New Stability Criterion and Pole Assignment for Switched Linear Systems

Dong-Hae Yeom, Ki-Hong Im, and Jin-Young Choi

Abstract: In this paper, we propose a new stability criterion and a controller design method for switched linear systems. The proposed stability criterion is applicable to each subsystem independently without the need to consider the overall system. The controller can be easily designed through geometric relations between eigenvalues of each subsystem matrix. The proposed methods provide a systematic and simple pole assignment approach for switched linear systems. Illustrative examples are given.

Keywords: Switched linear systems, stability criterion, pole assignment, robustness.

1. INTRODUCTION

In recent years, the stability of switched systems has received growing attention. A switched system is defined as the system that is converted to another system according to switching signals. Especially, when each subsystem is linear, the overall system is called switched linear system. The widespread application of such systems is motivated by the fact that high performance control systems can be realized by switching between relatively simple LTI systems. But even if all the subsystems are linear, the overall system may not necessarily be linear because the system has discontinuities at each switching instants. This means that general stability criterion and controller design methods for linear systems are no longer applicable because the stability of switched linear systems can not be guaranteed even when each subsystem is stable [11].

There are many issues concerning the stability and the controller design of switched systems. Among them, it is interesting to study the existence of a switching rule that ensures the stability of switched systems [11,19,22]. In this case, the system can be stabilized under a specific switching signal. However, it is more desirable to make switched systems stable under an arbitrary switching signal than under a specific switching signal. Thus, considerable research efforts [1,2,4,5,7,12,13] have been devoted to the study on arbitrarily switched systems. The stability problem of switched linear systems under an arbitrary switching signal is related to the existence of a common Lyapunov function. Although many efforts [15,18] have been devoted to this issue, finding a common Lyapunov function is still an open problem. In this paper, we present a new stability criterion and a controller design method for switched linear systems under an arbitrary switching signal.

The previous works related to the switched linear system under an arbitrary switching signal can be categorized into four main streams. First, Lie algebraic approach [1,2,13] is used to obtain sufficient conditions for stability. That is, if Lie algebra generated by given subsystems is abelian, nilpotent, or solvable, the switched linear system composed by such subsystems is globally uniformly asymptotically stable. However, in most cases, a designer can not avoid the cumbersome work checking whether or not all the combinations of each subsystem satisfy those conditions. The second approach to prove the stability of switched linear systems has been studied based on multiple Lyapunov functions approach [7]. The results of this study show that if there exist multiple Lyapunov functions which are well defined between switching times and non-increasing at every switching instant, the switched linear system is uniformly stable. Moreover, if the system satisfies the observability condition, it is uniformly asymptotically stable. However, the above approach fails to provide a guideline for obtaining non-increasing Lyapunov functions systematically. Thirdly, the switched Lyapunov function method and the less conservative LMI (Linear Matrix Inequality)-based conditions are developed in [4,23] for the stability analysis and the controller design of discrete-time switched linear systems. This approach is appropriate for application in a system consisting of a small number of states and subsystems due to the numerical complexity of LMI

Manuscript received April 11, 2005; accepted September 16, 2005. Recommended by Editorial Board member Jae Weon Choi under the direction of Editor-in-Chief Myung Jin Chung.

Dong-Hae Yeom, Ki-Hong Im, and Jin-Young Choi are with the School of Electrical and Computer Engineering, Seoul National University, Kwanak P.O. Box 34(#048), Seoul 151-742, Korea (e-mails: {dhyeom, khim, jychoi}@neuro.snu.ac.kr).

problem. The last approach is called dwelling time analysis [8]. The main idea of this approach is that a switched system is stable if all individual subsystems are stable and switching is sufficiently slow, so as to allow the transient effects to dissipate after each switch. However, this method is not applicable to fast switched systems since switching is not permitted for a dwelling time.

In this paper, we propose a new criterion to discriminate the stability of a switched linear system by only investigating each subsystem individually and provide a guideline for designing state feedback and output feedback controller. The proposed criterion is based on a theorem [19], which guarantees the stability of switched systems including nonlinear cases, if the Lyapunov functions defined on each region are non-increasing at every switching instant. To generate such non-increasing Lyapunov functions recursively, we adopt Man's theorem [3,14]. By using the proposed method, computational burden involved in checking for the stability is significantly reduced, comparing with the existing methods such as Lie algebra [1,2,13] and LMI approach [4,23] because the proposed criterion requires only one subsystem to be considered at a time. Unlike the multiple Lyapunov functions method [7] that requires the existence of non-increasing Lyapunov functions, the proposed method provides an explicit way to construct non-increasing Lyapunov functions. In addition, the proposed method can be applicable to fast switching system contrary to dwelling time approach [8].

To implement the proposed method, the state feedback gain for each subsystem is designed through eigenvalue assignment. However, it is not easy to assign those eigenvalues on desired position because the proposed stability criterion needs an additional requirement that the eigenvalues of the sum of closed-loop system matrix and its transpose must be placed in a specific region. To solve this problem, we introduce a scheme called disk stability criterion based on geometric relations between matrix sum and its eigenvalues on complex plane.

The application of this new design method can be easily extended to the case that additional subsystems are appended to the existing system. In this case, the only thing we should consider is to design feedback gains for added subsystems independently of the existing system. Our study also shows that the controlled switched linear system designed by the proposed method has robustness against uncertainties imposed on each subsystem matrix. Finally, the bounds of robustness are given in the form of a function that includes margins of disk criterion.

2. STABILITY OF SWITCHED SYSTEMS

In most cases, the stability of a switched system is proven by showing the existence of a common Lyapunov function [15,18]. In case that it becomes difficult to obtain a common Lyapunov function or that there is no common Lyapunov function, the stability of a switched system can be shown by using non-increasing multiple Lyapunov functions. In this section, we briefly present a theorem [19] worked by Pettersson and Lennartson which is used to prove the stability of a switched system.

First, suppose that $\Omega \subseteq \{\mathrm{R}^n \times \mathrm{D}\}$ of state space consisting of continuous and discrete variables is partitioned into p disjoint regions, which means that the partitioning satisfies $\Omega_1 \cup \cdots \cup \Omega_p = \Omega$ and $\Omega_i \cap \Omega_j = 0$, $i \neq j$, where $p \in \mathrm{P}$ may be infinity. And define the neighboring regions $\Lambda_{i,j}$, $i \in \mathrm{P}$, $j \in \mathrm{P}$, $i \neq j$ by

$$\Lambda_{i,j} = \left\{ \begin{array}{l} (x,d) \in \Omega \mid \exists t > 0 \text{ such that} \\ \big(x(t-\varepsilon), d(t-\varepsilon)\big) \in \Omega_i \text{ and} \\ \big(x(t+\varepsilon), d(t+\varepsilon)\big) \in \Omega_j,\ \varepsilon \to 0 \end{array} \right\}$$

which are sets of states passing from Ω_i to Ω_j. Note that $\mathrm{P} = \{1, 2, \cdots, p\}$ is an index set and that each element of the real-valued state vector $x \in \mathrm{R}^n$ is called continuous state variable and that each element of the discrete-valued state $d \in \mathrm{D}$ is denoted as discrete state variable.

The following theorem shows that if multiple Lyapunov functions are well defined on each region and non-increasing at each switching instant as well, the equilibrium point of the system is asymptotically stable in spite that the Lyapunov function is discontinuous.

Theorem 2.1 [19]: If there exists $V_k(x): cl\,\Omega_i \to \mathbb{R}$ and class K functions a, b, c such that

$$\begin{aligned} &a(\|x\|) \leq V_k(x) \leq b(\|x\|), && (x,d) \in \Omega_i, \\ &\dot{V}_k(x) \leq -c(\|x\|), && (x,d) \in \Omega_i, \\ &V_k(x) \geq V_{k+1}(x), && (x,d) \in \Lambda_{i,j}, \end{aligned} \tag{1}$$

where cl means the closure of a set, then the

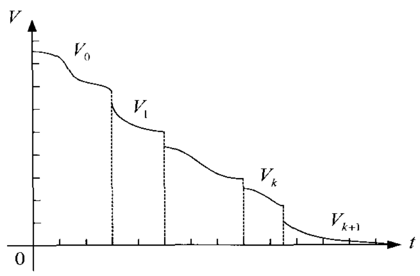


Fig. 1. Non-increasing Lyapunov functions.

equilibrium point is asymptotically stable in the sense of Lyapunov. If the assumptions hold globally and $a(\cdot)$ is a radially unbounded function, then the equilibrium point is globally asymptotically stable.

The first and second inequalities of (1) imply Lyapunov functions are well-defined on each region. And the third inequality of (1) implies the discontinuous Lyapunov functions are non-increasing at each switching instant. Through the above theorem, we can prove the stability of a system by using several discontinuous Lyapunov functions contrary to the traditional method where single continuous Lyapunov function is used. Fig. 1 depicts that if discontinuous multiple Lyapunov functions are non-increasing at each switching instant, the value of the functions converges to zero as time goes. Thus, the system with non-increasing multiple Lyapunov functions settles down to the equilibrium point.

3. NOVEL STABILITY CRITERION

In this section, we propose a new simple stability criterion for switched linear systems by using Theorem 2.1 and Man's theorem [3,14]. Consider a switched linear system as follows

$$\dot{x} = A_\sigma x, \tag{2}$$

where $x \in \mathrm{R}^n$ is a real valued vector, $\sigma : [0,\infty) \to \mathrm{P}$ is a piecewise constant switching signal, $\mathrm{P} = \{1,2,\cdots,p\}$ is a index set, and A_σ are Hurwitz matrices. Note that stability of each subsystem does not guarantee the stability of an overall system. In other words, the stability of a switched system consisting of stable subsystems depends on which switching signal is applied to the system. From this point of view, one can try to find a switching rule that ensures the stability of a switched system [22]. In this approach, which subsystem is activated depends on which region the trajectory of the solution of the given system is place in. The stability of the switched system is guaranteed only when the trajectory follows a path satisfying the switching rule. Thus, it is conservative that a switched system could be stabilized under a specific switching rule. In this paper, we investigate sufficient conditions that guarantee the stability of switched linear systems under an arbitrary switching signal.

First, we address Man's theorem that is initially intended to provide an upper bound to the solution of the Lyapunov matrix equation when the dominant eigenvalue of a matrix is known a priori. In this paper, however, this theorem is used to give a rule to generate positive definite matrices composing non-increasing Lyapunov functions at each switching instant.

Theorem 3.1 [3,14]: There exists a symmetric positive definite matrix M satisfying

$$A'M + MA + 2\delta N = 0,$$

where N is any symmetric positive definite matrix and δ is some positive scalar, for which $M < N$, if and only if

a) The symmetric matrix $(A + A' + 2\delta I)$ is negative definite.
b) The real parts of all the eigenvalues of the matrix A are less than $-\delta$. □

In the above theorem, the conditions become $A + A' + I < 0$ and $\mathrm{Re}[\lambda_i(A)] < -1/2$ when $\delta = 1/2$. By defining Lyapunov functions using M and N, we can obtain non-increasing Lyapunov functions. With that, we propose a new stability criterion for switched linear systems in the following theorem.

Theorem 3.2 [Novel Stability Criterion]: A switched linear system (2) is globally asymptotically stable if system matrices of each subsystem satisfy the following conditions.

a) All eigenvalues of the symmetric matrix $(A_\sigma + A'_\sigma)$ are less than -1.
b) The real parts of all eigenvalues of the matrix A_σ are less than -1/2.

Proof: Set a quadratic Lyapunov function candidate on region Ω_i as $V_k = x'M_k x$. The time derivative of the function along the trajectory yields $\dot{V}_k = x'(A'_i M_k + M_k A_i)x$. Because A_σ are Hurwitz from the fact that real part of all eigenvalues of the matrix are less than -1/2, there exist the positive definite matrix M_k as the solution of the Lyapunov matrix equation $A'_i M_k + M_k A_i + M_{k-1} = 0$ for any given positive definite matrix M_{k-1}. By Theorem 3.1 with $\delta = 1/2$, $M_{k-1} > M_k$, which means $M_{k-1} - M_k$ is a positive definite matrix. Whenever the trajectory of the solution shifts from region Ω_i to Ω_j, a new Lyapunov function is recursively defined as $V_{k+1} = x'M_{k+1}x$, where M_{k+1} is the solution of $A'_j M_{k+1} + M_{k+1} A_j + M_k = 0$. Therefore, the time derivative of a new Lyapunov function is negative definite because

$$\begin{aligned}\dot{V}_{k+1} &= x'(A'_j M_{k+1} + M_{k+1} A_j)x \\ &= -x'M_k x\end{aligned}$$

with $M_k > M_{k+1}$. By the extreme property of the Rayleigh quotient, it follows

$$\begin{aligned}&\lambda_{\min}(M_k)\|x\|_2^2 \le V_k(x) \le \lambda_{\max}(M_k)\|x\|_2^2, \\ &\dot{V}_k(x) \le -\lambda_{\min}(M_{k-1})\|x\|_2^2, \\ &\lambda_{\min}(M_{k+1})\|x\|_2^2 \le V_{k+1}(x) \le \lambda_{\max}(M_{k+1})\|x\|_2^2, \\ &\dot{V}_{k+1}(x) \le -\lambda_{\min}(M_k)\|x\|_2^2,\end{aligned}$$

where $\lambda_{\min}(\cdot)$, $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalue of a matrix, respectively. This means that Lyapunov functions are well defined in each region Ω_i, Ω_j, which meet the first and second conditions of (1) in Theorem 2.1. And for any consecutive region Ω_i, Ω_j, the corresponding Lyapunov functions are non-increasing because $M_k > M_{k+1}$ implies $V_k(x) > V_{k+1}(x)$, which meets the third condition of (1) in Theorem 2.1. Hence, by Theorem 2.1, the equilibrium point of a switched linear system (2) is asymptotically stable. Moreover, $\lambda_{\min}(M_k) \| x \|_2^2$ is a radially unbounded function as well as class K function. Therefore, (2) is globally asymptotically stable.

In the remaining part of the proof, we present the equivalence between the first condition of Theorem 3.2 and that of Theorem 3.1 when $\delta = 1/2$. The necessary and sufficient condition for a symmetric matrix to be negative definite is that all the eigenvalues of the matrix are less than zero. By the Ostrowski inequality (see Appendix A), the eigenvalues of the sum of symmetric matrices $X \in \mathrm{R}^{n \times n}$ and $Y \in \mathrm{R}^{n \times n}$ satisfy

$$\lambda_i(X+Y) = \lambda_i(X) + h, \; i = 1, 2, \cdots, n,$$

where $\lambda_{\min}(Y) \leq h \leq \lambda_{\max}(Y)$. In our case, substituting $A + A'$ and I for X, Y respectively, $(A + A' + I)$ is a negative definite matrix, if and only if

$$\lambda_i(A + A') + 1 < 0$$

for all i. □

Through the above theorem, we can discriminate stability of a switched linear system by just investigating each individual subsystem. That is, if each individual subsystem matrix A_σ satisfies two conditions in Theorem 3.2, there exist positive definite matrices M which compose non-increasing Lyapunov functions.

Compared with Lie algebraic method [1,2,13], the proposed criterion only require that each individual subsystem be checked. Unlike the multiple Lyapunov functions method [7], the proposed criterion provides an explicit way that generates positive definite matrices that compose non-increasing Lyapunov functions recursively. Compared with LMI approach based on a switched Lyapunov function [4,23], the proposed method is numerically simple. And, unlike dwelling time approach [8], the switched system satisfying the requirement of Theorem 3.2 is asymptotically stable regardless of how fast switching action between subsystems occurs. Furthermore, the criterion suggests a guideline for designing a state feedback control for each individual subsystem, which stabilizes the overall system.

The proposed method can be easily applied when additional subsystems are appended to the original system. For example, consider again a stabilized switched linear system given by

$$\dot{x} = A_\sigma x,$$

where $\sigma : [0, \infty) \to \mathrm{P} = \{1, 2, \cdots, p\}$ is a piecewise constant switching signal. Suppose that the index set P is extended to $\{1, 2, \cdots, p, p+1, \cdots, p+k\}$ corresponding to new added subsystems $\{A_{p+1}, \cdots, A_{p+k}\}$. In order to ensure the stability of the extended overall system, it is sufficient to check merely the property of each new subsystem by the proposed method in Theorem 3.2 without the need to consider the existing system. Note that the conditions in the above theorem are sufficient conditions, that is, we can not state that a switched system is unstable though the conditions are not fulfilled.

4. POLE ASSIGNMENT PROCEDURE

In this section, we discuss how to design a state feedback gain to satisfy the requirements in Theorem 3.2. Consider a switched linear system with control input as follows.

$$\dot{x} = A_\sigma x + B_\sigma u,$$

where $x \in \mathrm{R}^n, u \in \mathrm{R}^m$, and $\sigma : [0, \infty) \to \mathrm{P}$ is a piecewise constant switching signal. Suppose that each pair of (A_σ, B_σ) is controllable. If the control input is given as state feedback (i.e. $u = F_\sigma x$), the system becomes

$$\dot{x} = (A_\sigma + B_\sigma F_\sigma)x := G_\sigma x. \tag{3}$$

The same criterion in Theorem 3.2 is applied to G_σ in this case. The obstacle in determining the feedback gain is that the eigenvalues of $G_\sigma + G'_\sigma$ must be known. In general, the eigenvalues of the sum of two matrices are not given exactly in the form of eigenvalues of given matrices in spite that the eigenvalues of each matrix are known. For example, Horn's inequality [21] gives only the bounds, not the exact values of the eigenvalues of the sum of two matrices. What is worse, Horn's inequality can be applied only when the given matrices are symmetric or Hermitian. The following two theorems worked by Bauer and Householder deals with this issue. The first theorem states that the eigenvalues of the sum of the two matrices are restricted in some disks when one of the two matrices is simple (i.e. the matrix has distinct eigenvalues) and the norm of the other matrix is known.

Theorem 4.1 [10]: Let $X, Y \in \mathbb{R}^{n\times n}$, with X is a simple matrix. If X has eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ and ξ is an eigenvalue of $X+Y$, and if for a matrix norm induced by an absolute vector norm $\|\cdot\|_\nu$, we have

$$r = \| Y \| \inf_P \mathrm{K}(P),$$

where $\mathrm{K}(P) = \|P\| \|P^{-1}\|$ is the condition number of P and $\inf \mathrm{K}(P)$ is the infimum considered with respect to all P for which $P^{-1}XP$ is diagonal. Then ξ lies in at least one of the disks

$$\{z : |z-\lambda_i| \le r\},\ i=1,2,\cdots,n$$

on the complex z plane. □

In case that the given two matrices have distinct eigenvalues, the radius of the disks in the above theorem takes a more concrete form than the above theorem as follows.

Theorem 4.2 [10]: If, in addition to the hypotheses of Theorem 4.1, the matrix Y is simple with eigenvalues $\mu_1, \mu_2, \cdots, \mu_n$, then ξ lies in at least one of the disks

$$\left\{z : |z-\lambda_i| \le \inf_{P,R} \mathrm{K}(P^{-1}R) \max |\mu_i| \right\},\ i=1,2,\cdots,n,$$

where P and R diagonalize X and Y respectively, that is $X = PD_X P^{-1}, Y = RD_Y R^{-1}$, where D_X and D_Y are diagonal matrices. □

We are interested in the case that the given two matrices are transpose matrices each other because we need to find out the eigenvalues of $(G_\sigma + G'_\sigma)$. Our objective can be easily achieved by setting $G_\sigma = X, G'_\sigma = Y$ in Theorem 4.2. Then, we can replace μ_i and R with λ_j and $(P^{-1})'$, respectively. The resulting disks become

$$\left\{z : |z-\lambda_i| \le \inf_P \mathrm{K}\left(P^{-1}(P^{-1})'\right) \max |\lambda_j| \right\},$$

where $i,j=1,2,\cdots,n$. However, this result is not yet suitable for our purpose to make the eigenvalues of $(G_\sigma + G'_\sigma)$ less than -1 because the above disks containing the eigenvalues include the origin. This problem can be solved by translating the center of disks.

Corollary 4.3: If, in addition to the hypotheses of Theorem 4.2, G has eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, the eigenvalue of $G+G'$, ξ lies in at least one of the disks

$$\left\{z : |z-(\lambda_i-\alpha)| \le \inf_{P,R} \mathrm{K}(R'P) \max |\lambda_j+\alpha| \right\}$$

for any fixed complex number α, where $R = P\tilde{D}$ for any nonsingular diagonal matrix $\tilde{D}$ and $i,j=1,2,\cdots,n$.

Proof: Observe that P diagonalizes G if and only if P diagonalizes $\alpha I + G$ for any complex number α. As ξ is an eigenvalue of $G+G'$, there is a non-zero vector y for which

$$(-\alpha I + G + \alpha I + G')y = \xi y.$$

P diagonalizes G (i.e. $G = PDP^{-1}$), where $D = diag[\lambda_1, \lambda_2, \cdots, \lambda_n]$. However, the matrix P is not uniquely defined because for any matrix $R = P\tilde{D}$, G can be rewritten as $G = RDR^{-1}$, where $\tilde{D}$ is a nonsingular diagonal matrix. By replacing G and G' with PDP^{-1} and $(R^{-1})'DR'$, respectively, the above equation rewritten as

$$\left[-\alpha I + PDP^{-1} + \alpha I + (R^{-1})'DR'\right]y = \xi y,$$
$$P\left[-\alpha I + D + \alpha I + P^{-1}(R^{-1})'DR'P\right]P^{-1}y = \xi y.$$

Set

$$x = P^{-1}y \ne 0 \text{ and}$$
$$P\left[-\alpha I + D + \alpha I + P^{-1}(R^{-1})'DR'P\right]x = \xi Px.$$

Eliminating P on both side and transposing the terms related to diagonal yield

$$\begin{aligned}[\xi I - (D-\alpha I)]x &= \left[P^{-1}(R^{-1})'DR'P + \alpha I\right]x \\ &= \left[P^{-1}(R^{-1})'(D+\alpha I)R'P\right]x.\end{aligned}$$

Because

$$\begin{aligned}\inf \frac{\|[\xi I-(D-\alpha I)]x\|}{\|x\|} &\le \frac{\|[\xi I-(D-\alpha I)]x\|}{\|x\|} \\ &= \frac{\|P^{-1}(R^{-1})'(D+\alpha I)R'Px\|}{\|x\|} \\ &\le \|P^{-1}(R^{-1})'(D+\alpha I)R'P\| \\ &\le \|P^{-1}(R^{-1})'\| \|D+\alpha I\| \|R'P \\ &= \mathrm{K}(R'P) \max_{1\le j\le n} |\lambda_j+\alpha|,\end{aligned}$$

we have

$$\min_{1\le i\le n} |\xi-(\lambda_i-\alpha)| \le \mathrm{K}(R'P) \max_{1\le j\le n} |\lambda_j+\alpha|,$$

where $i,j=1,2,\cdots,n$. Since this is true for every P and R diagonalize G and G', respectively

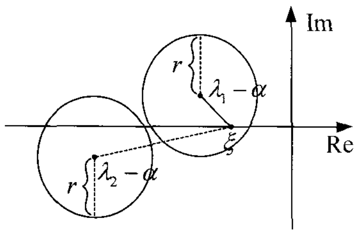


Fig. 2. Disks containing eigenvalues.

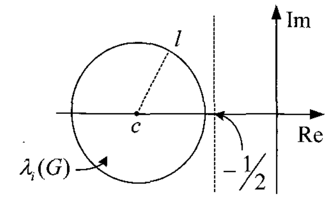


Fig. 3. $c+l<-1/2$ implies $\mathrm{Re}[\lambda_i(G)]<-1/2$.

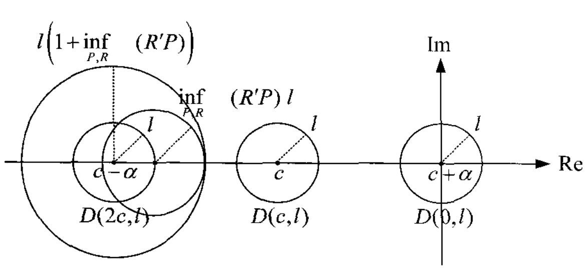


Fig. 4. Disk containing eigenvalues of $G+G'$.

$$\min_{1\le i\le n}|\xi-(\lambda_i-\alpha)|\le \inf_{P,R}\mathrm{K}(R'P)\max_{1\le j\le n}|\lambda_j+\alpha|:=r.$$

As shown in Fig. 2, the eigenvalue of $G+G'$ is located in the disk with radius r and center $\lambda_i-\alpha$ which minimizes $|\xi-(\lambda_i-\alpha)|$.

Hence, ξ lies in at least one of disks $|z-(\lambda_i-\alpha)|\le r$ of the complex z plane. □

Hence, the disks containing the eigenvalues of $G+G'$ can be translated so as to exclude the origin. Now, we will discuss the way to design the state feedback gain F_σ in (3) for each individual subsystem to satisfy the requirements in Theorem 3.2. Using Corollary 4.3, the relation between the range of the eigenvalues of G and the requirements $\mathrm{Re}[\lambda_i(G)]<-1/2$, $\lambda_i(G+G')<-1$ is revealed in the following theorem.

Theorem 4.4 [Disk Stability Criterion]: For matrix G whose eigenvalues lie in a disk with center c and radius l,

$$c+l<-\frac{1}{2},$$

$$1+\inf_{P,R}\mathrm{K}(R'P)<\frac{-2c-1}{l}$$

imply

$$\mathrm{Re}[\lambda_i(G)]<-\frac{1}{2},$$

$$\lambda_i(G+G')<-1,$$

respectively, where c is a negative constant and $G=PDP^{-1}=RDR^{-1}$, $R=P\tilde{D}$ for any nonsingular diagonal matrix $\tilde{D}$.

Proof: First, it is trivial that $c+l<-1/2$ implies $\mathrm{Re}[\lambda_i(G)]<-1/2$ (See Fig. 3).

Next, because the eigenvalues of G are equivalent to those of G', they lie in the same disk $D(c,l)$ with center c and radius l. We can obtain $D(c+\alpha,l)$ and $D(c-\alpha,l)$ by moving the center of $D(c,l)$ as $\pm\alpha$ along real axis. Here, set $\alpha=-c$, and $D(c+\alpha,l)$ becomes a disk with radius l and the origin as the center. By Corollary 4.3 the eigenvalues of $G+G'$ lie in at least one of the disks

$$\left\{z:|z-(\lambda_i-\alpha)|\le\inf_{P,R}\mathrm{K}(R'P)\max|\lambda_j+\alpha|\right\},$$

where $i,j=1,2,\cdots,n$, λ_i and λ_j are the eigenvalues of G and G', respectively. Because eigenvalues of G' is translated into $D(0,l)$, $\max|\lambda_j+\alpha|=l$. The center of the eigenvalues of $G+G'$ is located in $D(2c,l)$ and the range of the eigenvalues of $G+G'$ is restricted by $\inf\mathrm{K}(R'P)\,l$. Hence, the eigenvalues of $G+G'$ lie in the disk with center $2c$ and radius $l(1+\inf\mathrm{K}(R'P))$. See Fig. 4. Since every symmetric matrix has real eigenvalues, the eigenvalues of $G+G'$ lie on the intersection of the disk and the real axis. Therefore, on the real axis if the disk does not exceed -1, that is,

$$2c+l(1+\inf\mathrm{K}(R'P))<-1,$$

the eigenvalues of $G+G'$ are less than -1. □

By the above theorem, if the eigenvalues of each subsystem lie on an appropriate disk $D(c,l)$, the requirements in Theorem 3.2 is achieved, which guarantees the globally asymptotic stability of switched linear systems. However, the sufficient conditions of the above theorem do not provide an explicit relation because the value of $\inf\mathrm{K}(R'P)$ depends on c and l. We need to go through some trial and error in the procedure of finding a suitable value of c and l. The procedure of pole assignment for each subsystem is given in Table 1.

Remark 4.1: In general, it is not easy to find the exact quantity of $\inf\mathrm{K}(R'P)$. In practice, $\mathrm{K}(P)$ is used as the approximation of $\inf\mathrm{K}(R'P)$ as addressed below.

Table 1. Procedure of pole assignment for each subsystem.

Step 1: Select poles of the closed-loop subsystem G whose eigenvalues are restricted in the disk with center c and radius l, where $c+l$ should be less than -1/2.
Step 2: Compute the state feedback gain F from $G=A+BF$.
Step 3: Check the disk criterion whether $1+\inf \mathrm{K}(R'P)<(-2c-1)/l$.
Step 4: If the criterion is satisfied, $\mathrm{Re}[\lambda_i(G)]<-1/2$ and $\lambda_i(G+G')<-1$. Otherwise, go to Step 1 and try with another poles.

$$\begin{aligned}\inf \mathrm{K}(R'P) &= \inf \| R'P \| \| P^{-1}(R')^{-1} \| \\ &\leq \inf \| R' \| \| P \| \| P^{-1} \| \| (R')^{-1} \| \\ &= \inf \mathrm{K}(P)\mathrm{K}(R),\end{aligned}$$

where $R=P\tilde{D}$ for any nonsingular diagonal matrix $\tilde{D}$. For fixed P, there exists $\tilde{D}$ which makes $\mathrm{K}(R)$ small sufficiently. That is, $\mathrm{K}(R)\approx 1$ because the condition number of a matrix is greater than or equal to 1. Thus, $\inf \mathrm{K}(R'P)\leq \inf \mathrm{K}(P)\mathrm{K}(R)\approx \inf \mathrm{K}(P)$. Because matrix P is fixed, $\inf \mathrm{K}(P)=\mathrm{K}(P)$. Therefore, $\mathrm{K}(P)$ can be used as approximation of $\inf \mathrm{K}(R'P)$.

Remark 4.2: If the first trial fails, try with another combination of c and l in other to increase the value of $(-2c-1)/l$. In doing this, l should be selected as less value and the absolute value of c should be selected as larger value because $l>0,\ c<0$.

Remark 4.3: The requirements in Theorem 4.4 does not always satisfied by setting more less l and c, because condition number $\mathrm{K}(\cdot)$ does not have a fixed tendency to increase or to decrease according to the variation of l and c.

Remark 4.4: The requirements in Theorem 4.4 and Step 3 are sufficient conditions, that is, there exist combinations of c and l which satisfy $\mathrm{Re}[\lambda_i(G)]<-1/2$ and $\lambda_i(G+G')<-1$ in spite of the violation of the requirements of Theorem 4.4.

5. ROBUSTNESS

In this section, we discuss the robustness of switched linear systems with uncertainty. Suppose that a nominal switched system $\dot{x}=A_\sigma x+B_\sigma u$ is stabilized by means of $u=F_\sigma x$, the state feedback control through the proposed method in the previous section and that there is additive uncertainty ΔA_σ with respect to A_σ. It is clear that the system becomes more unstable as ΔA_σ becomes larger. In the following theorem, we give the bound of uncertainty at least to conserve the asymptotic stability when the system is perturbed by additive uncertainty imposed on each subsystem matrices.

Theorem 5.1: Suppose that there exists the state feedback gain F_σ such that

$$\mathrm{Re}[\lambda_i(G_\sigma)]<-\alpha_\sigma,$$
$$\lambda_i(G_\sigma+G'_\sigma)<-\beta_\sigma,$$

where $\alpha_\sigma\geq 1/2$, $\beta_\sigma\geq 1$, and G_σ are simple. If model uncertainty ΔA_σ is bounded by

$$\| \Delta A_\sigma \|_2 < \min\left\{\frac{\alpha_\sigma-1/2}{\inf \mathrm{K}(P_\sigma)},\ \frac{\beta_\sigma-1}{2}\right\},$$

where P_σ diagonalizes G_σ, then the switched system with model uncertainty

$$\dot{x}=(A_\sigma+\Delta A_\sigma)x+B_\sigma u$$

is globally asymptotically stable.

Proof: The closed-loop system with model uncertainty is denoted by $\tilde{G}_\sigma=G_\sigma+\Delta A_\sigma$. Thus,

$$\lambda_i(\tilde{G}_\sigma+\tilde{G}'_\sigma)=\lambda_i(X+Y),$$

where $G_\sigma=A_\sigma+B_\sigma F_\sigma$, $X=G_\sigma+G'_\sigma$ and $Y=\Delta A_\sigma+\Delta A'_\sigma$. Since X and Y are symmetric matrices, using Wely's inequality (see Appendix B) yields

$$\lambda_n(X)+\lambda_n(Y)\leq \lambda_i(X+Y)\leq \lambda_1(X)+\lambda_1(Y),$$

where the eigenvalues are ordered as $\lambda_1>\lambda_2>\cdots>\lambda_n$. Because $\lambda_i(G_\sigma+G'_\sigma)<-\beta_\sigma$,

$$\lambda_1(X)+\lambda_1(Y)<-\beta_\sigma+\lambda_1(\Delta A_\sigma+\Delta A'_\sigma).$$

By Fan-Hoffman's inequality (see Appendix C), the second term on the right side can be expressed in the form of singular values,

$$-\beta_\sigma+\lambda_1(\Delta A_\sigma+\Delta A'_\sigma)\leq -\beta_\sigma+2\eta_1(\Delta A_\sigma),$$

where the singular values are ordered as $\eta_1>\eta_2>\cdots>\eta_n$. Hence, by the definition of induced 2-norm

$$\lambda_i(\tilde{G}_\sigma+\tilde{G}'_\sigma)<-\beta_\sigma+2\| \Delta A_\sigma \|_2.$$

Thus,

$$\begin{aligned}&-\beta_\sigma+2\| \Delta A_\sigma \|_2\leq -1 \\ &\Rightarrow \lambda_i(\tilde{G}_\sigma+\tilde{G}'_\sigma)<-1.\end{aligned} \tag{4}$$

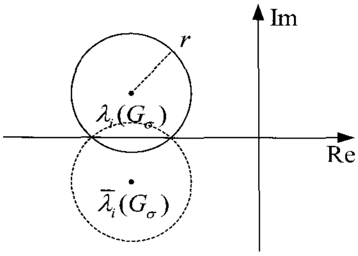


Fig. 5. Disk containing eigenvalues of $\tilde{G}_\sigma$.

On the other hand, by Theorem 4.1, if G_σ is a simple matrix, the eigenvalues of $\tilde{G}_\sigma = G_\sigma + \Delta A_\sigma$ lie in at least one of the disks, $\{z : |z - \lambda_i(G_\sigma)| \le r\}$, where $r = \inf \mathrm{K}(P_\sigma) \| \Delta A \|_2$ and P_σ diagonalizes G_σ . Fig. 5 presents the range of the eigenvalues of $\tilde{G}_\sigma$, where the upper bar denotes the complex conjugate. From this figure, we can obtain

$$\mathrm{Re}[\lambda_i(\tilde{G}_\sigma)] \le \mathrm{Re}[\lambda_i(G_\sigma)] + r$$
$$< -\alpha_\sigma + \inf \mathrm{K}(P_\sigma) \| \Delta A_\sigma \|_2 .$$

Therefore,

$$-\alpha_\sigma + \inf \mathrm{K}(P_\sigma) \| \Delta A_\sigma \|_2 < -1/2$$
$$\Rightarrow \mathrm{Re}[\lambda_i(\tilde{G}_\sigma)] < -1/2. \qquad (5)$$

From the above two results (4) and (5),

$$\| \Delta A_\sigma \|_2 < \min\left\{ \frac{\alpha_\sigma - 1/2}{\inf \mathrm{K}(P_\sigma)}, \frac{\beta_\sigma - 1}{2} \right\}$$

$$\text{implies} \begin{cases} \mathrm{Re}\left[\lambda_i(\tilde{G}_\sigma)\right] < -\dfrac{1}{2} \\ \lambda_i(\tilde{G}_\sigma + \tilde{G}'_\sigma) < -1 \end{cases} .$$

By Theorem 3.2, the switched system with model uncertainty which is bounded as above is globally asymptotically stable. □

Remark 5.1: It is hard to find the value of $\inf \mathrm{K}(P_\sigma)$. However, because $\inf \mathrm{K}(P_\sigma) \le \mathrm{K}(P_\sigma)$ implies

$$\frac{\alpha_\sigma - 1/2}{\inf \mathrm{K}(P_\sigma)} \ge \frac{\alpha_\sigma - 1/2}{\mathrm{K}(P_\sigma)},$$

we can choose the bound of $\| \Delta A_\sigma \|_2$ by replacing $\inf \mathrm{K}(P_\sigma)$ with $\mathrm{K}(P_\sigma)$. In this case, there is the disadvantage that the estimated tolerance of uncertainty decreases.

Remark 5.2: In the above theorem, the uncertainty bounds are proportional to the margins of criterion. Hence, if the nominal switched linear system is stabilized by larger α_σ and β_σ , the uncertainty bounds also increase.

6. ILLUSTRATIVE EXAMPLES

A simple example verifying the proposed method is given in this section. Consider the following a switched linear system with control input.

$$\begin{cases} \dot{x} = \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, & x \in \Omega_1 = \{x \mid x_1 x_2 \ge 0\} \\ \dot{x} = \begin{bmatrix} 2 & 1 \\ 3 & -4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, & x \in \Omega_2 = \{x \mid x_1 x_2 < 0\} \end{cases},$$

where each pair of the system matrix and the input matrix (here, vector) is controllable. The control inputs for each subsystem are defined as follows

$$u = \begin{cases} F_1 x, & x \in \Omega_1 \\ F_2 x, & x \in \Omega_2. \end{cases}$$

By the procedure of pole assignment in Section 6, we can obtain an appropriate state feedback gain for each subsystem as follows.

Step 1: Suppose that the eigenvalues of the closed-loop system corresponding to region Ω_1 lie in a disk with center -2 and radius 1 e.g. the eigenvalues are located at $\{-2 \pm i\}$.

Step 2: The feedback gain for this pole assignment is calculated by solving algebraic Riccati equation.

Step 3: Check the disk stability criterion given in Theorem 4.4 and Remark 4.1 as follows

$$1 + \mathrm{K}(P) < \frac{-2c - 1}{l},$$

where P diagonalizes G into D(i.e. $GP=PD$). Here, matrix P whose columns consist of the eigenvectors of G is used. In this case, the criterion is not satisfied because $\mathrm{K}(P) = 2.6180$, $c = -2$, and $l = 1$.

Step 4: Try with another combination of less $c = -2.5$ and $l = 0.5$ e.g. the eigenvalues of G are located at $\{-2.5 \pm 0.5i\}$. Then the criterion is satisfied because $\mathrm{K}(P) = 6.8541$.

The resulting feedback gain is $F_1 = [-1 \ -4.5]$. It should be noted that the criterion provides a guideline for selecting the next combination if the current trial fails. In the same manner, the feedback gain for the subsystem corresponding to region Ω_2 can be evaluated as F_2= [-4 –1.6667] when the eigenvalues of the system lie in a disk with center -3 and radius 1, that is, the eigenvalues are located at $\{-3 \pm i\}$. Then the resulting closed-loop systems,

$$\begin{cases} \dot{x} = \begin{bmatrix} -4 & -2.5 \\ 1 & -1 \end{bmatrix} x = G_1 x, & x \in \Omega_1 \\ \dot{x} = \begin{bmatrix} -2 & -0.6667 \\ 3 & -4 \end{bmatrix} x = G_2 x, & x \in \Omega_2 \end{cases} \qquad (6)$$

satisfy the requirements of Theorem 3.2 because

$$\lambda(G_1)=\{-2.5\pm 0.5i\},$$
$$\lambda(G_1+G_1')=\{-1.6459,\ -8.3541\},$$
$$\lambda(G_2)=\{-3\pm \mathrm{i}\},$$
$$\lambda(G_2+G_2')=\{-2.9268,\ -9.0732\}.$$

The numerical results are depicted in Figs. 6 and 7, where the solid and the dotted line represent x_1 and x_2 with initial point (10,-10), respectively. Fig. 8 shows the value of Lyapunov functions which are non-increasing at each switching instant. The switching signal occurs when the trajectory of the solution is passing from one region to another region. In this case, switching occurs at 0.24, 1.03, and 1.82 second and the Lyapunov functions are generated recursively as follows

$$V_0=x'M_0x,\ M_0\triangleq\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}\ \text{on}\ \Omega_2,$$

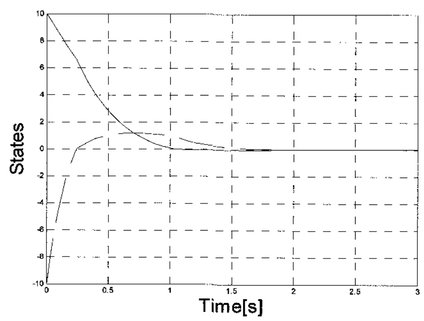


Fig. 6. Trajectories of states.

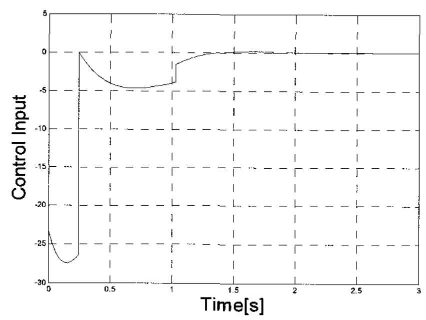


Fig. 7. Control input.

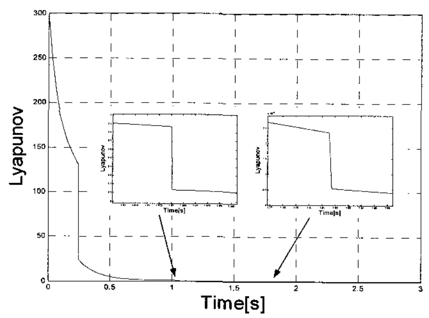


Fig. 8. Lyapunov function.

$$V_1=x'M_1x,\ M_1=\begin{bmatrix}0.4615 & -0.1385\\ -0.1385 & 0.8615\end{bmatrix}\ \text{on}\ \Omega_1,$$

where M_1 is the solution of $G'_1M_1+M_1G_1+M_0=0$,

$$V_2=x'M_2x,\ M_2=\begin{bmatrix}0.1093 & 0.0181\\ 0.0181 & 0.1213\end{bmatrix}\ \text{on}\ \Omega_2,$$

where M_2 is the solution of $G'_2M_2+M_2G_2+M_1=0$,

$$V_3=x'M_3x,\ M_3=\begin{bmatrix}0.0229 & -0.0147\\ -0.0147 & 0.0459\end{bmatrix}\ \text{on}\ \Omega_1,$$

where M_3 is the solution of $G'_1M_3+M_3G_1+M_2=0$.

The origin of the system (6) is asymptotically stable under an arbitrary switching signal. This means the stability of the system should be conserved regardless of an arbitrary partition of switching regions. Figs. 9 and 10 present the states and the control input with initial point (-10,10) when the partitioning of switching regions is $\Omega_1=\{x\,|\,(x_1-x_2)(x_1+x_2)<0\}$ and $\Omega_2=\{x\,|\,(x_1-x_2)(x_1+x_2)\geq 0\}$.

Figs. 11 and 12 present the states and the control input with initial point (10,-10) when the partitioning of switching regions is $\Omega_1=\{x\,|\,x^2_1+x^2_2\leq 1\}$ and $\Omega_2=\{x\,|\,x^2_1+x^2_2>1\}$.

Finally, the switched system (6) is robust against additive uncertainty on each subsystem matrix as mentioned in Section 5. The approximated limits of the robustness of the system are

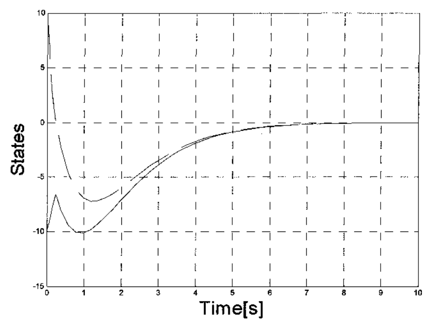


Fig. 9. Trajectories of states.

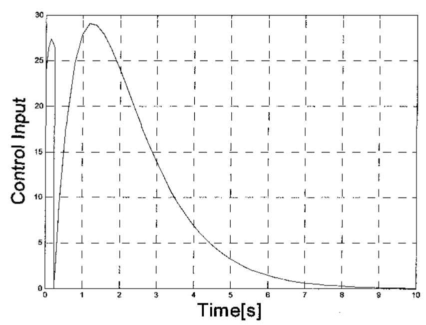


Fig. 10. Control input.

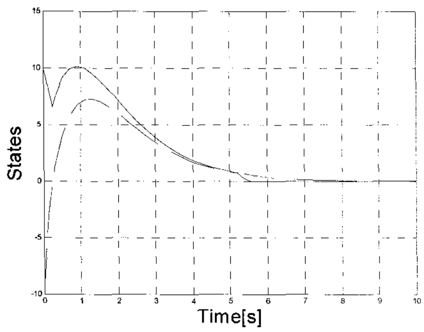


Fig. 11. Trajectories of states.

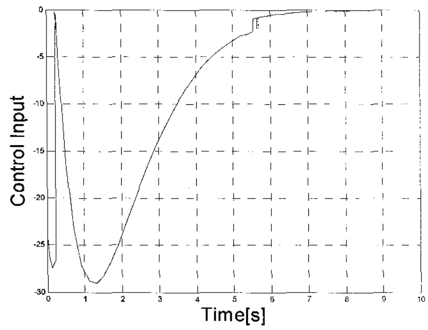


Fig. 12. Control input.

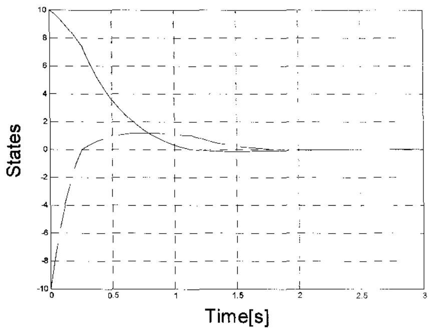


Fig. 13. Trajectory of states.

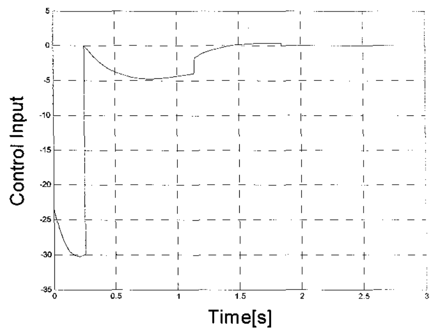


Fig. 14. Control input.

$$\| \Delta A_1 \|_2 < \min \left\{ \frac{\alpha_1 - 1/2}{\inf \mathrm{K}(P_1)}, \frac{\beta_1 - 1}{2} \right\} \approx 0.2918,$$

$$\| \Delta A_2 \|_2 < \min \left\{ \frac{\alpha_2 - 1/2}{\inf \mathrm{K}(P_2)}, \frac{\beta_2 - 1}{2} \right\} \approx 0.7419.$$

Hence, the stability of (6) with uncertainty is conserved as long as each additive uncertainty does not exceed the above limitations. The numerical results are depicted in Figs. 13 and 14.

The simulation circumstances are equivalent to the first case with region partition $\Omega_1 = \{x \mid x_1x_2 \geq 0\}$ and $\Omega_2 = \{x \mid x_1x_2 < 0\}$. In this case, there are uncertainties on each subsystem matrix as follows

$$\Delta A_1 = \begin{bmatrix} 0.21 & 0.17 \\ -0.2 & 0.15 \end{bmatrix}, \quad \Delta A_2 = \begin{bmatrix} 0.43 & -0.32 \\ -0.21 & 0.51 \end{bmatrix},$$

where the uncertainties are restricted in satisfactory norm bounds because $\| \Delta A_1 \|_2 = 0.2917$ and $\| \Delta A_1 \|_2 = 0.7412$.

7. CONCLUSIONS

We proposed a new stability criterion and provided a guideline for pole assignments for a switched linear system. Man's theorem was employed to generate Lyapunov functions recursively. The stability of the given system was proven by the fact that such Lyapunov functions were non-increasing at every switching instant. Thus, the proposed stability criterion is applicable to each subsystem independently without the need to consider the overall system.

Furthermore, the controller can be easily designed by the disk stability criterion using geometric relations between eigenvalues of each subsystem matrix. The proposed methods provide a systematic and simple pole assignment approach for a switched linear system

APPENDICES

A. Ostrowski inequality

Let X and Y be symmetric matrices of dimension n. Then the eigenvalues satisfy

$$\lambda_i(X+Y) = \lambda_i(X) + h,$$

where $\lambda_n(Y) \leq h \leq \lambda_1(Y)$ and the eigenvalues are ordered as $\lambda_1 > \lambda_2 > \cdots > \lambda_n$.

B. Weyl inequality

Let X and Y be symmetric matrices of dimension n. Then, the eigenvalues of a matrix sum satisfy

$$\lambda_{i+j-n}(X+Y) \geq \lambda_i(X) + \lambda_j(Y), \ i+j \geq n+1,$$
$$\lambda_{i+j-1}(X+Y) \leq \lambda_i(X) + \lambda_j(Y), \ i+j \leq n+1,$$

where $i, j = 1, 2, \cdots, n$.

C. Fan-Hoffman's inequality

For any square $n \times n$ matrix X the following holds

$$\lambda_i\left(\frac{X+X'}{2}\right) \leq \eta_i(X), \ 1 \leq i \leq n,$$

where λ and η represent the eigenvalues and singular values, respectively.

REFERENCES

[1] A. Agrachev and D. Liberzon, "Lie algebra conditions for exponential stability of switched systems," *Proc. of the 38th CDC*, pp. 2679-2684, 1999.

[2] A. Agrachev and D. Liberzon, "Lie algebraic stability criteria for switched systems," *SIAM Journal of Optimal Control*, vol. 40, no. 1, pp. 253-269, 2001.

[3] S. Barnett and F. Man, "Comments on a theorem on the Lyapunov matrix equation," *IEEE Trans. on Automatic Control*, vol. 15, no. 2, pp. 279-280, April 1970.

[4] J. Daafous, P. Riedinger, and C. Iung, "Stability analysis and control synthesis for switched system: A switched Lyapunov function approach," *IEEE Trans. on Automatic Control*, vol. 47, no. 11, pp. 1883-1887, 2002.

[5] L. Fang, H. Lin, and P. J. Antsaklis, "Stabilization and performance analysis for a class of switched systems," *Proc, of 43rd CDC*, 2004.

[6] Z. Gajic, *Lyapunov Matrix Equation in System Stability and Control*, Academic Press, pp. 245-246, 1995.

[7] J. Hespanha, "Extending LaSalle's invariance principle to switched linear systems," *Proc. of the 40th CDC*, pp. 2496-2501, 2001.

[8] J. Hespanha, "Stability of switched systems with average dwell-time," *Proc. of the 38th CDC*, pp. 2655-2660, 1999.

[9] J. Hespanha, "Control Using Logic and Switching," Tutorial on Supervisory Control, *Lecture Notes for the Tutorial Workshop of the 40th CDC*, 2001.

[10] P. Lancaster, and M. Tismenetsky, *The Theory of matrices*, Academic Press, pp.387-390, 1985.

[11] D. Liberzon, *Switching in Systems and Control*, Birkhauser, pp. 10-12, 2003.

[12] D. Liberzon, and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control System Magazine*, vol. 19, no. 5, pp. 59-70, 1999.

[13] D. Liberzon, J. P. Hespanha, and A. S. Morse, "Stability of switched systems : A lie algebraic condition," *Systems & Control Letters*, vol. 37, 1999.

[14] F. Man, "A theorem on the Lyapunov matrix equation," *IEEE Trans. on Automatic Control*, vol. 14, no. 3, pp. 306-306, June 1969.

[15] Y. Mori, T. Mori, and Y. Kuroe, "A solution of the common Lyapunov function problem for continuous time systems," *Proc. of the 36th CDC*, pp. 3530-3531, 1997.

[16] A. S. Morse, "Supervisory control of families of linear set-point controllers - part 1: Exact Matching," *IEEE Trans. on Automatic Control*, vol. 41, no. 10, pp. 1413-1431, 1996.

[17] A. S. Morse, "Towards a unified theory of parameter adaptive control - part II: Certainty equivalence and implicit tuning," *IEEE Trans. on Automatic Control*, vol. 37, no. 1, pp. 15-29, January 1992.

[18] T. Ooba, and Y. Funahashi, "On a common quadratic Lyapunov function for widely distant systems," *IEEE Trans. on Automatic Control*, vol. 42, no. 12, pp. 1697-1699, December 1997.

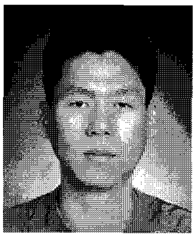
[19] S. Pettersson, *Analysis and Design of Hybrid Systems,* Thesis for the degree of doctor of philosophy, Chalmers Univ. of Technology, Göteborg Sweden, 1999.

[20] A. V. Savkin and R. J. Evans, *Hybrid Dynamical Systems*, Birkhauser, 2002.

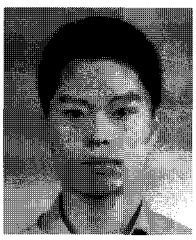
[21] R. P. Stanley, "Recent progress in algebraic combinatorics," *Bulletin of the American Mathematical Society*, vol. 40, no. 1, pp. 55-68, 2002.

[22] Z. Sun, "Stabilizability and Insensitivity of Switched Linear Systems," *IEEE Trans. Automatic Control*, vol. 49, no. 7, pp. 1133-1137, July 2004.

[23] G. Xie, and L. Wang, "New results on stabilizability of switched linear systems," *Proc. of the 42nd CDC*, 2003.



Dong-Hae Yeom received the B.S. degree in Electronic Engineering from Dong-a University in 1998, the M.S. degree in Electronics and Computer Engineering form Hanyang University in 2001. His research interests include nonlinear systems, and switching control.



Ki-Hong Im received the B.S. and M.S. degree in Mechanical Engineering from Pusan National University in 1999, 2001, respectively. His research interests include nonlinear systems, and adaptive control.

Jin-Young Choi received the B.S. M.S. and Ph.D. degree in Electrical Engineering and Computer Science from Seoul National University in 1982, 1884, 1993, respectively. His research interests include intelligent systems, adaptive control.