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# ON THE STABILITY OF A JENSEN TYPE FUNCTIONAL EQUATION ON GROUPS

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Abstract. In this paper we establish the stability of a Jensen type functional equation, namely $f(xy) - f(xy^{-1}) = 2f(y)$, on some classes of groups. We prove that any group $A$ can be embedded into some group $G$ such that the Jensen type functional equation is stable on $G$. We also prove that the Jensen type functional equation is stable on any metabelian group, $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, and $T(n, \mathbb{C})$.

## 1. Introduction

Given an operator $T$ and a solution class $\{u\}$ with the property that $T(u) = 0$, when does $\|T(v)\| \leq \varepsilon$ for an $\varepsilon > 0$ imply that $\|u - v\| \leq \delta(\varepsilon)$ for some $u$ and for some $\delta > 0$? This problem is called the stability of the functional transformation [28]. It happened in 1940 that the audience of the Mathematics Club of the University of Wisconsin had the pleasure to listen to the talk of S.M. Ulam presenting a list of unsolved problems. One of these problems can be considered as the starting point of a new line of investigation: The stability problem. This problem can be formulated as follows. If we replace a given functional equation by a functional inequality, then under what conditions we can state that the solutions of the inequality are close to the solutions of the equation. For instance, given a group $G_1$, a metric group $(G_2, d)$ and a positive number $\varepsilon$. The Ulam question is: does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$?

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In the case of a positive answer to this problem, we say that the homomorphisms $G_1 \to G_2$ are *stable* or that the Cauchy functional equation

$$f(x \cdot y) = f(x) \star f(y)$$

is stable for the pair $(G_1, G_2)$.

See S. M. Ulam[27] for a discussion of such problems, as well as D. H. Hyers[11, 12], D. H. Hyers and S. M. Ulam[16, 17], J. Aczél and J. Dhombres[1]. The first affirmative answer was given by D. H. Hyers[11] in 1941. We present his result in theorem below.

THEOREM 1.1. (Hyers[11]) *Let $E_1$ and $E_2$ be Banach spaces. If $f : E_1 \to E_2$ satisfies the inequality*

$$\| f(x+y) - f(x) - f(y) \| < \varepsilon \tag{1.1}$$

*for some $\varepsilon > 0$ and for all $x, y \in E_1$, then there exists a unique map $T : E_1 \to E_2$ such that*

$$T(x+y) - T(x) - T(y) = 0 \text{ for all } x, y \in E_1 \tag{1.2}$$

*and*

$$\| f(x) - T(x) \| < \varepsilon \text{ for all } x \in E_1. \tag{1.3}$$

If we carefully examine the proof of Hyers's theorem, the existence of the additive function $T$ uniformly approximating $f$, we easily recognize that the result remains true if we replace the additive group of the Banach space $E_1$ by a commutative semigroup. So we can conclude that the homomorphisms from an abelian semigroup into the additive group of a Banach space are stable.

After Hyers's result a great number of papers on the subject have been published, generalizing Ulam's problem and Hyers's theorem in various directions (see [10], [15]–[16] and [25]).

A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy the Jensen's functional equation if

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y) \tag{1.4}$$

for all $x, y \in \mathbb{R}$. Setting $\frac{1}{2}(x+y) = u$ and $\frac{1}{2}(x-y) = v$ we can rewrite the equation (1.4) as

$$2f(u) = f(u+v) + f(u-v).$$

The latter is equivalent to

$$f(xy) + f(xy^{-1}) = 2f(x) \tag{1.5}$$

when the domain of the function is replaced by an arbitrary group. The equation (1.5) was studied in the papers [2], [4] and [23]. The question of stability of equation (1.5) was investigated in [18]–[21] and [26]. In all these papers domain of $f$ is either an abelian group or some of its subsets. In [9], the present authors studied the stability of the equation (1.5) on arbitrary groups.

In the paper [24], the stability of the following Jensen type functional equation

$$2f\left(\frac{x-y}{2}\right) = f(x) - f(y) \tag{1.6}$$

was considered. Here again $f : \mathbb{R} \to \mathbb{R}$. Setting $\frac{1}{2}(x+y) = u$ and $\frac{1}{2}(x-y) = v$ we can rewrite the equation (1.6) as

$$2f(v) = f(u+v) - f(u-v).$$

The latter is equivalent to

$$f(xy) - f(xy^{-1}) = 2f(y) \tag{1.7}$$

and can be considered over an arbitrary group.

In the paper [24], the stability of equation (1.7) over a real normed space was considered. In the present paper we consider the stability of the *Jensen type* functional equation (1.7) over an arbitrary group.

## 2. Auxiliary results

Suppose that $G$ is an arbitrary group and $E$ is an arbitrary real Banach space.

DEFINITION 2.1. We will say that a function $f : G \to E$ is a $(G; E)$–*Jensen type function* if for any $x, y \in G$ we have

$$f(xy) - f(xy^{-1}) - 2f(y) = 0. \tag{2.1}$$

We denote the set of all $(G; E)$–Jensen type functions by $JT(G; E)$.

DEFINITION 2.2. We will say that a function $f : G \to E$ is a *(G;E)–quasijensen type function* if there is a $c > 0$ such that for any $x, y \in G$ we have

$$\|f(xy) - f(xy^{-1}) - 2f(y)\| \leq c. \tag{2.2}$$

It is clear that the set of $(G;E)$-quasijensen type functions is a real linear space. Denote it by $KJT(G;E)$. From (2.2) we obtain

$$\|f(y) - f(y^{-1}) - 2f(y)\| \le c,$$

therefore

$$\|f(y) + f(y^{-1})\| \le c. \tag{2.3}$$

Now letting $y$ for $x$ in (2.2), we get

$$\|f(y^2) - f(1) - 2f(y)\| \le c.$$

Hence

$$\|f(x^2) - 2f(x)\| \le c_2, \tag{2.4}$$

where $c_2 = c + \|f(1)\|$. Again substitution of $x = y^2$ in (2.2) yields

$$\|f(y^3) - f(y) - 2f(y)\| \le c$$

which is

$$\|f(y^3) - 3f(y)\| \le c. \tag{2.5}$$

Let $c$ be as in (2.2) and define the set $C$ as follows: $C = \{\, c_m \mid m \in \mathbb{N}\,\}$, where $c_1 = 0$, $c_2 = c + \|f(1)\|$, and $c_m = c + c_{m-2}$, if $m > 2$.

LEMMA 2.3. *Let $f \in KJT(G;E)$ such that*

$$\|\, f(xy) - f(xy^{-1}) - 2f(y)\,\| \le c.$$

*Then for any $x \in G$ and any $m \in \mathbb{N}$ the following relation holds:*

$$\|f(x^m) - mf(x)\| \le c_m. \tag{2.6}$$

*Proof.* The proof is by induction on $m$. For $m = 3$ the lemma is established. Suppose that for $m$ the lemma has been already established and let us verify it for $m+1$. Letting $x = y^m$ in (2.2), we have

$$\|f(y^{m+1}) - f(y^{m-1}) - 2f(y)\| \le c.$$

By induction hypothesis, we have

$$\|f(y^{m-1}) - (m-1)f(y)\| \le c_{m-1}$$

and hence,

$$\|f(y^{m+1}) - (m+1)f(y)\| \le c_{m+1} = c + c_{m-1}.$$

Now the lemma is proved.

LEMMA 2.4. *Let $f \in KJT(G;E)$. For any $m > 1$, $k \in \mathbb{N}$ and $x \in G$ we have*

$$\|f(x^{m^k}) - m^k f(x)\| \leq c_m(1 + m + \cdots + m^{k-1}) \tag{2.7}$$

*and*

$$\left\|\frac{1}{m^k} f(x^{m^k}) - f(x)\right\| \leq c_m. \tag{2.8}$$

*Proof.* The proof will be based on induction on $k$. If $k = 1$, then (2.7) follows from (2.6). Suppose (2.7) is true for $k$ and let us verify it for $k+1$. Substituting $x^m$ for $x$ in (2.7) implies

$$\|f(x^{m^{k+1}}) - m^k f(x^m)\| \leq c_m(1 + m + \cdots + m^{k-1}).$$

Now using (2.6) we obtain

$$\|m^k f(x^m) - m^{k+1} f(x)\| \leq c_m m^k$$

and hence

$$\|f(x^{m^{k+1}}) - m^{k+1} f(x)\| \leq c_m(1 + m + \cdots + m^k).$$

The latter implies

$$\left\|\frac{1}{m^{k+1}} f(x^{m^{k+1}}) - f(x)\right\| \leq c_m(1 + m + \cdots + m^k)\frac{1}{m^{k+1}} \leq c_m.$$

This completes the proof of the lemma.

From (2.8) it follows that for any $x \in G$ the set

$$\left\{\frac{1}{m^k} f(x^{m^k}) \,\middle|\, k \in \mathbb{N}\right\}$$

is bounded. Substituting $x^{m^n}$ in place of $x$ in (2.8), we obtain

$$\left\|\frac{1}{m^k} f(x^{m^{n+k}}) - f(x^{m^n})\right\| \leq c_m$$

Thus

$$\left\|\frac{1}{m^{n+k}} f(x^{m^{n+k}}) - \frac{1}{m^n} f(x^{m^n})\right\| \leq \frac{c_m}{m^n} \to 0 \quad as \quad n \to \infty.$$

From the latter, it follows that the sequence

$$\left\{\frac{1}{m^k} f(x^{m^k}) \,\middle|\, k \in \mathbb{N}\right\}$$

is a Cauchy sequence. Since the real Banach space $E$ is complete, the above sequence has a limit and we denote it by $\varphi_m(x)$. Thus

$$\varphi_m(x) = \lim_{k\to\infty} \frac{1}{m^k} f(x^{m^k}).$$

From (2.8), it follows that

$$\|\varphi_m(x) - f(x)\| \leq c_m, \quad \forall x \in G. \tag{2.9}$$

LEMMA 2.5. *Let $f \in KJT(G;E)$ such that*

$$\|f(xy) - f(xy^{-1}) - 2f(y)\| \leq c \quad \forall x, y \in G.$$

*Then for any $m \in \mathbb{N}$, we have $\varphi_m \in KJT(G;E)$.*

*Proof.* Indeed, by (2.9)

$$\begin{aligned}
&\|\varphi_m(xy) - \varphi_m(xy^{-1}) - 2\varphi_m(y)\| \\
&= \|\varphi_m(xy) - f(xy) - \varphi_m(xy^{-1}) + f(xy^{-1}) - 2\varphi_m(y) + 2f(y) \\
&\quad + f(xy) - f(xy^{-1}) - 2f(y)\| \\
&\leq \|\varphi_m(xy) - f(xy)\| + \|\varphi_m(xy^{-1}) - f(xy^{-1})\| \\
&\quad + 2\|\varphi_m(x) - f(x)\| + \|f(xy) - f(xy^{-1}) - 2f(y)\| \\
&\leq 4c_m + c.
\end{aligned}$$

This completes the proof of the lemma.

For any $x \in G$ we have the relation

$$\varphi_m(x^{m^k}) = m^k \varphi_m(x). \tag{2.10}$$

Indeed,

$$\begin{aligned}
\varphi_m(x^{m^k}) = \lim_{\ell\to\infty} \frac{1}{m^\ell} f((x^{m^k})^{m^\ell}) &= \lim_{\ell\to\infty} \frac{m^k}{m^{k+\ell}} f(x^{m^{k+\ell}}) \\
&= m^k \lim_{p\to\infty} \frac{1}{m^p} f(x^{m^p}) = m^k \varphi_m(x).
\end{aligned}$$

LEMMA 2.6. *If $f \in KJT(G;E)$, then $\varphi_2 = \varphi_m$ for any $m \geq 2$.*

*Proof.* By Lemma 2.5, we have $\varphi_2, \varphi_m \in KJT(G;E)$. Hence the function

$$g(x) = \lim_{k\to\infty} \frac{1}{m^k} \varphi_2(x^{m^k})$$

is well-defined and is a $(G;E)$–quasijensen type function.

It is clear that $g(x^{m^k}) = m^k g(x)$ and $g(x^{2^k}) = 2^k g(x)$ for any $x \in G$ and any $k \in \mathbb{N}$. From (2.9), it follows that there are $d_1, d_2 \in \mathbb{R}_+$ such that for all $x \in G$

$$(2.11) \qquad \|\varphi_2(x) - g(x)\| \le d_1 \quad \text{and} \quad \|\varphi_m(x) - g(x)\| \le d_2.$$

Hence $g \equiv \varphi_2$ and $g \equiv \varphi_m$ and we obtain $\varphi_2 \equiv \varphi_m$.

DEFINITION 2.7. By $(G; E)$-*pseudojensen type function* we will mean a $(G; E)$-quasijensen type function $f$ such that $f(x^n) = nf(x)$ for any $x \in G$ and any $n \in \mathbb{N}$.

The space of $(G; E)$-pseudojensen type function will be denoted by $PJT(G; E)$.

LEMMA 2.8. *For any $f \in KJT(G; E)$, the function*

$$\widehat{f}(x) = \lim_{k \to \infty} \frac{1}{2^k} f(x^{2^k})$$

*is well-defined and is a $(G; E)$-pseudojensen function such that for any $x \in G$*

$$\|\widehat{f}(x) - f(x)\| \le c_2.$$

*Proof.* By Lemma 2.5, $\widehat{f}$ is a $(G; E)$-quasijensen type function. Now by Lemma 2.6, we have $\widehat{f}(x^m) = \varphi_m(x^m) = m\varphi_m(x) = m\widehat{f}(x)$. Thus $\varphi_m(x) = \widehat{f}(x)$ and hence $\varphi_2(x) = \widehat{f}(x)$ by Lemma 2.6. From equality $\widehat{f} = \varphi_2$ we have $\|\widehat{f}(x) - f(x)\| = \|\varphi_2(x) - f(x)\| \le c_2$.

REMARK 2.9. If $f \in PJT(G; E)$, then:

(1) $f(x^{-n}) = -nf(x)$ for any $x \in G$ and $n \in \mathbb{N}$;
(2) if $y \in G$ is an element of finite order then $f(y) = 0$;
(3) if $f$ is a bounded function on $G$, then $f \equiv 0$.

*Proof.* Suppose for some $c > 0$ the following relation holds

$$\|f(xy) - f(xy^{-1}) - 2f(y)\| \le c.$$

From (2.3) it follows that

$$\|f(y^k) + f(y^{-k})\| \le c, \ \forall y \in G, \ \forall k \in \mathbb{N}.$$

The last inequality is equivalent to $k\|f(y) + f(y^{-1})\| \le c$ or $\|f(y) + f(y^{-1})\| \le \frac{c}{k}$ for all $y \in G$ and all $k \in \mathbb{N}$. The latter implies $f(y^{-1}) = -f(y)$. Thus for any $n \in \mathbb{N}$, we have

$$f(y^{-n}) = f((y^n)^{-1}) = -f(y^n) = -nf(y).$$

Hence, the assertion 1 is established.

Similarly we verify the assertions 2 and 3.

We denote by $B(G;E)$ the space of all bounded mappings on a group $G$ that take values in $E$.

THEOREM 2.10. *For an arbitrary group $G$ the following decomposition holds*

$$KJT(G;E) = PJT(G;E) \oplus B(G;E).$$

*Proof.* It is clear that $PJT(G;E)$ and $B(G;E)$ are subspaces of $KJT(G;E)$, and $PJT(G;E) \cap B(G;E) = \{0\}$. Hence the subspace of $KJT(G;E)$ generated by $PJT(G;E)$ and $B(G;E)$ is their direct sum. That is $PJT(G;E) \oplus B(G;E) \subseteq KJT(G;E)$. Let us verify that $KJT(G;E) \subseteq PJT(G;E) \oplus B(G;E)$. Indeed, if $f \in KJT(G;E)$, then by Lemma 2.8 we have $\widehat{f} \in PJT(G;E)$ and $\widehat{f} - f \in B(G;E)$.

DEFINITION 2.11. Let $E$ be a Banach space and $G$ be a group. A mapping $f : G \to E$ is said to be a $(G;E)$*–quasiadditive mapping* of a group $G$ if the set $\{f(xy) - f(x) - f(y) \mid x, y \in G\}$ is bounded.

DEFINITION 2.12. By a $(G;E)$*–pseudoadditive mapping* of a group $G$ we mean its $(G;E)$–quasiaddtive mapping $f$ that satisfies $f(x^n) = nf(x)$ for all $x \in G$ and for all $n \in \mathbb{Z}$.

DEFINITION 2.13. A *quasicharacter* of a group $G$ is a real-valued function $f$ on $G$ such that the set $\{f(xy) - f(x) - f(y) \mid x, y \in G\}$ is bounded.

DEFINITION 2.14. By a *pseudocharacter* of a group $G$ we mean its quasicharacter $f$ that satisfies $f(x^n) = nf(x)$ for all $x \in G$ and all $n \in \mathbb{Z}$.

The set of all $(G;E)$–quasiadditive mappings is a vector space (with respect to the usual operations of addition of functions and their multiplication by numbers), which will be denoted by $KAM(G;E)$. The subspace of $KAM(G;E)$ consisting of $(G;E)$–pseudoadditive mappings will be denoted by $PAM(G;E)$ and the subspace consisting of additive mappings from $G$ to $E$ will be denoted by $\mathrm{Hom}(G;E)$. We say that a $(G;E)$–pseudoadditive mapping $\varphi$ of the group $G$ is *nontrivial* if $\varphi \notin \mathrm{Hom}(G;E)$.

The space of quasicharacters will be denoted by $KX(G)$, the space of pseudocharacters will be denoted by $PX(G)$, and the the space of real additive characters on $G$ will be denoted by $X(G)$.

REMARK 2.15. If a group $G$ has nontrivial pseudocharacter, then for any Banach space $E$ there is nontrivial $(G;E)$–pseudoadditive mapping.

*Proof.* Let $f$ be a nontrivial pseudocharacter of the group $G$ and $e \in E$ such that $e \neq 0$. Consider a mapping $\varphi : G \to E$ such that $\varphi(x) = f(x) \cdot e$. It easy to see that $\varphi$ is nontrivial $(G; E)$–pseudoadditive mapping.

In [7] and [8], some classes of groups having nontrivial pseudocharacters are considered.

THEOREM 2.16. *For any group $G$ the following relations hold:*

(1) $KAM(G; E) \subseteq KJT(G; E), PAM(G; E) \subseteq PJT(G; E)$, *and* $\mathrm{Hom}(G; E) \subseteq JT(G; E)$;
(2) *If* $f \in PJT(G; E)$, *and* $f(xy) = f(yx)$ *for any* $x, y \in G$, *then* $f \in PAM(G; E)$.
(3) *If* $f \in PJT(G; E)$, *and for some* $a, b \in G$ *we have* $ab = ba$, *then* $f(ab) = f(a) + f(b)$.

*Proof.* (1) Let $f \in KAM(G; E)$ and $c > 0$ such that $\|f(xy) - f(x) - f(y)\| \leq c$ for all $x, y \in G$. Then we have

$$\begin{aligned}
&\|f(xy) - f(xy^{-1}) - 2f(y)\| \\
&= \|f(xy) - f(x) - f(y) - f(xy^{-1}) + f(x) + f(y^{-1})\| \\
&= \|f(xy) - f(x) - f(y) - (f(xy^{-1}) - f(x) - f(y^{-1}))\| \\
&\leq \|f(xy) - f(x) - f(y)\| + \|f(xy^{-1}) - f(x) - f(y^{-1})\| \leq 2c,
\end{aligned}$$

that is, $KAM(G; E) \subseteq KJT(G; E)$. Hence, $PAM(G; E) \subseteq PJT(G; E)$.

(2) Let $f \in PJT(G; E)$, $c > 0$ such that $\|f(xy) - f(xy^{-1}) - 2f(y)\| \leq c$ and $f(xy) = f(yx)$ for all $x, y \in G$. Then we have

$$\begin{aligned}
&2\|f(xy) - f(x) - f(y)\| \\
&= \|f(xy) - f(xy^{-1}) - 2f(y) + f(xy) - f(yx^{-1}) - 2f(x)\| \\
&\leq \|f(xy) - f(xy^{-1}) - 2f(y)\| + \|f(yx) - f(yx^{-1}) - 2f(x)\| \leq 2c.
\end{aligned}$$

Hence $\|f(xy) - f(x) - f(y)\| \leq c$ and $f \in PAM(G; E)$.

(3) Let $A$ be the subgroup of $G$ generated by elements $a$ and $b$. From the previous item we have $PJT(A; E) = PAM(A; E)$. Then for some $c > 0$ and for any $n \in \mathbb{N}$, we get

$$\begin{aligned}
n \| f(ab) - f(a) - f(b) \| &= \| f((ab)^n) - f(a^n) - f(b^n) \| \\
&= \| f(a^n b^n) - f(a^n) - f(b^n) \| \leq c.
\end{aligned}$$

The latter is possible only if $f(ab) - f(a) - f(b) = 0$.

COROLLARY 2.17. *If $G$ is an abelian group, then $PJT(G; E) = \mathrm{Hom}(G; E)$.*

## 3. Stability

Suppose that $G$ is a group and $E$ is a real Banach space.

DEFINITION 3.1. We shall say that the equation (2.1) *is stable* for the pair $(G; E)$ if for any $f : G \to E$ satisfying functional inequality

$$\|f(xy) - f(xy^{-1}) - 2f(y)\| \leq c \quad \forall x, y \in G$$

for some $c > 0$ there is a solution $j$ of the functional equation (2.1) such that the function $j(x) - f(x)$ belongs to $B(G; E)$.

It is clear that the equation (2.1) is stable on $G$ if and only if $PJT(G; E) = JT(G; E)$. From Corollary 2.17 it follows that the equation (2.1) is stable on any abelian group. We will say that a $(G; E)$-pseudojensen function $f$ is nontrivial if $f \notin JT(G; E)$.

THEOREM 3.2. *Let $E_1$, $E_2$ be a Banach spaces over reals. Then the equation* (2.1) *is stable for the pair $(G; E_1)$ if and only if it is stable for the pair $(G; E_2)$.*

*Proof.* Let $E$ be a Banach space and $\mathbb{R}$ be the set of reals. Suppose that the equation (2.1) is stable for the pair $(G; E)$. Suppose that (2.1) is not stable for the pair $(G, \mathbb{R})$, then there is a nontrivial real-valued pseudojensen type function $f$ on $G$. Now let $e \in E$ and $\|e\| = 1$. Consider the function $\varphi : G \to E$ given by the formula $\varphi(x) = f(x) \cdot e$. It is clear that $\varphi$ is a nontrivial pseudojensen type $E$–valued function, and we obtain a contradiction.

Now suppose that the equation (2.1) is stable for the pair $(G, \mathbb{R})$, that is, $PJT(G, \mathbb{R}) = JT(G, \mathbb{R})$. Denote by $E^*$ the space of linear bounded functionals on $E$ endowed by functional norm topology. It is clear that for any $\psi \in PJT(G, E)$ and any $\lambda \in E^*$ the function $\lambda \circ \psi$ belongs to the space $PJT(G, \mathbb{R})$. Indeed, for some $c > 0$ and any $x, y \in G$ we have $\|\psi(xy) - \psi(xy^{-1}) - 2\psi(y)\| \leq c$. Hence

$$\begin{aligned}|\lambda \circ \psi(xy) - \lambda \circ \psi(xy^{-1}) - \lambda \circ 2\psi(y)| &= |\lambda(\psi(xy) - \psi(xy^{-1}) - 2\psi(y))| \\ &\leq c\|\lambda\|.\end{aligned}$$

Obviously, $\lambda \circ \psi(x^n) = n\lambda \circ \psi(x)$ for any $x \in G$ and for any $n \in \mathbb{N}$. Hence the function $\lambda \circ \psi$ belongs to the space $PJT(G, \mathbb{R})$. Let $f : G \to H$ be a nontrivial pseudojensen type mapping. Then there are $x, y \in G$ such that $f(xy) - f(xy^{-1}) - 2f(y) \neq 0$. Hahn–Banach Theorem implies that there is a $\ell \in E^*$ such that $\ell(f(xy) - f(xy^{-1}) - 2f(y)) \neq 0$, and we see that $\ell \circ f$ is a nontrivial pseudojensen type real–valued function on $G$. This contradiction proves the theorem.

In what follows the space $KJT(G, \mathbb{R})$ will be denoted by $KJT(G)$, the space $PJT(G, \mathbb{R})$ will be denoted by $PJT(G)$, the space $JT(G, \mathbb{R})$ will be denoted by $JT(G)$.

COROLLARY 3.3. *The equation* (2.1) *over a group $G$ is stable if and only if $PJT(G) = JT(G)$.*

Due to the previous theorem we may simply say that the equation (2.1) is stable or not stable.

REMARK 3.4. For any group $G$ and any Banach space $E$ the following relation $PAM(G; E) \cap JT(G; E) = \mathrm{Hom}(G; E)$ holds.

*Proof.* It is clear that $\mathrm{Hom}(G; E) \subseteq PAM(G; E) \cap JT(G; E)$.

Lemma 1 from [6] asserts that if $f \in PAM(G; E)$, then for any $x, y \in G$ we have $f(xy) = f(yx)$.

Suppose that $f \in PAM(G; E) \cap JT(G; E)$. Since $f \in JT(G; E)$, the map $f$ satisfies

$$f(xy) - f(xy^{-1}) - 2f(y) = 0. \tag{3.1}$$

Interchanging $x$ with $y$ in (3.1), we have

$$f(yx) - f(yx^{-1}) - 2f(x) = 0.$$

Taking into account the relations

$$f(yx) = f(xy) \text{ and } f(yx^{-1}) = -f(xy^{-1}),$$

we get

$$f(xy) + f(xy^{-1}) - 2f(x) = 0. \tag{3.2}$$

Adding (3.1) and (3.2), we obtain $2f(xy) - 2f(x) - 2f(y) = 0$. Hence $f(xy) = f(x) + f(y)$ and $f \in \mathrm{Hom}(G; E)$, so

$$PAM(G; E) \cap JT(G; E) = \mathrm{Hom}(G; E). \tag{3.3}$$

REMARK 3.5. If a group $G$ has nontrivial pseudocharacter, then the equation (2.1) is not stable on $G$.

*Proof.* Let $\varphi$ be a nontrivial pseudocharacter of $G$. Suppose that there is $j \in JT(G)$ such that the function $\varphi - j$ is bounded. Then there is a $c > 0$ such that $|\varphi(x) - j(x)| \leq c$ for any $x \in G$. Hence for any $n \in \mathbb{N}$ we have $c \geq |\varphi(x^n) - j(x^n)| = n|\varphi(x) - j(x)|$ and we see that the latter is possible if $\varphi(x) = j(x)$. So, $\varphi \in PX(G) \cap JT(G)$. Hence, $\varphi \in X(G)$ and we come to a contradiction with the assumption about $\varphi$.

Let $G$ be an arbitrary group. For $a, b, c \in G$, we set $[a, b] = a^{-1}b^{-1}ab$ and $[a, b, c] = [[a, b], c]$.

DEFINITION 3.6. We shall say that $G$ is *metabelian* if for any $x, y, z \in G$ we have $[[x, y], z] = 1$.

It is clear that if $[x, y] = 1$, then $[[x, y], z] = 1$, and hence any abelian group is metabelian.

Our next goal is to proof a stability theorem for any metabelian group. Consider the group $H$ over two generators $a, b$ and the following defining relations:

$$[b, a]a = a[b, a], \quad b[b, a] = [b, a]b.$$

If we set $c = [b, a]$ we get the following representation of $H$ in terms of generators and defining relations:

$$H =< a, b, c \mid c = [b, a], \quad [c, a] = [c, b] = 1 > . \tag{3.4}$$

It is well known that each element of $H$ can be uniquely represented as $g = a^m b^n c^k$, where $m, n, k \in \mathbb{Z}$. The mapping

$$g = a^m b^n c^k \to \begin{bmatrix} 1 & n & k \\ 0 & 1 & m \\ 0 & 0 & 1 \end{bmatrix}$$

is an isomorphism between $H$ and $UT(3, \mathbb{Z})$.

LEMMA 3.7. *Let $f \in PJT(H)$ and $f(c) = 0$, then $f \in X(H)$.*

*Proof.* Let $x = a^m b^n c^k$ and $y = a^{m_1} b^{n_1} c^{k_1}$ be two elements from $H$, then from the representation (3.4) it follows

$$xy = a^{m+m_1} b^{n+n_1} c^{m_1 n+k+k_1}, \quad yx = a^{m+m_1} b^{n+n_1} c^{mn_1+k+k_1}.$$

Hence by Theorem 2.16 we have

$$f(xy) = f(a^{m+m_1} b^{n+n_1}) + f(c^{m_1 n+k+k_1}) = f(a^{m+m_1} b^{n+n_1}),$$

$$f(yx) = f(a^{m+m_1} b^{n+n_1}) + f(c^{mn_1+k+k_1}) = f(a^{m+m_1} b^{n+n_1}).$$

Thus $f(xy) = f(yx)$ for any $x, y \in H$. By Theorem 2.16 we obtain that $f \in PX(H)$. From the representation (3.4) it follows that the subgroup of $H$ generated by element $c$ is the commutator subgroup of $H$. Lemma 2 from [6] establishes that if $G$ is a group and $\varphi \in PX(G)$ such that $\varphi|_{G'} \equiv 0$, then $\varphi \in X(G)$. Here $G'$ is the commutator subgroup of $G$. Hence, $f \in X(H)$.

LEMMA 3.8. *Let $f \in PJT(H)$, then $f(c) = 0$.*

*Proof.* Let $x = a^m b^n c^k$, $y = a^{m_1} b^{n_1} c^{k_1}$, then

$$xy^{-1} = a^m b^n c^k c^{-k_1} b^{-n_1} a^{-m_1} = a^{m-m_1} b^{n-n_1} c^{m_1 n_1 - m_1 n + k - k_1}.$$

Hence by Theorem 2.16, we obtain

$$\begin{aligned}
& f(xy) - f(xy^{-1}) - 2f(y) \\
&= f(a^{m+m_1} b^{n+n_1} c^{nm_1+k+k_1}) \\
&\quad - f(a^{m-m_1} b^{n-n_1} c^{m_1 n_1 - nm_1 + k - k_1}) - 2f(a^{m_1} b^{n_1} c^{k_1}) \\
&= f(a^{m+m_1} b^{n+n_1}) + f(c^{nm_1+k+k_1}) \\
&\quad - f(a^{m-m_1} b^{n-n_1}) - f(c^{m_1 n_1 - nm_1 + k - k_1}) \\
&\quad - 2f(a^{m_1} b^{n_1}) - 2f(c^{k_1}) \\
&= f(a^{m+m_1} b^{n+n_1}) - f(a^{m-m_1} b^{n-n_1}) - 2f(a^{m_1} b^{n_1}) \\
&\quad + f(c^{nm_1+k+k_1}) - f(c^{m_1 n_1 - nm_1 + k - k_1}) - 2f(c^{k_1}) \\
&= f(a^{m+m_1} b^{n+n_1}) - f(a^{m-m_1} b^{n-n_1}) - 2f(a^{m_1} b^{n_1}) \\
&\quad + f(c^{nm_1+k+k_1-m_1 n_1+nm_1-k+k_1-2k_1}) \\
&= f(a^{m+m_1} b^{n+n_1}) - f(a^{m-m_1} b^{n-n_1}) - 2f(a^{m_1} b^{n_1}) \\
&\quad + f(c^{2nm_1-m_1 n_1})
\end{aligned}$$

Hence the set

$$M = \Big\{ f(a^{m+m_1} b^{n+n_1}) - f(a^{m-m_1} b^{n-n_1}) - 2f(a^{m_1} b^{n_1}) + f(c^{2nm_1-m_1 n_1}) \,\Big|\, m, n, k, m_1, n_1 \in \mathbb{Z} \Big\}$$

is bounded. Let us set $n_1 = n = 2l$, then for some $\Delta$, we have

$$|f(a^{m+m_1} b^{n+n_1}) - f(a^{m+m_1}) - 2f(b^n)| \leq \Delta, \tag{3.5}$$

$$|f(a^{m_1} b^n) - f(a^{m_1}) - 2f(b^l)| \leq \Delta. \tag{3.6}$$

Taking into account these two relations, we see that the set

$$M_1 = \Big\{ f(a^{m+m_1} b^{n+n_1}) - f(a^{m-m_1} b^{n-n_1}) - 2f(a^{m_1} b^{n_1}) \,\Big|\, m, n, k, m_1, n_1 \in \mathbb{Z} \Big\}$$

is bounded. Now from boundedness of the sets $M$ and $M_1$ it follows that the set

$$\left\{ f(c^{2nm_1 - m_1 n}) = f(c^{nm_1}) \,\middle|\, n, m_1 \in \mathbb{Z} \right\}$$

is bounded too. But it is possible only if $f(c) = 0$.

LEMMA 3.9. $PJT(H) = X(H)$.

*Proof.* The proof follows from Lemma 3.7 and Lemma 3.8.

THEOREM 3.10. *The equation* (2.1) *is stable on any metabelian group.*

*Proof.* Let $G$ be a metabelian group and $f \in PJT(G)$. If $x, y \in G$, then there is a homomorphism $\tau$ of $H$ into $G$ such that $\tau(a) = x$ and $\tau(b) = y$. Obviously, the function $f^*(g) = f(\tau(g))$ belongs to $PJT(H)$. Now if $f(xy) - f(xy^{-1}) - 2f(y) \neq 0$, then $f^*(ab) - f^*(ab^{-1}) - 2f^*(b) \neq 0$ and we arrive at a contradiction with the previous Lemma 3.9. Thus $f \in JT(G)$ and $PJT(G) = JT(G)$. Therefore the equation (2.1) is stable on $G$.

## 4. Some classic groups $GL(n, \mathbb{C}), SL(n, \mathbb{C}), T(n, \mathbb{C})$

For any group $G$ denote by $G^2$ its subset $\{x^2 \mid x \in G\}$.

THEOREM 4.1. *Let $G$ be a group such that $G = G^2$, then $PJT(G) = PX(G)$.*

*Proof.* Let $f \in PJT(G)$. For some $c > 0$ and any $x, y \in G$, we have

$$|f(xy^2) - f(xyy^{-1}) - 2f(y)| \leq c,$$

hence

$$|f(xy^2) - f(x) - 2f(y)| = |f(xy^2) - f(x) - f(y^2)| \leq c. \tag{4.1}$$

Let $x, z$ be an arbitrary elements from $G$, then for some $y \in G$ we have $z = y^2$. Now from (4.1) it follows $|f(xz) - f(x) - f(z)| = |f(xy^2) - f(x) - f(y^2)| \leq c$. Hence $f \in PX(G)$.

THEOREM 4.2. *Let $G$ denote the group $GL(n, \mathbb{C}), SL(n, \mathbb{C})$ or $T(n, \mathbb{C})$. Then the equation* (2.1) *is stable over $G$.*

*Proof.* Let $G$ be one of the groups $GL(n, \mathbb{C}), SL(n, \mathbb{C})$ or $T(n, \mathbb{C})$. For any $x \in G$ there is $y \in G$ such that $y^2 = x$. By Theorem 4.1, we have $PJT(G) = PX(G)$. Let us show that $PX(G) = X(G)$. The group $T(n, \mathbb{C})$ is solvable, hence by Theorem 1 from [6] we have $PJT(T(n, \mathbb{C})) =$

*Proof.* Let $G$ be one of the groups $GL(n,\mathbb{C}), SL(n,\mathbb{C})$ or $T(n,\mathbb{C})$. For any $x \in G$ there is $y \in G$ such that $y^2 = x$. By Theorem 4.1, we have $PJT(G) = PX(G)$. Let us show that $PX(G) = X(G)$. The group $T(n,\mathbb{C})$ is solvable, hence by Theorem 1 from [6] we have $PJT(T(n,\mathbb{C})) = PX(T(n,\mathbb{C})) = X(T(n,\mathbb{C}))$. Consider the group $SL(n,\mathbb{C})$. It is well known that the group $SL(n,\mathbb{C})$ is generated by the set of elementary matrices, and that every elementary matrix is conjugate with it's inverse. Hence, if $f \in PX(SL(n,\mathbb{C}))$ and $x$ an elementary matrix, then $f(x) = 0$. It is well known that for any $n \in \mathbb{N}$ there exists $k(n)$ such that every element from $SL(n,\mathbb{C})$ can be represented as product no more then $k(n)$ elementary matrices.

If $|f(xy) - f(x) - f(y)| \leq c$ for all $x, y \in SL(n,\mathbb{C})$, then for any $g \in SL(n,\mathbb{C})$ we have $|f(g)| \leq k(n)c$, and we see that $f$ is a bounded function of $SL(n,\mathbb{C})$. Therefore $f \equiv 0$ and $PX(SL(n,\mathbb{C})) = 0$. It is well known that $SL(n,\mathbb{C})$ is commutator subgroup of $GL(n,\mathbb{C})$. By Lemma 2 from [6] it follows that if a pseudocharacter of a group $G$ is zero on its commutator subgroup $G'$ then this pseudocharacter is a character of $G$. Hence, we get $PX(GL(n,\mathbb{C})) = X(GL(n,\mathbb{C}))$. So in any cases we have $PX(G) = X(G)$ and the equation (2.1) is stable over $G$.

REMARK 4.3. Note that the Jensen type functional equation (2.1) is not stable on the group $G$ if $G$ is either $GL(2,\mathbb{Z})$ or $SL(2,\mathbb{Z})$. This is due to the fact that $SL(2,\mathbb{Z})$ has a nontrivial pseudocharacter (see Remark 3.5). Thus, in general, the equation (2.1) is not stable on groups $GL(n,\mathbb{Z})$ and $SL(n,\mathbb{Z})$.

## 5. The theorem of embedding

DEFINITION 5.1. Let $G$ be a group, $f \in PJT(G;E)$, and $b$ an automorphism of $G$. We will say that $f$ is invariant relative to $b$ if for any $x \in G$ the relation $f(x^b) = f(x)$ holds. If the latter relation is valid for any $b \in B$, where $B$ is a group of automorphism of $G$, then we will say that $f$ is invariant relative to $B$.

From now on, the set of pseudojensen type functions on $G$ invariant relative to $B$ will be denoted by $PJT(G,B;E)$ and if $E = R$, then the space $PJT(G,B;R)$ will be denoted $PJT(G,B)$.

THEOREM 5.2. *Let $H$ and $A$ be a groups such that $A$ is an abelian group and $H = H^2, A = A^2$. Let $Q = A \cdot H$ be a semidirect product of groups $A$ and $H$, $A$ acts by automorphism on $H$, and $H \lhd Q$. Then $PJT(Q) = PX(Q) = X(A) \oplus PX(H,A)$ and $X(Q) = X(A) \oplus X(H,A)$.*

*Proof.* Suppose that $f \in PJT(Q)$ and for some $c > 0$ and for any $x, y \in Q$, we have

$$|f(xy) - f(xy^{-1}) - 2f(y)| \leq c. \tag{5.1}$$

We can assume that $f\big|_A \equiv 0$. Indeed, the restriction of $f$ to $A$ is an element of the space $PJT(A)$. Hence by Corollary 2.17 it is an element of the space $X(A)$. Let $\varphi = f \circ \tau$, where $\tau : Q \to A$ a natural epimorphism with $\ker \tau = H$. It is clear $\varphi \in X(Q)$. Hence in order to show that $f \in PX(Q)$ it is necessary and sufficient to show that $\pi = f - \varphi \in PX(Q)$. But it is clear that $\pi\big|_A \equiv 0$. So we can assume $f\big|_A \equiv 0$.

Let $a, b \in A$ , $u, v \in H$. Then we have

$$|f(uaa) - f(uaa^{-1}) - 2f(a)| \leq c.$$

Hence

$$|f(ua^2) - f(u)| \leq c.$$

Since $A = A^2$ we get that for any $a \in A$ the following relation

$$|f(ua) - f(u)| \leq c,$$

or

$$|f(au^a) - f(u)| \leq c.$$

It follows that

$$|f(au) - f(u^{a^{-1}})| \leq c. \tag{5.2}$$

For any $b \in A$ and $v \in H$, we have $2f(bv) = f((bv)^2) = f(b^2v^bv)$. Taking into account (5.2), we get

$$|f(b^2v^bv) - f(v^{bb^{-2}}v^{b^{-2}})| \leq c$$

or

$$|2f(bv) - f(v^{b^{-1}}v^{b^{-2}})| \leq c.$$

From the latter inequality and (5.2), we get

$$|2f(v^{b^{-1}}) - f(v^{b^{-1}}v^{b^{-2}})| \leq 3c.$$

Now by Theorem 4.1, we obtain

$$|2f(v^{b^{-1}}) - f(v^{b^{-1}}) - f(v^{b^{-2}})| \leq 4c \tag{5.3}$$

or

$$|f(v^{b^{-1}}) - f(v^{b^{-2}})| \leq 4c. \tag{5.4}$$

Let us put $d = b^{-1}$ and $w = v^d$. Now from (5.4) it follows that for any $d \in A$ and any $w \in H$ the following relation

$$|f(w) - f(w^d)| \leq 4c \tag{5.5}$$

holds. Changing $w$ by $w^n$ in the last relation we get

$$|f(w^n) - f((w^n)^d)| \leq 4c.$$

Hence

$$|f(w) - f(w^d)| = \tfrac{1}{n}|f(w^n) - f((w^n)^d)| \leq \tfrac{1}{n}4c.$$

And we see that for any $d \in A$ and any $w \in H$ the following relation

$$f(w) = f(w^d) \tag{5.6}$$

holds. Now from (5.2), we get

$$| f(au) - f(u) | \leq c. \tag{5.7}$$

Let $\psi = f\big|_H$, then from the relation (5.6) we get $\psi^d = \psi$. Or $\psi \in PX(H,A)$.

Now let us show that $f \in PX(Q)$. Let $x = au$ and $y = bv$. Taking into account (5.6) and (5.7) we have

$$\begin{aligned}|f(xy) - f(x) - f(y)| &= |f(abu^bv) - f(au) - f(bv)| \\ &= |f(abu^bv) - f(u^bv) - f(au) + f(u) - f(bv) \\ &\quad + f(v) + f(u^bv) - f(u) - f(v)| \\ &\leq |f(abu^bv) - f(u^bv)| + |f(au) + f(u)| \\ &\quad + |f(bv) + f(v)| + |f(u^bv) - f(u) - f(v)| \\ &\leq 4c.\end{aligned}$$

So, $f \in PX(Q)$ and $PJT(Q) = PX(Q)$. Now by Theorem 2 from [5] we obtain $PX(Q) = X(A) \oplus PX(H,A)$. The latter relation implies $X(Q) = X(A) \oplus X(H,A)$.

Let $A$ and $B$ be an arbitrary groups. For each $b \in B$ denote by $A(b)$ a group that is isomorphic to $A$ under isomorphism $a \to a(b)$. Denote by $D = A^{(B)} = \prod_{b \in B} A(b)$ the direct product of groups $A(b)$. It is clear that if $a_1(b_1)a_2(b_2)\cdots a_k(b_k)$ is an element of $D$, then for any $b \in B$, the mapping

$$b^* : a_1(b_1)a_2(b_2)\cdots a_k(b_k) \to a_1(b_1b)a_2(b_2b)\cdots a_k(b_kb)$$

is an automorphism of $D$ and $b \to b^*$ is an embedding of $B$ into $\operatorname{Aut} D$.

Hence, we can form a semidirect product $G = B \cdot D$. This group is called *the wreath product* of the groups $A$ and $B$, and will be denoted by $G = A \wr B$. We will identify the group $A$ with subgroup $A(1)$ of $D$, where $1 \in B$. Hence, we can assume that $A$ is a subgroup of $D$.

LEMMA 5.3. *Any group $G$ can be embedded into a group $H$ such that $H^2 = H$.*

*Proof.* The group $H$ can be constructed by using amalgamated free product (see [22]) or by using wreath product (see [3]).

THEOREM 5.4. *Let $G$ be an arbitrary group. Then $G$ can be embedded into a group $Q$ such that $PJT(Q) = JT(Q) = X(Q)$. Hence the equation (2.1) stable over $Q$.*

*Proof.* Let us fix an arbitrary infinite Abelian group $A$ such that $A = A^2$. Let us choose a group $H$ satisfying Lemma 5.3.

Let us verify that the equation (2.1) is stable on $Q = H \wr A$. Denote by $D$ the subgroup of $Q$ generated by $H(b)$, $b \in A$. The group $D$ satisfies condition $D^2 = D$. By Theorem 5.2 we have $PJT(Q) = PX(Q) = X(A) \oplus PX(D, A)$.

Let us verify that $PX(D, A) = X(D, A)$. Suppose that $f \in PX(D, A)$ Let $b_i$ for $i \in \mathbb{N}$ be distinct elements from $A$. Let $a, \alpha \in H$. Consider elements $u_k = a(b_1)a(b_2)\cdots a(b_k)$ and $v_k = \alpha(b_1)\alpha(b_2)\cdots\alpha(b_k)$. Then by Corollary 2.17, for any $k \in \mathbb{N}$, we have

$$|f(u_k v_k) - f(u_k) - f(v_k)| = \left|\sum_{i=1}^{k}[f(a\alpha(b_i)) - f(a(b_i)) - f(\alpha(b_i))]\right|.$$

By formula (5.6), we have $f(d(b_i)) = f(d^{b_i}(1)) = f(d(1))$ for any $d \in A$ and for any $i \in \mathbb{N}$. Let $r = f(a\alpha) - f(a) - f(\alpha)$. Hence $r = f(a\alpha(b_i)) - f(a(b_i)) - f(\alpha(b_i))$ for any $i \in N$. Therefore

$$\begin{aligned} |f(u_k v_k) - f(u_k) - f(v_k)| &= \left|\sum_{i=1}^{k}[f(a\alpha(b_i)) - f(a(b_i)) - f(\alpha(b_i))]\right|. \\ &= \left|k[f(a\alpha(1)) - f(a(1)) - f(\alpha(1))]\right|. \\ &= k\,|r|. \end{aligned}$$

Further we have

$$|f(u_k v_k) - f(u_k) - f(v_k)| \le c.$$

Hence

$$k\,|r| < c$$

and

$$|r| \leq c\frac{1}{k} \quad \forall k \in \mathbb{N}.$$

The latter is possible only if $r = 0$. Thus $f(a\alpha) - f(a) - f(\alpha) = 0$ and $f \in X(D, A)$. Hence $PJT(Q) = X(A) \oplus X(D, A)$. And we see that $PJT(Q) = X(Q)$. So the equation is stable (2.1) on the group $Q$.

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