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UNITARY INTERPOLATION ON $AX = Y$ IN A TRIDIAGONAL ALGEBRA ALG$\mathcal{L}$

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Abstract. Given operators X and Y acting on a separable complex Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A such that $AX = Y$. We show the following : Let Alg$\mathcal{L}$ be a subspace lattice acting on a separable complex Hilbert space $\mathcal{H}$ and let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators acting on $\mathcal{H}$. Then the following are equivalent:

(1) There exists a unitary operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $AX = Y$.

(2) There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $|\alpha_j| = 1$ and $y_{ij} = \alpha_j x_{ij}$ for $j \in \mathbb{N}$.

1. Introduction

Let $\mathcal{A}$ be a subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all operators acting on a Hilbert space $\mathcal{H}$ and let X and Y be operators acting on $\mathcal{H}$. An *interpolation question* for $\mathcal{A}$ asks for which X and Y is there a bounded operator $A \in \mathcal{A}$ such that $AX = Y$. An n-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison[8]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance[9]: his result was extended by Hopenwasser[2] to the case that $\mathcal{U}$ is a CSL-algebra. Munch[10] obtained conditions for interpolation in case A is

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required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser[3] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation n-vectors, although necessity was not proved in that paper.

We establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. The symbol Alg$\mathcal{L}$ is the algebra of all bounded operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let x and y be two vectors in a Hilbert space $\mathcal{H}$. Then $\langle x, y\rangle$ means the inner product of the vectors x and y. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

2. Results

Let $\mathcal{H}$ be a separable complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots\}$. Let $x_1, x_2, \cdots, x_n$ be vectors in $\mathcal{H}$.
Then $[x_1, x_2, \cdots, x_n]$ means the closed linear subspace generated by the vectors $x_1, x_2, \cdots, x_n$. Let $\mathcal{L}$ be the subspace lattice generated by the subspaces $[e_{2k-1}]$, $[e_{2k-1}, e_{2k}, e_{2k+1}]$ $(k = 1, 2, \cdots)$. Then the algebra Alg$\mathcal{L}$ is called a tridiagonal algebra which was introduced by F. Gilfeather and D. Larson[1]. These algebras have been found to be useful counterexample to a number of plausible conjectures.

Let $\mathcal{A}$ be the algebra consisting of all bounded operators acting on $\mathcal{H}$ of the form

$$\begin{pmatrix} * & * & & & \\ & * & & & \\ & * & * & * & \\ & & & * & \\ & & & * & \ddots \end{pmatrix}$$

with respect to the orthonormal basis $\{e_1, e_2, \cdots\}$, where all non-starred entries are zero. It is easy to see that Alg$\mathcal{L}$=$\mathcal{A}$.

We consider interpolation problems for the above tridiagonal algebra Alg$\mathcal{L}$.

Lemma 1. Let $A = (a_{ij})$ be an operator in the tridiagonal algebra Alg$\mathcal{L}$. Then the following are equivalent:

(1) $A = (a_{ij})$ is unitary.

(2) A is a diagonal operator with $|a_{ii}| = 1$ for all $i \in \mathbb{N}$.

Proof. Suppose that $A = (a_{ij})$ is unitary. Since $AA^* = A^*A = I$, $a_{ij} = 0$ for all $i \neq j$ and $|a_{ii}| = 1$. So A is a diagonal operator with $|a_{ii}| = 1$ for all $i \in \mathbb{N}$. The converse is clear. □

Theorem 2. Let Alg$\mathcal{L}$ be the tridiagonal algebra on a separable complex Hilbert space $\mathcal{H}$ and let $X = (x_{ij})$ and $Y = (y_{ij})$ be operators acting on $\mathcal{H}$. Then the following are equivalent:

(1) There exists a unitary operator $A = (a_{ij})$ in Alg$\mathcal{L}$ such that $AX = Y$.

(2) There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $|\alpha_i| = 1$ and $y_{ij} = \alpha_i x_{ij}$ for all $i, j \in \mathbb{N}$.

Proof. Suppose that $A = (a_{ij})$ is a unitary operator in Alg$\mathcal{L}$ such that $AX = Y$. By Lemma 1, A is a diagonal operator with $|a_{ll}| = 1$ for all $l \in \mathbb{N}$. Let $\alpha_l = a_{ll}$ for $l = 1, 2, \cdots$. Since $AX = Y$, $y_{ij} = a_{ii}x_{ij} = \alpha_i x_{ij}$ for $i, j = 1, 2, \cdots$.

Conversely, assume that there is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $|\alpha_j| = 1$ and $y_{ij} = \alpha_i x_{ij}$ for $i, j = 1, 2, \cdots$. Let $A = (a_{jj})$ be a diagonal operator with $a_{jj} = \alpha_j$ for each $j \in \mathbb{N}$. Since $\{\alpha_n\}$ is bounded, A is a bounded operator and unitary. Since $y_{ij} = \alpha_i x_{ij}$ for all $i, j = 1, 2, \cdots$, $AX = Y$. □

Theorem 3. Let n be a fixed natural number $(n \geq 2)$. Let Alg$\mathcal{L}$ be the tridiagonal algebra on a separable complex Hilbert space $\mathcal{H}$ and let $X_i = (x_{jk}^{(i)})$ and $Y_i = (y_{jk}^{(i)})$ be operators acting on $\mathcal{H}$ for $i = 1, 2, \cdots, n$, where n is a fixed natural number. Then the following are equivalent:

(1) There exists a unitary operator $A = (a_{jk})$ in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$.

(2) There is a bounded sequence $\{\alpha_m\}$ in $\mathbb{C}$ such that $|\alpha_k| = 1$ and $y_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ for all $i = 1, 2, \cdots, n$ and $j, k \in \mathbb{N}$.

Proof. Suppose that $A = (a_{jk})$ is a unitary operator in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots, n$. By Lemma 1, A is a diagonal operator with $|a_{kk}| = 1$ for all k in $\mathbb{N}$. Let $\alpha_k = a_{kk}$ for $k = 1, 2, \cdots$. Then $\{\alpha_m\}$ is bounded. Since $AX_i = Y_i$, $y_{jk}^{(i)} = a_{jj} x_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ for $i = 1, 2, \cdots, n$ and $j, k = 1, 2, \cdots$.

Conversely, assume that there is a bounded sequence $\{\alpha_m\}$ in $\mathbb{C}$ such that $|\alpha_k| = 1$ and $y_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ for $i = 1, 2, \cdots, n$ and $j, k = 1, 2, \cdots$. Let A be a diagonal operator with diagonal $\{\alpha_m\}$. Since $\{\alpha_m\}$ is bounded, A is a bounded operator. Since $y_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ for $i = 1, 2, \cdots, n$ and $j, k = 1, 2, \cdots$, $AX_i = Y_i$. □

By the similar way with the above, we have the following.

Theorem 4. Let Alg$\mathcal{L}$ be the tridiagonal algebra on a separable complex Hilbert space $\mathcal{H}$ and let $X_i = (x_{jk}^{(i)})$ and $Y_i = (y_{jk}^{(i)})$ be operators acting on $\mathcal{H}$ for $i = 1, 2, \cdots$. Then the following are equivalent:

(1) There exists a unitary operator $A = (a_{jk})$ in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \cdots$.

(2) There is a bounded sequence $\{\alpha_n\}$ in $\mathbb{C}$ such that $|\alpha_k| = 1$ and $y_{jk}^{(i)} = \alpha_j x_{jk}^{(i)}$ for all $i, j, k \in \mathbb{N}$.

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