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ON SOME GRONWALL TYPE INEQUALITIES FOR A SYSTEM INTEGRAL EQUATION

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Abstract. In this paper we consider analogous of Gronwall-type inequalities involving iterated integrals in the inequality (1.2) for functions when the function u in the right-hand side of the inequality (1.2) is replaced by the function u^p for some p. These inequalities are effective tools in the study of a system of an integral equation. We also provide some integral inequalities involving iterated integrals.

1. Introduction

Let $u : [\alpha, \alpha + h] \to R$ be a continuous real-valued function satisfying the inequality

$$0 \leq u(t) \leq \int_{\alpha}^{t} [a + bu(s)]\, ds \quad \text{for} \quad t \in [\alpha, \alpha + h],$$

where a, b are nonnegative constants. Then $u(t) \leq ahe^{bh}$ for $t \in [\alpha, \alpha + h]$. This result was proved by T. H. Gronwall[6] in the year 1919, and is the prototype for the study of several integral inequalities of Volterra type, and also for obtaining explicit bounds of the unknown function. The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of various types (see Gronwall[6] and Guiliano[7]). Some applications of this result to the study of stability of the solution of linear and nonlinear differential equations may be found in Bellman[2]. Some

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applications to existence and uniqueness theory of differential equations may be found in Nemyckii-Stepanov[8] and Bihari[3]. During the past few years several authors (see references below and some of the references cited therein) have established several Gronwall type integral inequalities in two or more independent real variables. Of course, such results have application in the theory of partial differential equations and Volterra integral equations.

Bainov and Simeonov obtained the following interesting Gronwall-type inequality, which appear in [1, p.106]: *Let $u(t)$ and $a(t)$ be continuous functions in $J = [\alpha, \beta]$, let $b_k(t, s_1, \ldots, s_k)$ be nonnegative continuous functions for $\alpha \le s_k \le \cdots \le s_1 \le t \le \beta$, and suppose that*

$$u(t) \le a(t)+\sum_{k=1}^{n} \int_{\alpha}^{t} \left(\int_{\alpha}^{s_1} \cdots \left(\int_{\alpha}^{s_{k-1}} b_k(t, s_1, \ldots, s_k) u(s_k)\, ds_k \right) \cdots \right) ds_1$$

for $t \in J$. Then $u(t) \le \mu(t)$, $t \in [\alpha, \beta]$, where $\mu(t)$ is a solution of the equation

(1.1)

$$u(t) = a(t)+\sum_{k=1}^{n} \int_{\alpha}^{t} \left(\int_{\alpha}^{s_1} \cdots \left(\int_{\alpha}^{s_{k-1}} b_k(t, s_1, \ldots, s_k) u(s_k)\, ds_k \right) \cdots \right) ds_1$$

for $t \in J$.

Finding an exact solution of a linear system of an integral equation (1.1) often proves to be very difficult. Therefore such solutions are estimated as, e.g., in the following theorem, which also appear in [1, p.107].

THEOREM 1.1. *Let $u(t)$ and $a(t)$ be nonnegative continuous functions in $J = [\alpha, \beta]$, with $a(t)$ nondecreasing in J, and let $f_i(t, s)$, $i = 1, \ldots, n$, be nonnegative continuous functions for $\alpha \le s \le t \le \beta$ which are nondecreasing in t for fixed $s \in J$. If*

$$\begin{aligned} u(t) \le a(t) + \int_{\alpha}^{t} f_1(t, t_1) \bigg(\int_{\alpha}^{t_1} f_2(t_1, t_2) \cdots \\ \times \left(\int_{\alpha}^{t_{n-1}} f_n(t_{n-1}, t_n) u(t_n)\, dt_n \right) \cdots \bigg) dt_1 \end{aligned} \tag{1.2}$$

for $t \in J$, then $u(t) \le R_1(t, t)$, for $t \in J$, where $R_1(T, t)$ can be succes-

sively determined from the formulas

$$R_n(T,t) = a(T)\exp\bigg(\int_\alpha^t \sum_{i=1}^n f_i(T,s)\,ds\bigg),$$

$$R_k(T,t) = E_k(T,t)\bigg[a(T) + \int_\alpha^t f_k(T,s)\frac{R_{k+1}(T,s)}{E_k(T,s)}\,ds\bigg],$$

$$E_k(T,t) = \exp\left(\int_\alpha^t \bigg[\sum_{i=1}^{k-1} f_i(T,\tau) - f_k(T,\tau)\bigg]\,d\tau\right)$$

for $k = n-1, \dots, 1, \alpha \le t \le T \le \beta$.

In this paper we consider analogous of inequalities involving iterated integrals in the inequality (1.2) for functions when the function u in the right-hand side of the inequality (1.2) is replaced by the function u^p for some p. We also provide some integral inequalities involving iterated integrals.

2. Some nonlinear inequalities

In this section we state and prove some new nonlinear integral inequalities involving iterated integrals. The following two results are known in [1].

LEMMA 2.1. *Let $b(t)$ and $f(t)$ be continuous function for $t \ge \alpha$, let $v(t)$ be a differentiable function for $t \ge \alpha$, and suppose*

$$v'(t) \le b(t)v(t) + f(t), \quad t \ge \alpha$$

and $v(\alpha) \le v_0$. Then, for $t \ge \alpha$,

$$v(t) \le v_0 \exp\bigg(\int_\alpha^t b(s)\,ds\bigg) + \int_\alpha^t f(s)\exp\bigg(\int_s^t b(\tau)\,d\tau\bigg)\,ds.$$

LEMMA 2.2. *Let $v(t)$ be a positive differential function satisfying the inequality*

$$v'(t) \le b(t)v(t) + k(t)v^p(t), \quad t \in J = [\alpha, \beta],$$

where the functions b and k are continuous in J, and $p \geq 0, p \neq 1$, is a constant. Then

$$v(t) \leq \exp\left(\int_\alpha^t b(s)\,ds\right)\left[v^q(\alpha) + q\int_\alpha^t k(s)\exp\left(-q\int_\alpha^s b(\tau)\,d\tau\right)ds\right]^{1/q}$$

for $t \in [\alpha, \beta_1)$, where β_1 is chosen so that the expression between [...] is positive in the subinterval $[\alpha, \beta_1)$.

The following nonlinear inequalities involving iterated integrals of the Gronwall-type holds.

THEOREM 2.3. *Let $u(t)$ be nonnegative continuous function in $J = [\alpha, \beta]$ and $a(t)$ be positive nondecreasing continuous function in J, and let $f_i(t,s)$, $i = 1, \dots, n$, be nonnegative continuous functions for $\alpha \leq s \leq t \leq \beta$ which are nondecreasing in t for fixed $s \in J$. If*

(2.1)
$$u(t) \leq a(t) + \int_\alpha^t f_1(t,t_1)\left(\int_\alpha^{t_1} f_2(t_1,t_2)\cdots \times\left(\int_\alpha^{t_{n-1}} f_n(t_{n-1},t_n)u^p(t_n)\,dt_n\right)\cdots\right)dt_1$$

for $t \in J$, where $p \geq 0, p \neq 1$, is a constant. Then $u(t) \leq Y_1(t,t)$, where $Y_1(T,t)$ can be successively determined from the formulas

$$Y_n(T,t) = \exp\left(\int_\alpha^t \sum_{i=1}^{n-1} f_i(T,s)\,ds\right) \times\left[a^q(T) + q\int_\alpha^t f_n(T,s)\exp\left(-q\int_\alpha^s \sum_{i=1}^{n-1} f_i(T,\tau)\,d\tau\right)ds\right]^{1/q}$$

for $t \in [\alpha, \beta_1)$, with $q = 1 - p$ and β_1 is chosen so that the expression between [...] is positive in the subinterval $[\alpha, \beta_1)$, and

(2.2)
$$Y_k(T,t) = E_k(T,t)\left[a(T) + \int_\alpha^t f_k(T,s)\frac{Y_{k+1}(T,s)}{E_k(T,s)}\,ds\right],$$
$$E_k(T,t) = \exp\left(\int_\alpha^t \left[\sum_{i=1}^{k-1} f_i(T,\tau) - f_k(T,\tau)\right]d\tau\right),$$

for $k = n-1, \dots, 1, \alpha \leq t \leq T \leq \beta$.

Proof. Fix $T \in (\alpha, \beta]$. For $\alpha \le t \le T$ we obtain form (2.1), (2.3)

$$u(t) \le a(T) + \int_\alpha^t f_1(T,t_1) \times \left(\int_\alpha^{t_1} f_2(T,t_2) \cdots \left(\int_\alpha^{t_{n-1}} f_n(T,t_n) u^p(t_n)\, dt_n\right) \cdots\right) dt_1.$$

Now we introduce the functions

$$m_1(t) = a(T) + \int_\alpha^t f_1(T,t_1) \times \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{n-1}} f_n(T,t_n) u^p(t_n)\, dt_n\right) \cdots\right) dt_1,$$

$$m_k(t) = m_{k-1}(t) + \int_\alpha^t f_k(T,t_k) \times \left(\int_\alpha^{t_k} \cdots \left(\int_\alpha^{t_{n-1}} f_n(T,t_n) m_{k-1}^p(t_n)\, dt_n\right) \cdots\right) dt_k,$$

for $t \in [\alpha, T]$ and $k = 2, \dots, n$. Then the inequality (2.3) implies that $m_k(\alpha) = a(T), k = 1, \dots, n$, and

$$u(t) \le m_1(t) \le \cdots \le m_n(t),\ t \in [\alpha, T].$$

Thus, induction with respect to k gives

(2.4)
$$m_k'(t) \le \left(\sum_{i=1}^{k-1} f_i(T,t) - f_k(T,t)\right) m_k(t) + f_k(T,t) m_{k+1}(t),$$

(2.5)
$$m_n'(t) \le \left(\sum_{i=1}^{n-1} f_i(T,t)\right) m_n(t) + f_n(T,t) m_n^p(t)$$

for $t \in [\alpha, T], k = 1, 2, \dots, n-1$. Lemma 2.2 and the inequality (2.5) imply that

$$m_n(t) \le \exp\left(\int_\alpha^t \sum_{i=1}^{n-1} f_i(T,s)\, ds\right) \times \left[a^q(T) + q\int_\alpha^t f_n(T,s) \exp\left(-q\int_\alpha^s \sum_{i=1}^{n-1} f_i(T,\tau)\, d\tau\right) ds\right]^{1/q}$$
$$= Y_n(T,t)$$

for $\alpha \le t \le T \le \beta_1$. Applying the lemma 2.1 to inequality (2.4) for $k = n-1, \dots, 2, 1$, we obtain

$$m_k(t) \le E_k(T,t)\left(a(t) + \int_\alpha^t f_k(T,s)\frac{Y_{k+1}(T,s)}{E_k(T,s)}\,ds\right) = Y_k(T,t),$$

where the function $E_k(T,t)$ is defined by (2.2). Hence we get

$$u(t) \le m_1(t) \le Y_1(T,t)$$

for $\alpha \le t \le T \le \beta$, which implies the result $u(t) \le Y_1(t,t)$ for $T = t$.

In the same manner we can prove the following theorem.

THEOREM 2.4. *Let $u(t)$ and $a(t)$ be nonnegative continuous function in $J = [\alpha, \beta]$, with $a(t)$ be nondecreasing in J, and let $f_i(t,s)$, $i = 1, \dots, n$, be nonnegative continuous functions for $\alpha \le s \le t \le \beta$ which are nondecreasing in t for fixed $s \in J$. If*

$$\begin{aligned} u(t) \le a(t) &+ \int_\alpha^t f_1(t,t_1) \\ &\times \left(\int_\alpha^{t_1} f_2(t_1,t_2)\cdots\left(\int_\alpha^{t_{n-1}} f_n(t_{n-1},t_n)u^p(t_n)\,dt_n\right)\cdots\right)dt_1 \end{aligned}$$

for $t \in J$, where $p > 1$, is a constant. Then $u(t) \le \widehat{R}_1(t,t)$, where $\widehat{R}_1(T,t)$ can be successively determined from the formulas

$$\widehat{R}_n(T,t) = a(T)\left[1 - (p-1)a^{p-1}(T)\int_\alpha^t \left(\sum_{i=1}^n f_i(T,s)\right)ds\right]^{\frac{1}{1-p}}$$

for $t \in [\alpha, \beta_p)$, with

$$\beta_p = \sup\left\{t \in (\alpha,\beta) : (p-1)a^{p-1}(T)\int_\alpha^t \left(\sum_{i=1}^n f_i(T,s)\,ds < 1\right)\right\},$$

and

$$\widehat{R}_k(T,t) = E_k(T,t)\left[a(T) + \int_\alpha^t f_k(T,s)\frac{\widehat{R}_{k+1}(T,s)}{E_k(T,s)}\,ds\right],$$

$$E_k(T,t) = \exp\left(\int_\alpha^t \left[\sum_{i=1}^{k-1} f_i(T,\tau) - f_k(T,\tau)\right]d\tau\right)$$

for $k = n-1, \dots, 1, \alpha \le t \le T \le \beta$.

Proof. We start from the relation (2.3)–(2.4), which can be obtained as in theorem 2.3. Thus, induction with respect to k gives

$$m'_k(t) \leq \left(\sum_{i=1}^{k-1} f_i(T,t) - f_k(T,t)\right) m_k(t) + f_k(T,t)m_{k+1}(t) \tag{2.6}$$

for $t \in [\alpha, T], k = 1, 2, \ldots, n-1$, and

$$\begin{aligned} m'_n(t) &= m'_{n-1}(t) + f_n(T,t)m^p_{n-1}(t) \\ &\leq \sum_{i=1}^{k-2} f_i(T,t)m_{n-1}(t) + f_{n-1}(T,t)m_n(t) + f_n(T,t)m^p_{n-1}(t) \\ &\leq \left(\sum_{i=1}^{n} f_i(T,t)\right) m^p_n(t), \end{aligned}$$

that is,

$$m'_n(t) \leq R(T,t)m_n(t), \tag{2.7}$$

where

$$R(T,t) = \left(\sum_{i=1}^{n} f_i(T,t)\right) m^{p-1}_n(t).$$

Lemma 2.1 and the inequality (2.7) imply that

$$m_n(t) \leq a(T) \exp\left(\int_\alpha^t R(T,s)\, ds\right) \tag{2.8}$$

for $\alpha \leq t \leq T$. From (2.8) we successively obtain

$$\begin{aligned} m^{p-1}_n(t) &\leq a^{p-1}(T) \exp\left(\int_\alpha^t (p-1)R(T,s)\, ds\right), \\ R(T,t) &\leq \left(\sum_{i=1}^{n} f_i(T,t)\right) a^{p-1}(T) \exp\left(\int_\alpha^t (p-1)R(T,s)\, ds\right), \\ Z(T,t) &\leq (p-1)\left(\sum_{i=1}^{n} f_i(T,t)\right) a^{p-1}(T) \exp\left(\int_\alpha^t Z(T,s)\, ds\right), \end{aligned}$$

where $Z(T,t) = (p-1)R(T,t)$. Consequently, we have

$$Z(T,t) \exp\left(-\int_\alpha^t Z(T,s)\, ds\right) \leq (p-1)\left(\sum_{i=1}^{n} f_i(T,t)\right) a^{p-1}(T)$$

or

$$\frac{d}{ds}\left[-\exp\left(-\int_\alpha^t Z(T,\tau)\,d\tau\right)\right] \le (p-1)\left(\sum_{i=1}^n f_i(T,s)\right)a^{p-1}(T).$$

Integrating this from α to t yields

$$1-\exp\left(-\int_\alpha^t Z(T,\tau)\,d\tau\right) \le (p-1)a^{p-1}(T)\int_\alpha^t\left(\sum_{i=1}^n f_i(T,s)\right)ds,$$

from which we conclude that

$$\exp\left(\int_\alpha^t R(T,\tau)\,d\tau\right) \le \left[1-(p-1)a^{p-1}(T)\int_\alpha^t\left(\sum_{i=1}^n f_i(T,s)\right)ds\right]^{\frac{1}{1-p}}.$$

This and (2.8) imply that

$$m_n(t) \le a(T)\left[1-(p-1)a^{p-1}(T)\int_\alpha^t\left(\sum_{i=1}^n f_i(T,s)\right)ds\right]^{\frac{1}{1-p}} = \widehat{R_n}(T,t).$$

Applying the lemma 2.1 to inequality (2.6) for $k = n-1,\dots,2,1$, we obtain

$$m_k(t) \le E_k(T,t)\left(a(t)+\int_\alpha^t f_k(T,s)\frac{\widehat{R}_{k+1}(T,s)}{E_k(T,s)}\,ds\right) = \widehat{R}_k(T,t),$$

where the function $E_k(T,t)$ is defined by (2.2). Hence we get

$$u(t) \le m_1(t) \le \widehat{R}_1(T,t)$$

for $\alpha \le t \le T \le \beta$, which implies the result $u(t) \le \widehat{R}_1(t,t)$ for $T = t$.

THEOREM 2.5. *Let $u(t)$ be nonnegative continuous function in $J = [\alpha,\beta]$, and let $f_i(t,s)$, $i = 1,\dots,n$, be nonnegative continuous functions for $\alpha \le s \le t \le \beta$ which are nondecreasing in t for fixed $s \in J$. If*

$$(2.9)\qquad \begin{aligned} u(t) \le a + \int_\alpha^t f_1(t,t_1)\Bigg(\int_\alpha^{t_1} f_2(t_1,t_2)\cdots \\ \times\left(\int_\alpha^{t_{n-1}} f_n(t_{n-1},t_n)u(t_n)\log u(t_n)\,dt_n\right)\cdots\Bigg)dt_1 \end{aligned}$$

for $t \in J$, *where* $a > 1$, *is a constant. Then* $u(t) \le a^{Q_1(t,t)}$, *where* $Q_1(T,t)$ *can be successively determined from the formulas*

$$Q_n(T,t) = \exp\left(\int_\alpha^t \sum_{i=1}^{n-1} f_i(T,s)\,ds\right) \times \left[a^q(T) + q\int_\alpha^t f_n(T,s)\exp\left(-q\int_\alpha^s \sum_{i=1}^{n-1} f_i(T,\tau)\,d\tau\right)ds\right]^{1/q}$$

for $t \in [\alpha, \beta_1)$, *with* $q = 1-p$ *and* β_1 *is chosen so that the expression between* [...] *is positive in the subinterval* $[\alpha, \beta_1)$, *and*

$$Q_k(T,t) = E_k(T,t)\left[a(T) + \int_\alpha^t f_k(T,s)\frac{Q_{k+1}(T,s)}{E_k(T,s)}\,ds\right],$$

$$E_k(T,t) = \exp\left(\int_\alpha^t \left[\sum_{i=1}^{k-1} f_i(T,\tau) - f_k(T,\tau)\right]d\tau\right)$$

for $k = n-1, \dots, 1, \alpha \le t \le T \le \beta$.

Proof. Fix $T \in (\alpha, \beta]$. For $\alpha \le t \le T$ we obtain form (2.9),

$$u(t) \le a + \int_\alpha^t f_1(T,t_1) \times \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{n-1}} f_n(T,t_n)u(t_n)\log u(t_n)\,dt_n\right)\cdots\right)dt_1.$$

Now we introduce the functions

$$m_1(t) = a + \int_\alpha^t f_1(T,t_1) \times \left(\int_\alpha^{t_1} \cdots \left(\int_\alpha^{t_{n-1}} f_n(T,t_n)u(t_n)\log u(t_n)\,dt_n\right)\cdots\right)dt_1$$

for $t \in [\alpha, T]$. Then $m_1(\alpha) = a$, the the function $m_1(t)$ is nondecreasing in $t \in J$, $u(t) \le m_1(t)$, and

$$m_1'(t) \le B(T,t)m_1(t),$$

where

$$B(T,t) = f_1(T,t)\Big(\int_\alpha^t f_2(T,t_2)\cdots \times \left(\int_\alpha^{t_{n-1}} f_n(T,t_n)\log m_1(t_n)\,dt_n\right)\cdots dt_2\Big).$$

That is,

$$\frac{m_1'(t)}{m_1(t)} \le B(T,t). \tag{2.10}$$

By taking $t = t_1$ in (2,10) and then integrating it from α to $t, t \in [\alpha, \beta]$ we obtain

$$\log m_1(t) \le \log a + \int_\alpha^t B(T,t_1)\,dt_1. \tag{2.11}$$

Now by a suitable application of Theorem 1.1 to (2.11), we get

$$\log m_1(t) \le \log a \cdot Q_1(T,t) = \log a^{Q_1(T,t)}, \tag{2.12}$$

where $Q_1(T,t)$ can be successively determined from the formulas

$$Q_n(T,t) = a\exp\left(\int_\alpha^t \sum_{i=1}^n f_i(T,s)\,ds\right),$$

$$Q_k(T,t) = E_k(T,t)\left[a + \int_\alpha^t f_k(T,s)\frac{Q_{k+1}(T,s)}{E_k(T,s)}\,ds\right],$$

$$E_k(T,t) = \exp\left(\int_\alpha^t \left[\sum_{i=1}^{k-1} f_i(T,\tau) - f_k(T,\tau)\right]d\tau\right)$$

for $k = n-1, \ldots, 1$, $\alpha \le t \le T \le \beta$. From (2.12) we observe that

$$m_1(t) \le a^{Q_1(T,t)}. \tag{2.13}$$

Therefore by using (2.13) in $u(t) \le m_1(t)$, we have the result

$$u(t) \le a^{Q_1(t,t)} \quad \text{for} \quad t = T.$$

3. Some companion inequalities

The following companion of the Gronwall-type inequality also holds.

THEOREM 3.1. *Let $u(t), a(t), b(t), f_i, i = 1, \dots, n,$ and $g_j, j = 1, \dots, n-1$, be nonnegative continuous functions in $J = [\alpha, \beta]$, with $a(t)$ and $b(t)$ are nondecreasing in J, and suppose that*

$$
\begin{aligned}
u(t) \le a(t) + b(t)\Bigg[&\int_\alpha^t f_1(t_1)u(t_1)\,dt_1 \\
&+ \int_\alpha^t g_1(t_1)\left(\int_\alpha^{t_1} f_2(t_2)u(t_2)\,dt_2\right)dt_1 + \cdots \\
&+ \int_\alpha^t g_1(t_1)\left(\int_\alpha^{t_1} g_2(t_2)\cdots\left(\int_\alpha^{t_{n-2}} g_{n-1}(t_{n-1})\right.\right. \\
&\times \left.\left.\left(\int_\alpha^{t_{n-1}} f_n(t_n)u(t_n)\,dt_n\right)dt_{n-1}\right)\cdots\right)dt_1\Bigg],
\end{aligned}
\tag{3.1}
$$

for $t \in [\alpha, \beta]$. Then

$$
u(t) \le a(t)\exp\left(b(t)\int_\alpha^t B(t_1)dt_1\right)
\tag{3.2}
$$

for $t \in [\alpha, \beta]$, where

$$
\begin{aligned}
B(t) = {}& f_1(t) \\
&+ g_1(t)\Bigg[\int_\alpha^t f_2(t_2)\,dt_2 + \int_\alpha^t g_2(t_2)\left(\int_\alpha^{t_2} f_3(t_3)\,dt_3\right)dt_2 \\
&+ \cdots \\
&+ \int_\alpha^t g_2(t_2)\left(\int_\alpha^{t_2} g_3(t_3)\cdots\left(\int_\alpha^{t_{n-2}} g_{n-1}(t_{n-1})\right.\right. \\
&\times\left.\left.\left(\int_\alpha^{t_{n-1}} f_n(t_n)\,dt_n\right)dt_{n-1}\right)\cdots\right)dt_2\Bigg].
\end{aligned}
\tag{3.3}
$$

Proof. Fix $T \in (\alpha, \beta]$. For $\alpha \le t \le T$ we obtain form (3.1),

$$
\begin{aligned}
u(t) \le a(T) + b(T)\Bigg[&\int_\alpha^t f_1(t_1)u(t_1)\,dt_1 + \cdots \\
&+ \int_\alpha^t g_1(t_1)\left(\int_\alpha^{t_1} g_2(t_2)\cdots\left(\int_\alpha^{t_{n-2}} g_{n-1}(t_{n-1})\right.\right. \\
&\times\left.\left.\left(\int_\alpha^{t_{n-1}} f_n(t_n)u(t_n)\,dt_n\right)\cdots\right)\right)dt_1\Bigg].
\end{aligned}
\tag{3.4}
$$

We denote the right-hand side of (3.4) by $m(t)$ for $t \in [\alpha, T]$. Then $m(\alpha) = a(T)$, the the function $m(t)$ is nondecreasing in $t \in J$, $u(t) \leq m(t)$, and

$$m'(t) \leq b(T)B(t)m(t), \tag{3.5}$$

where $B(t)$ is defined by (3.3). Now by a suitable application of Lemma 2.1 to (3.5), we get

$$m(t) \leq a(T)\exp\left(b(T)\int_\alpha^t B(t_1)dt_1\right) \tag{3.6}$$

for $\alpha \leq t \leq T \leq \beta$. Therefore by using (3.6) in $u(t) \leq m(t)$ and for $t = T$ we have the required inequality in (3.2) .

An essential element in the investigation of the integral inequalities in the following theorem is the application of the result of Theorem 3.1.

THEOREM 3.2. *Let $u(t), b(t)$, $f_i, i = 1, \ldots, n$, and $g_j, j = 1, \ldots, n-1$, be nonnegative continuous functions in $J = [\alpha, \beta]$, with $b(t)$ is nondecreasing in J, and suppose that*

$$\begin{aligned} u(t) \leq a + b(t)\Bigg[&\int_\alpha^t f_1(t_1)u(t_1)\log u(t_1)\,dt_1 \\ &+ \int_\alpha^t g_1(t_1)\left(\int_\alpha^{t_1} f_2(t_2)u(t_2)\log u(t_2)\,dt_2\right)dt_1 \\ &+ \cdots \\ &+ \int_\alpha^t g_1(t_1)\left(\int_\alpha^{t_1} g_2(t_2)\cdots\left(\int_\alpha^{t_{n-2}} g_{n-1}(t_{n-1})\right.\right. \\ &\times\left.\left.\left(\int_\alpha^{t_{n-1}} f_n(t_n)u(t_n)\log u(t_n)\,dt_n\right)dt_{n-1}\right)\cdots\right)dt_1\Bigg] \end{aligned} \tag{3.7}$$

for $t \in [\alpha, \beta]$, where $a > 1$ is a constant. Then

$$u(t) \leq a^{\exp(b(t)\int_\alpha^t B(t_1)dt_1)} \tag{3.8}$$

for $t \in [\alpha, \beta]$, where $B(t)$ is defined by (3.3).

Proof. Fix $T \in (\alpha, \beta]$. For $\alpha \le t \le T$ we obtain form (3.7),

(3.9)
$$
\begin{aligned}
u(t) \le a + b(T)\Bigg[&\int_\alpha^t f_1(t_1)u(t_1)\log u(t_1)\,dt_1 \\
&+\cdots \\
&+\int_\alpha^t g_1(t_1)\bigg(\int_\alpha^{t_1} g_2(t_2)\cdots\bigg(\int_\alpha^{t_{n-2}} g_{n-1}(t_{n-1}) \\
&\quad\times\bigg(\int_\alpha^{t_{n-1}} f_n(t_n)u(t_n)\log u(t_n)\,dt_n\bigg)\,dt_{n-1}\bigg)\cdots\bigg)\,dt_1\Bigg].
\end{aligned}
$$

We denote the right-hand side of (3.4) by $m_1(t)$ for $t \in [\alpha, T]$. Then $m_1(\alpha) = a$, the the function $m_1(t)$ is nondecreasing in $t \in J$, $u(t) \le m_1(t)$, and

$$m_1'(t) \le b(T)[f_1(t)\log m_1(t) + g_2(t)B_1(t)]m_1(t), \tag{3.10}$$

where

$$
\begin{aligned}
B_1(t) = &\int_\alpha^t f_2(t_2)\log m_1(t_2)\,dt_2 \\
&+\int_\alpha^t g_2(t_2)\bigg(\int_\alpha^{t_2} f_3(t_3)\log m_1(t_3)\,dt_3\bigg)dt_2 + \cdots \\
&+\int_\alpha^t g_2(t_2)\bigg(\cdots\int_\alpha^{t_{n-2}} g_{n-1}(t_{n-1}) \\
&\quad\times\bigg(\int_\alpha^{t_{n-1}} f_n(t_n)\log m_1(t_n)\,dt_n\bigg)\,dt_{n-1}\cdots\bigg)\,dt_2.
\end{aligned}
$$

Condition (3.10) imply that

$$\frac{m_1'(t)}{m_1(t)} \le b(T)[f_1(t)\log m_1(t) + g_2(t)B_1(t)]. \tag{3.11}$$

By taking $t = t_1$ in (3.11) and then integrating it from α to $t, t \in [\alpha, \beta]$ we have

$$\log m_1(t) \le \log a + b(T)\int_\alpha^t [f_1(t_1)\log m_1(t_1) + g_2(t_1)B_1(t_1)]\,dt_1. \tag{3.12}$$

Now by a suitable application of Theorem 3.1 to (3.12), we get

$$\log m_1(t) \le \log a \cdot \exp\bigg(b(T)\int_\alpha^t B(t_1)dt_1\bigg) \tag{3.13}$$

for $\alpha \le t \le T \le \beta$, where the function $B(t)$ is defined by (3.3). Therefore by using (3.13) in $u(t) \le m_1(t)$ and for $t = T$ we have the required inequality in (3.8) .

By a similar reasoning to the proof of Theorem 2.3 and Theorem 2.4, we also can prove the following results.

THEOREM 3.3. *Let $u(t), b(t), f_i, i = 1, \dots, n$, and $g_j, j = 1, \dots, n-1$, be nonnegative continuous functions in $J = [\alpha, \beta]$, with $b(t)$ is nondecreasing in J, $a(t) > 0$ is nondegreasing continuous function in J, and suppose that*

$$\begin{aligned} u(t) \le a(t) + b(t)\Bigg[\int_\alpha^t & f_1(t_1)u^p(t_1)\,dt_1 \\ &+ \int_\alpha^t g_1(t_1)\left(\int_\alpha^{t_1} f_2(t_2)u^p(t_2)\,dt_2\right)dt_1 + \cdots \\ &+ \int_\alpha^t g_1(t_1)\left(\int_\alpha^{t_1} g_2(t_2)\cdots\left(\int_\alpha^{t_{n-2}} g_{n-1}(t_{n-1})\right.\right. \\ &\left.\left.\times\left(\int_\alpha^{t_{n-1}} f_n(t_n)u^p(t_n)\,dt_n\right)dt_{n-1}\right)\cdots\right)dt_1\Bigg] \end{aligned} \tag{3.14}$$

for $t \in [\alpha, \beta]$, where $p \ge 0, p \ne 1$ is a constant. Then

$$u(t) \le \left[a^q(t) + qb(t)\int_\alpha^t [f_1(t_1) + g_1(t_1)B(t_1)]dt_1\right]^{1/q}, \tag{3.15}$$

for $t \in [\alpha, \beta_1)$, where $B(t)$ is defined in (3.3), and $q = 1 - p$, β_1 is chosen so that the expression between [...] is positive in the subinterval $[\alpha, \beta_1)$.

Proof. Fix $T \in (\alpha, \beta]$. For $\alpha \le t \le T$ we obtain form (3.14),

$$\begin{aligned} u(t) \le a(T) + b(T)\Bigg[\int_\alpha^t & f_1(t_1)u^p(t_1)\,dt_1 \\ &+ \cdots \\ &+ \int_\alpha^t g_1(t_1)\left(\int_\alpha^{t_1} g_2(t_2)\cdots\left(\int_\alpha^{t_{n-2}} g_{n-1}(t_{n-1})\right.\right. \\ &\left.\left.\times\left(\int_\alpha^{t_{n-1}} f_n(t_n)u^p(t_n)\,dt_n\right)\cdots\right)\right)dt_1\Bigg]. \end{aligned} \tag{3.16}$$

We denote the right-hand side of (3.16) by $m(t)$ for $t \in [\alpha, T]$. Then $m(\alpha) = a(T)$, the the function $m(t)$ is nondecreasing in $t \in J$, $u(t) \leq m(t)$, and

$$m'(t) \leq b(T)[f_1(t) + b_1(t)B(t)]m^p(t), \tag{3.17}$$

where $B(t)$ is defined by (3.3). Now by a suitable application of Lemma 2.2 to (3.17), we get

$$m(t) \leq \left[a^q(T) + qb(T)\int_\alpha^t [f_1(t_1) + g_1(t_1)B(t_1)]dt_1\right]^{1/q} \tag{3.18}$$

for $\alpha \leq t \leq T \leq \beta_p$. Therefore by using (3.18) in $u(t) \leq m(t)$ and for $t = T$ we have the required inequality in (3.15) .

THEOREM 3.4. *Let $u(t), a(t), b(t)$, $f_i, i = 1, \ldots, n$, and $g_j, j = 1, \ldots, n-1$, be nonnegative continuous functions in $J = [\alpha, \beta]$, with $a(t)$ and $b(t)$ are nondecreasing in J, and suppose that*

$$\begin{aligned} u(t) \leq a(t) + b(t)\Bigg[&\int_\alpha^t f_1(t_1)u^p(t_1)\,dt_1 \\ &+ \int_\alpha^t g_1(t_1)\left(\int_\alpha^{t_1} f_2(t_2)u^p(t_2)\,dt_2\right)dt_1 \\ &+ \cdots \\ &+ \int_\alpha^t g_1(t_1)\left(\int_\alpha^{t_1} g_2(t_2)\cdots\left(\int_\alpha^{t_{n-2}} g_{n-1}(t_{n-1})\right.\right. \\ &\quad\times \left.\left.\left(\int_\alpha^{t_{n-1}} f_n(t_n)u^p(t_n)\,dt_n\right)dt_{n-1}\right)\cdots\right)dt_1\Bigg] \end{aligned} \tag{3.19}$$

for $t \in [\alpha, \beta]$, where $p > 1$ is a constant. Then

$$u(t) \leq a(t)\left[1-(p-1)b(t)a^{p-1}(t)\int_\alpha^t [f_1(t_1)+g_1(t_1)B(t_1)]dt_1\right]^{\frac{1}{1-p}} \tag{3.20}$$

for $t \in [\alpha, \beta_p)$, where $\beta_p = \sup\{t \in (\alpha, \beta) : (p-1)b(t)a^{p-1}(t)\int_\alpha^t [f_1(t_1) + g_1(t_1)B(t_1)]dt_1 < 1\}$.

Proof. We start from the relation (3.16)–(3.17), which can be ob tained as in theorem 3.3. Thus gives

$$m'(t) \le b(T)[f_1(t) + g_1(t)B(t)]m^p(t), \tag{3.21}$$

where $B(t)$ is defined by (3.3). That is

$$m'(t) \le R(T,t)m(t), \tag{3.22}$$

where

$$R(T,t) = b(T)[f_1(t) + g_1(t)B(t)]m^{p-1}(t).$$

Lemma 2.1 and the inequality (3.22) imply that

$$m(t) \le a(T)\exp\biggl(\int_\alpha^t R(T,s)\,ds\biggr) \tag{3.23}$$

for $\alpha \le t \le T$. From (3.23) we successively obtain

$$m^{p-1}(t) \le a^{p-1}(T)\exp\biggl(\int_\alpha^t (p-1)R(T,s)\,ds\biggr),$$

$$R(T,t) \le b(T)[f_1(t) + g_1(t)B(t)]a^{p-1}(T)\exp\biggl(\int_\alpha^t (p-1)R(T,s)\,ds\biggr),$$

$$Z(T,t) \le (p-1)b(T)[f_1(t) + g_1(t)B(t)]a^{p-1}(T)\exp\biggl(\int_\alpha^t Z(T,s)\,ds\biggr),$$

where $Z(T,t) = (p-1)R(T,t)$. Consequently, we have

$$Z(T,t)\exp\biggl(-\int_\alpha^t Z(T,s)\,ds\biggr) \le (p-1)b(T)[f_1(t) + g_1(t)B(t)]a^{p-1}(T).$$

or

$$\frac{d}{dt_1}\biggl[-\exp\biggl(-\int_\alpha^t Z(T,s)\,ds\biggr)\biggr] \le (p-1)b(T)[f_1(t) + g_1(t)B(t)]a^{p-1}(T)$$

Integrating this from α to t yields

$$\begin{aligned} &1 - \exp\biggl(-\int_\alpha^t Z(T,s)\,ds\biggr) \\ &\le (p-1)a^{p-1}(T)b(T)\int_\alpha^t [f_1(t_1) + g_1(t_1)B(t_1)]\,dt_1, \end{aligned}$$

from which we conclude that

$$\exp\left(\int_\alpha^t R(T,s)\,ds\right)$$
$$\leq \left[1-(p-1)a^{p-1}(T)b(T)\int_\alpha^t [f_1(t_1)+g_1(t_1)B(t_1)]\,dt_1\right]^{\frac{1}{1-p}}.$$

This and (3.23) imply that
(3.24)

$$m(t) \leq a(T)\left[1-(p-1)a^{p-1}(T)b(T)\int_\alpha^t [f_1(t_1)+g_1(t_1)B(t_1)]\,dt_1\right]^{\frac{1}{1-p}}$$

for $\alpha \leq t \leq T \leq \beta_p$. Therefore by using (3.24) in $u(t) \leq m(t)$ and for $t = T$, we have the required inequality in (3.20) .

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