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A NOTE ON MULTIPLICATIVE AND ALMOST MULTIPLICATIVE LINEAR MAPS

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Abstract. This note is a verification on the relations between multiplicative and almost multiplicative linear maps; and the continuity of almost multiplicative linear maps.

1. Introduction

In 1952, E. A. Michael asked his famous question about the continuity of multiplicative linear functionals on complete locally convex topological algebras. From that time mathematicians have studied this interesting problem in various directions. The theorem [2.4] is a well known affirmative answer for the case of complete normed algebras. The last result in this point of view is proved in [1] where it is claimed that every multiplicative linear functional is automatically continuous on complete metrizable FLM algebras .

In [4] Jarosz extends this problem for the maps between Banach algebras and proves the continuity of multiplicative linear maps from a Banach algebra A to a semi-simple Banach algebra B, and in [3] he discusses the continuity of ϵ-multiplicative linear functionals on Banach algebras.

In this note we recall some previous well-known theorems in section 2 and then we have a discussion on almost multiplicative maps. Some more

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results on multiplicative and almost multiplicative linear functionals are proved in this section.

2. Definitions and related theorems

In this section we recall some theorems and related definitions.

Definition 2.1. *Let A and B be Banach algebras, $\varphi : A \longrightarrow B$ be a linear map and $\epsilon > 0$. We say φ is ϵ-multiplicative if for all $x, y \in A$,*
$||\varphi(xy) - \varphi(x)\varphi(y)|| \leq \epsilon||x||||y||$.

Definition 2.2. *Let A and B be Banach algebras, $\varphi : A \longrightarrow B$ be a linear map. We say φ is almost multiplicative if there exists an $\epsilon > 0$ such that for all $x, y \in A$, $||\varphi(xy) - \varphi(x)\varphi(y)|| \leq \epsilon||x||||y||$.*

Almost linear functionals and ϵ- multiplicative linear functionals are similarly defined.

Theorem 2.3. *Let X be a normed linear space. Then $dimX < \infty$ if and only if every linear functional on X be continuous.*

Proof. [4] □

Theorem 2.4. *Let A be a Banach algebra and let $\varphi : A \longrightarrow C$ be a multiplicative linear functional. Then φ is continuous and $||\varphi|| = 1$.*

Proof. [2] □

There are many examples which says that we can not drop the completeness of normed algebras.

Theorem 2.5. *(Gleason, Kahane, and Zelazko). Let A have a unit element and let φ be a linear functional on A. The following conditions are equivalent:*

(1) $\varphi(1) = 1$ *and* $Ker(\varphi) \subseteq Sing(A)$

(2) $\varphi(a) \in Sp(a) \quad (\forall a \in A)$
(3) φ *is multiplicative.*

Proof. [2] □

Theorem 2.6. *Let* $T : A \longrightarrow B$ *be a linear multiplicative map from a Banach algebra* A *into a semi-simple Banach algebra* B*. Then* T *is continuous.*

Proof. [4] □

In this theorem, B is commutative. The following example shows that we can not drop the commutativity.

Example 2.7. *Suppose* A *is a Banach space with infinite dimension. By trivial definition for product on* A*,* A *will be a Banach algebra. Suppose* f *is a discontinuous linear functional. By Theorem [2.3] such an* f *exists. Define* $\varphi : A \longrightarrow M_2(C)$ *by*

$$\varphi(a) = \begin{bmatrix} 0 & f(a) \\ 0 & 0 \end{bmatrix}$$

for every $a \in A$*. This* φ *is a discontinuous linear multiplicative map on* A*.*

Proposition 2.8. *Let* $(A, (p_k))$ *be a locally m-convex topological algebra and* $\varphi : A \longrightarrow \Phi$ *be a linear multiplicative functional. Let also* $N_k = \mathrm{N}(p_k)$ *be the null space of* p_k *and* $A_k = A/N_k$ *be the quotient algebra. If there exists* k *such that* $(A_k, |.|)$ *is complete and* $N_k \subseteq Ker(\varphi)$*, then* φ *is continuous.*

Proof. Let $(A_k, |.|)$ be a Banach algebra and define $f : A_k \longrightarrow \Phi$ by $f(x + N_k) = \varphi(x)$. This f is a multiplicative linear functional and so it is continuous. The set

$$\{|\varphi(x)| : p_k(x) < 1\} = \{|f(x + N_k)| : |x + N_k| < 1\}$$

is bounded. So φ is bounded on a neighborhood of zero; and hence φ is continuous. □

3. Almost multiplicative linear maps

Obviously if φ is ϵ-multiplicative, then for all $\delta > \epsilon$ it is δ-multiplicative. The following proposition shows that if φ is ϵ-multiplicative and $\delta < \epsilon$, then φ is not necessarily δ-multiplicative.

Proposition 3.1. *If A is a Banach algebra, $T : A \longrightarrow C$ is a linear multiplicative functional, and $S : A \longrightarrow C$ is a linear map such that $||S|| \leq \epsilon$, then $T + S$ is not multiplicative where it is $(3\epsilon + \epsilon^2)$-multiplicative.*

Proof. For a proof, see [4]. □

Lemma 3.2. *Let A be a Banach algebra, $\epsilon > 0$ and $\varphi : A \longrightarrow C$ be a linear functional such that for all $x, y \in A$, $|\varphi(xy) - \varphi(x)\varphi(y)| < \epsilon(||x|| \pm ||y||)$. Then φ is multiplicative.*

Proof. If for all $x, y \in A$, $|\varphi(xy) - \varphi(x)\varphi(y)| < \epsilon(||x|| - ||y||)$, then For every $a \in A$, $\varphi(a^2) = (\varphi(a))^2$. Now let for all $x, y \in A$ we have $|\varphi(xy) - \varphi(x)\varphi(y)| < \epsilon(||x|| + ||y||)$. Then for every $a \in A$:

$$|\varphi(a^2) - (\varphi(a))^2| \leq 2\epsilon||a||$$

Let $x = 2^n a$. We have:

$$|\varphi(a^2) - (\varphi(a))^2| \leq 2\epsilon 2^{-n}||a||$$

and therefore $\varphi(a^2) = (\varphi(a))^2$. So A is a Jordan algebra and by [2; Proposition 6.16] φ is multiplicative. □

Proposition 3.3. *Every ϵ- multiplicative functional F is continuous, and $||F|| \leq 1 + \epsilon$.*

Proof. [3] □

Theorem 3.4. *Every linear almost multiplicative map from a Banach algebra A to a semi-simple Banach algebra B is continuous.*

Proof. Suppose $T : A \to B$ is linear and $\epsilon-$multiplicative. For any multiplicative linear functional $F : B \longrightarrow C$, FoT is a linear functional and,

$$|FoT(ab) - FoT(a)FoT(b)| \leq ||F|||T(ab) - T(a)T(b)| \leq \epsilon||a||||b||.$$

Therefore FoT is a linear almost multiplicative functional. So by theorem [3.3] , it is continuous. Let (a_n) be a sequence with $a = \lim a_n$, and $b = \lim T(a_n)$. We show that $b = T(a)$. For any multiplicative linear functional, such as F:

$$F(b) = F(\lim T(a_n)) = \lim F(T(a_n)) = FoT(a)$$

So, $F(b - T(a)) = 0$. Since B is semi-simple, so $b = T(a)$.

Hence from the closed Graph Theorem, T is continuous. □

Theorem 3.5. *Let A be a Banach algebra with unit and $\varphi : A \longrightarrow \Phi$ be a linear functional. If*

1) $ker(\varphi) \subseteq sing(A)$

or

2)$\varphi(a) \in SP(\varphi(1)a)$, for any $a \in A$

then φ is almost multiplicative.

Proof. Let $ker(\varphi) \subseteq sing(A)$ and $\psi = \frac{\varphi}{\varphi(1)}$. Then $ker(\psi) = ker(\varphi)$, and $\psi(1) = 1$. Hence ψ is continuous multiplicative linear functional and $|\psi(x)| \leq ||x||$ for all $x \in A$. Therefore $\psi(a) \in SP(a)$ and so $\varphi(a) \in SP(\varphi(1)a)$.

Conversely let for all $x \in A$, $\varphi(a) \in SP(\varphi(1)a)$. Since φ is nonzero, for every $x \in A$, $\frac{\varphi(x)}{\varphi(1)} \in SP(x)$. So $\psi(x) \in SP(x)$, and $ker(\varphi) = ker(\psi) \subseteq sing(A)$.

Now for $x, y \in A$,

$$|\varphi(xy) - \varphi(x)\varphi(y)| \leq |1 - \varphi(1)||\varphi(1)|||x||||y||$$

Which means that φ is almost multiplicative. □

Here we give an example to show that, in theorem [3.5], φ may be almost multiplicative, where (1) and (2) do not hold. In Theorem [3.5] if $ker(\varphi) \subseteq sing(A)$, then $\varphi(xy)\varphi(1) = \varphi(x)\varphi(y)$. By the following example we show that this relation is not always true.

Example 3.6. *Let* $X = \{\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} : a_{ij} \epsilon C\}$, *and* $T : X \longrightarrow C$ *be a functional with:*

$$T(\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}) = a_{11},$$

then T *is linear multiplicative. Let* $S : X \longrightarrow C$ *be a functional with:*

$$S(\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}) = a_{11} + a_{21},$$

then S *is linear but is not multiplicative .*
Now let define $\varphi : X \longrightarrow C$ *with:*

$$\varphi(A) = (T + S)(A).$$

The relation $\varphi(xy)\varphi(1) = \varphi(x)\varphi(y)$ *is not always true. For example let* $A = \begin{bmatrix} 2 & 0 \\ -2 & 4 \end{bmatrix}$, *and* $B = \begin{bmatrix} 3 & 0 \\ -3 & -4 \end{bmatrix}$, *then* $\varphi(AB)\varphi(1) \neq \varphi(A)\varphi(B)$.

References

[1] E.Ansari-Piri. Topics on fundamental topological algebras. Honam Mathematical Journal. Vol. 23, No. 1, July 2001.

[2] F.F.Bonsall. J. Duncan. Complete Normed Algebras. Springer-verlag 1973.

[3] K.Jarosz. Perturbation of Banach Algebras. Lecture Notes in Mathematics. Springer- verlag 1985.

[4] K.Jarosz. Automatic Continuity Selected Examples. Bull. Southern Math. 1995.

[5] P. Semrl. Almost multiplicative functions and almost linear multiplicative functionals. Aeqationes Mathematicae. 63(2002) 180-192.

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