=a'$.

Differentiating (3.4) and using (3.1), we find

$$\{a+\alpha^2(4\lambda-3h)\}\,\nabla\alpha=-\alpha\,\{\alpha^2+(2n+1)\alpha\,(h-\lambda)\}\,\nabla\lambda. \tag{3.5}$$

From (3.5) and Remark 3.2, we have

$$\{a+\alpha^2(4\lambda-3h)+\lambda\alpha^2+(2n+1)\alpha\lambda\,(h-\lambda)\}\xi\alpha=0.$$

If $\xi\alpha\neq 0$, then by (3.4) and this, we have

$$2(2n+1)(h-\lambda)\lambda+3\alpha\,(3\lambda-2h)=0, \tag{3.6}$$

which enables us to obtain

$$3(3\lambda-2h)\nabla\alpha=-\,\{3\alpha+2(2n+1)(h-\lambda)\}\,\nabla\lambda,$$

or, using Remark 3.2,

$$3(3\lambda-2h)\alpha=-\lambda\,\{3\alpha+2(2n+1)(h-\lambda)\}\,.$$

This, connected to (3.6), gives

$$0=\alpha\lambda=\beta>0.$$

It is contradictory. Consequently, we have $\xi\alpha=0$ on Ω. Thus, (3.5) implies that

$$\{\alpha+(2n+1)(h-\lambda)\}\,\xi\lambda=0.$$

If $\xi\lambda\neq 0$, then $\alpha+(2n+1)(h-\lambda)=0$ on this subset and hence $\nabla\alpha=0$ because of (3.1), a contradiction by Lemma 3.1. Thus we conclude that

$$\xi\lambda=0,\qquad \xi\rho=0.$$

Because of (2.12) and (2.13), it follows that

$$W\alpha=0,\qquad W\lambda=0.$$

Putting

$$-b=\alpha^2+(2n+1)\alpha\,(h-\lambda)\,, \tag{3.7}$$

we have $b\neq 0$. Because if not, then $\alpha+(2n+1)(h-\lambda)=0$. This leads to $\nabla\alpha=0$ because of (3.1), a contradiction.

Thus, we have from (3.5)

$$\alpha\nabla\lambda = \frac{a + \alpha^2(4\lambda - 3h)}{b}\nabla\alpha, \tag{3.8}$$

which shows that

$$\nabla\beta = \left\{\frac{a + \alpha^2(4\lambda - 3h)}{b} + \lambda\right\}\nabla\alpha. \tag{3.9}$$

On the other hand, since we have $\xi h = 0, \xi\alpha = 0$ and (3.1), we can write (2.11) as

$$A\nabla\alpha - h\nabla\alpha + \frac{1}{2}\nabla\beta = (h-\lambda)(2\lambda - \alpha)U. \tag{3.10}$$

From (3.9) and (3.10), we have

$$A\nabla\alpha + \left\{\frac{a + \alpha^2(4\lambda - 3h)}{b} + \frac{1}{2}\lambda - h\right\}\nabla\alpha = (h-\lambda)(2\lambda - \alpha)U, \tag{3.11}$$

or, using (2.8)

$$A^2\nabla\alpha + \left\{\frac{a + \alpha^2(4\lambda - 3h)}{b} + \frac{1}{2}\lambda - h\right\}A\nabla\alpha = (h-\lambda)^2(2\lambda - \alpha)U,$$

By the way, we have from (2.21)

$$A^2\nabla\alpha + 2(\lambda - h)A\nabla\alpha + (h-\lambda)^2\nabla\alpha = 0,$$

where we have used Remark 2.1 and the fact that $\xi\alpha = W\alpha = 0$.

From the last two equations, it follows that

$$\left\{\frac{a + \alpha^2(4\lambda - 3h)}{b} + h - \frac{3}{2}\lambda\right\}A\nabla\alpha - (h-\lambda)^2\nabla\alpha = (h-\lambda)^2(2\lambda - \alpha)U,$$

which together with (3.1) and (3.11) gives

$$\sigma\nabla\alpha = \tau U, \tag{3.12}$$

where we have put

$$\sigma = 2\left\{\frac{a + \alpha^2(4\lambda - 3h)}{b} + h - \frac{3}{2}\lambda\right\}\left\{\frac{a + \alpha^2(4\lambda - 3h)}{b} + \frac{1}{2}\lambda - h\right\} + \frac{c}{4},$$

$$\tau = (h-\lambda)(2\lambda - \alpha)\left\{\frac{a + \alpha^2(4\lambda - 3h)}{b} - \lambda\right\}.$$

Differentiating (3.12) covariantly and taking the skew-symmetric part obtained, we find

$$(X\sigma)\,Y\alpha - (Y\sigma)\,X\alpha = (X\tau)\,u(Y) - (Y\tau)\,u(X) + \tau du(X,Y).$$

Putting $Y = \xi$ in this and using $\xi\alpha = \xi\lambda = \xi h = 0$, we find

$$(2\lambda - \alpha)\left\{\frac{a + \alpha^2\,(4\lambda - 3h)}{b} - \lambda\right\} du\,(\xi, X) = 0,$$

or, using (2.22),

$$(3.13) \qquad (2\lambda - \alpha)\left\{a + \alpha^2\,(4\lambda - 3h) - b\lambda\right\}\{\nabla\alpha - (3\lambda - 2h)U\} = 0.$$

We consider on the case where $a + \alpha^2\,(4\lambda - 3h) - b\lambda = 0$, then by (3.8) we have

$$\alpha\nabla\lambda = \lambda\nabla\alpha.$$

Using (3.4) and (3.7), it follows from this that

$$2\,(2n+1)\,(h - \lambda)\,\lambda + 3\,(3\lambda - 2h)\,\alpha = 0,$$

which together with $\alpha\nabla\lambda = \lambda\nabla\alpha$ gives

$$2\,(2n+1)\,(h - \lambda)\,\lambda + 3\,(3\lambda - 2h)\,\alpha + 3\lambda\alpha = 0$$

because of Lemma 3.1. From the last equation and (3.13), we have $\alpha\lambda = 0$, a contradiction. So we have

$$(3.14) \qquad a + \alpha^2\,(4\lambda - 3h) - b\lambda \neq 0$$

on Ω. Thus (3.13) implies that

$$(2\lambda - \alpha)\,\{\nabla\alpha - (3\lambda - 2h)U\} = 0.$$

If $2\lambda = \alpha$, then $\beta = \alpha\lambda = \frac{1}{2}\alpha^2$, which connected to (3.10) gives

$$A\nabla\alpha = \left(h - \frac{\alpha}{2}\right)\nabla\alpha.$$

From this and (3.11), we have

$$b\lambda = a + \alpha^2\,(4\lambda - 3h)\,,$$

which produces a contradiction because of (3.14). So, we are led to

$$2\lambda - \alpha \neq 0$$

on Ω. Thus we have

$$\nabla \alpha = (3\lambda - 2h)U. \tag{3.15}$$

From this and (3.8), we have

$$\nabla h = \frac{a + \alpha^2 (4\lambda - 3h)}{\alpha b}(3\lambda - 2h)U$$

by virtue of (3.1).

Differentiating (3.15) covariantly and using the last equation, and taking the skew-symmetric part, we find $(3\lambda - 2h)du = 0$, which together with (3.15) and Lemma 3.1 implies that $du = 0$. Thus, owing to Remark 2.2, we see that Ω is void.

Combining Theorem KNT and the above arguments, we conclude that

Theorem 3.3. *Let M be a real hypersurface in a nonflat complex space form $M_n(c)$ which satisfies $R_\xi \phi = \phi R_\xi$ and at the same time $R_\xi S = S R_\xi$. If $g(SA\xi, A\xi)$ is constant, then M is a Hopf hypersurface. Further, M is locally congruent to one of* (A_1), (A_2) *type if $c > 0$, or* (A_0), (A_1), (A_2) *type if $c < 0$ provided that $\eta(A\xi) \neq 0$.*

Remark 3.4. Replacing the hypothesis $g(SA\xi, A\xi) =$ const. in Theorem 3.3 by $g(R_\xi A\xi, A\xi) =$ const., we can, using the quite same method as that used in Theorem 3.3, verify that ξ is a principal curvature vector.

Thus, we have

Theorem 3.5. *Let M be a real hypersurface in $M_n(c)$ which satisfies $R_\xi \phi = \phi R_\xi$ and $R_\xi S = S R_\xi$. If $g(R_\xi A\xi, A\xi)$ is constant, then M is the same types at that of Theorem 3.3.*

4. Real hypersurfaces satisfying

$R_\xi \phi = \phi R_\xi$ **and** $\nabla_\xi R_\xi = 0$. Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. Then we have (2.1). Differentiating (2.1) covariantly, we find

$$\begin{aligned} g((\nabla_X R_\xi) Y, Z) &= -\frac{c}{4}\{\eta(Z) g(\nabla_X \xi, Y) + \eta(Y) g(\nabla_X \xi, Z)\} \\ &\quad + (X\alpha) g(AY, Z) + \alpha g((\nabla_X A) Y, Z) \\ &\quad - g(A\xi, Z)\{g((\nabla_X A)\xi, Y) - g(A\phi AY, X)\} \\ &\quad - g(A\xi, Y)\{g((\nabla_X A)\xi, Z) - g(A\phi AZ, X)\}, \end{aligned}$$

which together with (1.1) and (1.11) implies that

$$\begin{aligned} g((\nabla_\xi R_\xi) Y, Z) = &-\tfrac{c}{4}\{u(Y)\eta(Z) + u(Z)\eta(Y)\} + (\xi\alpha) g(AY, Z) \\ &+ \alpha g((\nabla_\xi A) Y, Z) - g(A\xi, Z)\{3u(Y) + Y\alpha\} \\ &- g(A\xi, Y)\{3u(Z) + Z\alpha\}. \end{aligned} \tag{4.1}$$

Now, suppose that $A\phi = \phi A$ is satisfied. Then we have $A\xi = \alpha\xi$, namely, $U = 0$ and hence α is constant (see, [6]). Thus, (1.10) turns out to be

$$(\nabla_X A)\xi + A\phi AX - \alpha\phi AX = 0.$$

This, together with $A\phi = \phi A$ and the Codazzi equation (1.3), yields $\nabla_\xi A = 0$. Using these facts, (4.1) becomes $\nabla_\xi R_\xi = 0$. Further, we easily, making use of (2.1), verify that $A\phi = \phi A$ implies $R_\xi \phi = \phi R_\xi$.

Conversely, we assume that $R_\xi \phi = \phi R_\xi$ and $\nabla_\xi R_\xi = 0$. Then we have (2.2) and

$$\begin{aligned} \alpha(\nabla_\xi A) X + (\xi\alpha) AX &= \frac{c}{4}\{u(X)\xi + \eta(X) U\} + \eta(AX)\{3AU + \nabla\alpha\} \\ &\quad + \{3g(AU, X) + X\alpha\} A\xi \end{aligned}$$

by virtue of (4.1). This, together with (1.11), yields

$$\alpha AU + \frac{c}{4} U = 0, \tag{4.2}$$

which shows that $\alpha \neq 0$ on Ω. So a function λ given by $\beta = \alpha\lambda$ is defined. Replacing X by U in (2.2) and taking account of (1.4) and (4.2), we find

$$A^2\xi = \rho A\xi + \frac{c}{4}\xi \tag{4.3}$$

because of $\alpha \neq 0$, where we have put $\alpha\rho = \alpha\lambda - \frac{c}{4}$. From (2.1) and (4.3) we see that $R_\xi A = AR_\xi$, which connected with $R_\xi\phi = \phi R_\xi$ implies that $\Omega = \emptyset$, that is, $U = 0$ and hence $\alpha\,(A\phi - \phi A) = 0$ (cf. [4] and [5]).

Thus we have

Theorem 4.1. *Let M be a real hypersurface in a complex space form $M_n\,(c)\,, c \neq 0$. Then the followings are equivalent:*

(1) *$A\phi = \phi A$ holds on M.*

(2) *$\nabla_\xi R_\xi = 0$ and $R_\xi\phi = \phi R_\xi$ hold on M provided that $\eta(A\xi) \neq 0$.*

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