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GROUP ACTIONS IN A REGULAR RING

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ABSTRACT. Let R be a ring with identity, X the set of all nonzero, nonunits of R and G the group of all units of R. We will consider two group actions on X by G, the regular action and the conjugate action. In this paper, by investigating two group actions we can have some results as follows: First, if G is a finitely generated abelian group, then the orbit $O(x)$ under the regular action on X by G is finite for all nilpotents $x \in X$. Secondly, if F is a field in which 2 is a unit and $F \setminus \{0\}$ is a finitley generated abelian group, then F is finite. Finally, if G in a unit-regular ring R is a torsion group and 2 is a unit in R, then the conjugate action on X by G is trivial if and only if G is abelian if and only if R is commutative.

1. Introduction and basic definitions

Let R be a ring with identity 1, X the set of all nonzero, nonunits of R and G the group of all units of R. In this paper, we will consider two group actions of G on X. We call the action, $((g,x) \longrightarrow gx)$ from $G \times X$ to X, regular action and the action, $((g,x) \longrightarrow gxg^{-1})$ from $G \times X$ to X, conjugate action. If $\phi : G \times X \longrightarrow X$ is one of the above actions, then for each $x \in X$, we define the *orbit* of x by $O(x) = \{\phi(g,x) : g \in G\}$. We also define the *stablizer* of x by $\mathrm{Stab}(x) = \{g \in G : \phi(g,x) = x\}$. Recall that G is *transitive* on X (or G acts transitively on X) if there is an $x \in X$ with $O(x) = X$ and the group action on X by G is *trivial* if $O(x) = \{x\}$ for all $x \in X$.

We define the *index* of a nilpotent $x \in R$ by the positive integer n such that $x^n = 0$ and $x^{n-1} \neq 0$ and denote it by $\mathrm{ind}(x)$. In particular, the additive zero 0 in R is nilpotent of index 1. We define the *index*

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of R by the supremum of the indices of all nilpotents of R and denote it by $\operatorname{ind}(R)$. If $\operatorname{ind}(R)$ is finite, then we say that R has a *bounded index*. A ring R is *von-Newmann regular* (or simply *regular*) (resp. *unit-regular*) provided that for any $a \in R$ there exists an element $r \in R$ (resp. $u \in G$) such that $a = ara$ (resp. $a = aua$). A ring R is *strongly regular* provided that for any $a \in R$ there exists an element $r \in R$ such that $a = ra^2$. Also a regular ring R is *abelian* provided all idempotents in R is central. It is known in [1] that R is abelian regular ring if and only if R is strongly regular and that an abelian regular ring is unit-regular. It is also known by [1, Corollary 7.1] that if R is regular of bounded index, then R is unit-regular.

Through this paper, unless stated otherwise, R is a ring with identity 1, G is the group of all units of R and X is the set of all nonzero, nonunits in R. Also for each $x \in X$, $O(x)$ is considered as an orbit of x under the given group action. Let $N(R)$ be the set of all nilpotents of R. In [3], If R is a ring such that X is a finite union of orbits under the regular action, then the Jacobson radical J is a nil ideal of R, and so $J \subseteq N(R)$.

2. Regular action in regular rings

The following theorem has been proved in [3]:

THEOREM 2.1. *Let R be a ring such that X is a finite union of orbits under the regular action on X by G. Then X is the set of all right zero-divisors of R. Moreover, if X is a nonempty finite set, then R is a finite ring.*

LEMMA 2.2. *Let R be a ring such that X is a union of n orbits under the regular action on X by G. Then $n + 1 \geq \operatorname{ind}(R)$.*

Proof. Let $x \in R$ be any nilpotent such that $\operatorname{ind}(x) = m$ $(m \geq 2)$. Consider $O(x^i)$ and $O(x^j)$, where $i > j$. Then $O(x^i) \cap O(x^j) = \emptyset$. Indeed, if $O(x^i) \cap O(x^j) \neq \emptyset$, then $x^i = gx^j$ for some $g \in G$. Thus $0 = x^m = x^i x^{m-i} = gx^j x^{m-i} = gx^{m-i+j}$, and so $x^{m-i+j} = 0$, a contradiction. Hence $O(x^i) \cap O(x^j) = \emptyset$, where $i > j$, which implies that $O(x)$, $O(x^2)$, $\ldots$, $O(x^{m-1})$ are disjoint orbits. Therefore, $n \geq m-1$, and then $n+1 \geq \operatorname{ind}(R)$.

In general, the converse of Lemma 3.2 is not true by taking an example of a ring $R = \mathbb{Z}_4 \oplus \mathbb{Z}$. Indeed, $\operatorname{ind}(R) = 2$ but X is not a union of finite number of orbits under the regular action on X by G.

COROLLARY 2.3. *Let R be a ring such that X is a union of a finite number of orbits under the regular action on X by G. Then R is regular if and only if R is unit-regular.*

Proof. It follows from Lemma 2.2 and [1, Corollary 7.11].

PROPOSITION 2.4. *Let R be a ring such that $X \neq \emptyset$. If the regular action on X by G is transitive, then for all $y, z \in X$, $yz = 0$.*

Proof. By Lemma 2.2, $2 \geq \text{ind}(R)$. Since $X \neq \emptyset$, $2 = \text{ind}(R)$ and so there exists $x \in X$ with $x^2 = 0$. Since the regular action on X by G is transitive, $X = O(x)$. For all $y, z \in X$, $y = gx, z = hx$ for some $g, h \in G$, and then $yz = (gx)(hx) = g(xh)x = g(kx)x = (gk)x^2 = 0$ for some $k \in G$.

REMARK 1. (1) By Propositon 2.4, we can observe that if R is a ring such that $X \neq \emptyset$ and the regular action on X by G is transitive, then $X \cup \{0\} = N(R)$ and for all $y, z \in X$, $y + z \in X$. Since for all $y, z \in X$, $yz = 0$, R is a local ring, i.e., $J = X \cup \{0\}$.
(2) In general, we can observe that if R is a ring such that X is a union of n orbits under the regular action on X by G, then there exists $x \in X$ with $\text{ind}(x) = n + 1$ if and only if R is a local ring.
(3) In any regular ring R with $X \neq \emptyset$, there is no transitive regular action on X by G.

EXAMPLE 1. Let p be any prime number, k be any positive integer and let $\mathbb{Z}_{p^k} = \{0, 1, \ldots, p^k - 1\}$ be the ring of integers modulo p^k. Then $X = O(p) \cup O(p^2) \cdots \cup O(p^{k-1}) = J \setminus \{0\}$ is a union of $k - 1$ orbits $O(p), \ldots, O(p^{k-1})$ under the regular action, and so $\mathbb{Z}_{p^k}$ is a local ring.

EXAMPLE 2. Let p be any prime number and let $\mathbb{Z}_{p^2} = \{0, 1, \ldots, p^2 - 1\}$ be the ring of integers modulo p^2. Then $N(\mathbb{Z}_{p^2}) = \{0, p, 2p, \ldots, (p-1)p\} = X \cup \{0\}$ and so in $\mathbb{Z}_{p^2}$ there is a transitive regular action on X by G, i.e., $X = O(p)$.

PROPOSITION 2.5. *Let R be a ring. If $g \in G$ is of a finite order such that $O(1-g) = \{1-g\}$ under the regular action on X by G, then $\text{ind}(1-g) = 2$ and $g + g^{-1} = 2$.*

Proof. Since $g \in G$ is of a finite order, $1 - g \in X$ for all $g(\neq 1) \in G$. If $O(1-g) = \{1-g\}$ for some $g \in G$ under the regular action on X by G, then $g(1-g) = 1 - g$, and so $(1-g)^2 = 0$. Hence $\text{ind}(1-g) = 2$, and then $g + g^{-1} = 2$.

EXAMPLE 3. Let n be any positive integer and let $\mathbb{Z}_{4n} = \{0, 1, \ldots, 4n-1\}$ be the ring of integers modulo $4n$. Then under the regular action on X by G, $O(2n) = \{2n\}$ and $1-2n \in G$ is an involution. Thus $(1-2n)+(1-2n)^{-1} = (1-2n)+(1-2n) = 2$.

PROPOSITION 2.6. *A ring R is strongly regular if and only if $O(x) = O(x^2)$ for all $x \in X$ under the regular action on X by G.*

Proof. Suppose that R is strongly-regular and let $x \in X$ be arbitrary. Since strongly regular ring is unit-regular, there exists $g \in G$ such that $x = xgx$. Since strongly regular ring is also abelian-regular, the idempotent gx is central, and so $x = x(gx) = (gx)x = gx^2$. Hence for all $x \in X$, $O(x) = O(x^2)$ under the regular action on X by G. Conversely, suppose that $O(x) = O(x^2)$ for all $x \in X$ under the regular action on X by G. Let $a \in R$ be an arbitrary element. If $a \in G$, then $a^{-1}a^2 = a$. If $a \in X$, then by assumption, $O(a^2) = O(a)$ and so $ga^2 = a$ for some $g \in G$. Hence R is strongly regular.

PROPOSITION 2.7. *Let R be a ring whose characteristic is not 2. If $x(\neq 0) \in N(R)$ with $\mathrm{ind}(x) = n$ and $1 + x \in G$ is of finite order k, then $kx^{n-1} = 0$ and $2k^i x^{n-i-1} = 0$ for all $i = 1, \ldots, n-2$.*

Proof. Since $x(\neq 0) \in N(R)$ with $\mathrm{ind}(x) = n$, $x^n = 0 \neq x^{n-1}$. Since $1 + x \in G$ has order k, $1 = (1+x)^k = 1 + kx + \binom{k}{2}x^2 + \binom{k}{3}x^3 + \cdots + x^k$. Thus we have an equation $kx + \binom{k}{2}x^2 + \binom{k}{3}x^3 + \cdots + x^k = 0$–(1). By multiplying x^{n-2} to both sides of (1), we can have $kx^{n-1} = 0$ since $x^n = 0 \neq x^{n-1}$. Next, by multiplying $2kx^{n-3}$ to both sides of (1), we can have $2kx^{n-2} = 0$ since $x^n = 0$ and $kx^{n-1} = 0$. By mathematical induction on n and by multiplying $2k^{i-1}x^{n-i-2}$ to both sides of (1), we can have $2k^i x^{n-i-1} = 0$ for all $i = 1, \ldots, n-2$.

EXAMPLE 4. Let $\mathbb{Z}_{288} = \{0, 1, \ldots, 287\}$ be the ring of integers modulo 288. Let $6 \in N(\mathbb{Z}_{288})$ be an element of $\mathrm{ind}(6) = 5$. Note that the order of 7 ($= 1 + 6$) is 12. Then we can have $12 \cdot 6^4 = 2 \cdot 12 \cdot 6^3 = 2 \cdot 12^2 \cdot 6^2 = 2 \cdot 12^3 \cdot 6 = 0$.

REMARK 2. (1) Let R be a ring whose characteristic is 2. If $x(\neq 0) \in N(R)$ with $\mathrm{ind}(x) = n$ and $1 + x \in G$ is of finite order k, then k is even and $x^k = 0$. Indeed, if k is odd, then we have $x + x^{k-1} + x^k = 0$ from an equation $(1+x)^k = 1 + x + x^{k-1} + x^k = 1$, and so $x^{n-1} = 0$, a contradiction. Hence k is even and then we have $x^k = 0$ from an equation $(1+x)^k = 1 + x^k = 1$.

(2) Let R be a ring with the characteristic 0 in which there is no left (or right) zero-divisors and let $M_m(R)$ $(m \geq 2)$ be a full matrix ring of $m \times m$ over R. Then if $x(\neq 0) \in M_m(R)$ is nilpotent, then the order of $1 + x$ is not finite. In fact, assume that there exists a nilpotent $x(\neq 0) \in M_m(R)$ such that $\text{ind}(x) = n$ and $1 + x \in G$ is of finite order k. Since $x^{n-1} \neq 0$ and $kx^{n-1} = 0$ by Proposition 2.7, there exists $a(\neq 0) \in R$ such that $ka = 0$. Since $a \neq 0$ and R has no left (or right) zero divisors, we have $k \cdot 1 = 0$, which is a contradiction to the assumption that the characteristic of R is 0.

COROLLARY 2.8. *Let R be a ring such that $X \neq \emptyset$ and the regular action on X by G is transitive. If the order of $1 + x \in G$ is finite for some $x \in X$, then the order of $1 + y \in G$ is equal to the order of $1 + x$ for all $y \in X$, i.e., $1 + J$ is a torsion group.*

Proof. Let k be the order of $1 + x$. Since the regular action on X by G is transitive, $X \cup \{0\} = J$, $y^2 = 0$ for all $y \in X$ and $y = gx$ for some $g \in G$ by Proposition 2.4. Since the order of $1 + x$ is k and $x^2 = 0$, $kx = 0$ by Proposition 2.7. Note that $kx = 0$ if and only if $ky = 0$ for all $y \in X$ and also the order of $1 + x$ is equal to the one of $1 + y$ for all $y \in X$. Hence $1 + J$ is a torsion group.

EXAMPLE 5. Let p be any prime number. By Example 2, in a ring $\mathbb{Z}_{p^2}$ there is a transitive regular action on X by G. Note that the order of $1 + y$ is p for all $y \in X$.

THEOREM 2.9. *Let R be a ring in which G is a finitely generated abelian group. If $x(\neq 0) \in N(R)$, then the orbit $O(x)$ under the regular action on X by G is finite.*

Proof. If $O(x) = \{x\}$ or $G = \{1\}$, then $O(x) = \{x\}$, and so $O(x)$ is finite. Thus suppose that $O(x) \neq \{x\}$ and $G \neq \{1\}$. Then $|O(x)| > 1$ and $\text{Stab}(x)$ is a proper subgroup of G. Let $H = \text{Stab}(x)$ and let $S = \{a_1, a_2, \ldots, a_k\}$ be the set of generators of G. Since $x(\neq 0) \in N(R)$, $x^n = 0$ and $x^{n-1} \neq 0$ for some positive integer n. Thus $(1 + x^{n-1})x = x$ implies that $1 + x^{n-1} \in H$ and so $H \neq \{1\}$. Since H is a proper subgroup of G, H is generated by $\{a_1^{s_1}, a_2^{s_2}, \ldots, a_k^{s_k}\}$ for some nonnegative integers $s_1, s_2, \ldots, s_k$ but not all $s_i = 1$. Let $g = \prod_{i=1}^{k} a_i^{t_i} \in G$ be arbitrary. Then $gx = \left(\prod_{i=1}^{k} a_i^{t_i}\right) x = \left(\prod_{s_i \geq 2} a_i^{t_i}\right) x$. For each $s_i \geq 2$, by the division algorithm for $\mathbb{Z}$, $t_i = r_i + q_i s_i$ for some $r_i, q_i \in \mathbb{Z}$ where $s_i - 1 \geq r_i \geq 0$. Thus for all $g \in G$, $gx = \left(\prod_{s_i \geq 2} a_i^{t_i}\right) x = \left(\prod_{s_i \geq 2} a_i^{r_i}\right) x$, and so $O(x)$ is finite.

Corollary 2.10. *Let R be a ring such that $X \neq \emptyset$ and the regular action on X by G is transitive. If G is a finitely generated abelian group, then R is finite. In addition, if 2 is a unit in G, then R is commutative.*

Proof. R is finite by Theorem 2.1 and Theorem 2.9. Also if 2 is a unit in G, then R is commutative by [2, Theorem 2.11].

Theorem 2.11. *Let R be a ring such that G is a finitely generated abelian group and 2 is a unit in G. If $e \in X$ is idempotent, then $O(e)$ under the regular action is finite.*

Proof. The proof is similar to the one of Theorem 2.9. If $O(e) = O(e^2) = \{e\}$ or $G = \{1\}$ for idempotent $e \in X$, then $O(e) = \{e\}$, and so $O(e)$ is finite. Suppose that $O(e) \neq \{e\}$ and $G \neq \{1\}$. Then $|O(e)| > 1$ and Stab(e) is a proper subgroup of G. Let $H =$ Stab(e) and let $S = \{a_1, a_2, \dots, a_k\}$ be the set of generators of G. Since $e \in X$ is idempotent and 2 is a unit in G, $2e - 1 (\neq 1) \in G$. Thus $(2e-1)e = e$ implies that $2e - 1 \in H$ and so $H \neq \{1\}$. Since H is a proper subgroup of G, H is generated by $\{a_1^{s_1}, a_2^{s_2}, \dots, a_k^{s_k}\}$ for some nonnegative integers $s_1, s_2, \dots, s_k \geq 0$ but not all $s_i = 1$. Let $g = \prod_{i=1}^k a_i^{t_i} \in G$ be arbitrary. Then $ge = \left(\prod_{i=1}^k a_i^{t_i}\right) e = \left(\prod_{s_i \geq 2} a_i^{t_i}\right) e$. For each $s_i \geq 2$, by the division algorithm for $\mathbb{Z}$, $t_i = r_i + q_i s_i$ for some $r_i, q_i \in \mathbb{Z}$, where $s_i - 1 \geq r_i \geq 0$. Thus for all $g \in G$, $gx = \left(\prod_{s_i \geq 2} a_i^{t_i}\right) e = \left(\prod_{s_i \geq 2} a_i^{r_i}\right) e$, and so $O(e)$ is finite.

Corollary 2.12. *Let R be a unit-regular ring. If G is a finitely generated abelian group and 2 is a unit in G, then every orbit under the regular action is a finite set.*

Proof. By [4, Lemma 2.3], every orbit is $O(e)$ for some idempotent $e \in X$ and so is $O(e)$ is a finite set by Theorem 2.11.

Corollary 2.13. *Let R be a regular ring such that $X \neq \emptyset$ and G is a finitely generated abelian group and 2 is a unit in G. Then the following are equivalent:*

(1) *X is a union of finite number of orbits under the regular action on X by G;*
(2) *X is finite;*
(3) *R is finite commutative.*

Proof. (1) $\Rightarrow$ (2). Suppose that X is a union of finite number of orbits under the regular action on X by G. Since $\mathrm{ind}(R)$ is finite, a regular ring R is unit-regular by Corollary 2.3. Since G is abelian group, R is commutative by [4, Theorem 3.2]. By Corollary 2.12, every orbit under the regular action on X by G is finite. Since there exists a finite number of orbits under the regular action on X by G, X is finite.
(2) $\Rightarrow$ (3). It follows from Theorem 2.1.
(3) $\Rightarrow$ (1). It is clear.

It is well-known in the field theory that if F is a finite field, then $F \setminus \{0\}$ is a cyclic group. But in [4, Theorem 3.7] the converse could be true in case that 2 is a unit in F. In general, we have the following Theorem:

THEOREM 2.14. *Let F be a field in which 2 is a unit. If $F \setminus \{0\}$ is a finitely generated abelian group, then F is a finite field.*

Proof. Consider a ring $R = F \times F$. Then R is a unit-regular ring and (2, 2) is a unit in R. Since $F \setminus \{0\}$ is a finitely generated abelian group, the group of units of R is a finitely generated abelian group. Take an idempotent $(1,0) \in R$. Then the orbit $O((1,0))$ is equal to $\{(g,0) : g \in F \setminus \{0\}\}$ under the regular action on X by G. By Theorem 2.11, $O((1,0))$ is a finite set, i.e., $|O((1,0))| = |F \setminus \{0\}|$. Hence F is a finite field.

3. Conjugate action in regular rings

We begin with the following Lemma:

LEMMA 3.1. *Let R be a ring such that G is a torsion group. If the conjugate action on X by G is trivial, then G is abelian.*

Proof. Let $g, h \in G$ be arbitrary. Since the order of g is finite, $1 - g \in X$. Since the conjugate action on X by G is trivial, the orbit $O(1-g) = \{1-g\}$, i.e., $h(1-g)h^{-1} = 1-g$ and so $gh = hg$. Hence G is abelian.

Note that the converse of Lemma 3.1 is not true by the following example:

EXAMPLE 6. Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{Z}_2 \right\}$. Then R is a non-commutative ring but $G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ is an abelian group. The orbit of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is equal to $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} \neq \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}$, and so the conjugate action on X by G is not trivial.

PROPOSITION 3.2. *Let R be a unit-regualr ring such that G is a torsion group and 2 is a unit. Then the following are equivalent:*

(1) *The conjugate action on X by G is trivial;*
(2) *G is abelian;*
(3) *R is commutative.*

Proof. (1) $\Rightarrow$ (2). It follows from the Lemma 3.1.
(2) $\Rightarrow$ (3). It follows from [4, Theorem 3.2].
(3) $\Rightarrow$ (1). It is clear.

PROPOSITION 3.3. *Let R be a ring such that the conjugate action on X by G is transitive. If $N(R) \neq \{0\}$, then for all $y, z \in X$, $yz = 0$.*

Proof. Since $N(R) \neq \{0\}$, there exists a nonzero $x \in N(R)$ such that $\text{ind}(x) = n$ for some positive integer n. Then $O(x^2) = 0$. Indeed, if $O(x^2) \neq 0$, then $O(x) = O(x^2) = X$ since the conjugate action on X by G is transitive, and so $x^2 = gxg^{-1}$ for some $g \in G$, which implies that $\text{ind}(x) < n$, a contradiction. Next, for all $y, z \in X$, $y = hxh^{-1}, z = kxk^{-1}$ for some $h, k \in G$, and so $y^2 = hx^2h^{-1} = 0$, $z^2 = kx^2k^{-1} = 0$. Note that for all $y, z \in X$, $y + z \in X$ and then $0 = (y+z)^2 = yz + zy$, and so $yz = -zy$. Hence $yz = (hxh^{-1})(kxk^{-1}) = hx(h^{-1}kx)k^{-1} = -h(h^{-1}kx)xk^{-1} = -kx^2k^{-1} = 0$.

REMARK 3. By Proposition 3.3, if R is a ring such that the conjugate action on X by G is transitive and $N(R) \neq \{0\}$, then R is a local ring and $J^2 = (0)$.

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