

ON C -BOCHNER CURVATURE TENSOR OF A CONTACT METRIC MANIFOLD

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ABSTRACT. We prove that a (κ, μ) -manifold with vanishing E -Bochner curvature tensor is a Sasakian manifold. Several interesting corollaries of this result are drawn. Non-Sasakian (κ, μ) -manifolds with C -Bochner curvature tensor B satisfying $B(\xi, X) \cdot S = 0$, where S is the Ricci tensor, are classified. $N(\kappa)$ -contact metric manifolds M^{2n+1} , satisfying $B(\xi, X) \cdot R = 0$ or $B(\xi, X) \cdot B = 0$ are classified and studied.

1. Introduction

In [4], Blair, Koufogiorgos and Papantoniou introduced the class of contact metric manifolds M with contact metric structures (φ, ξ, η, g) , in which the curvature tensor R satisfies the equation

$$R(X, Y)\xi = (\kappa I + \mu h)R_0(X, Y)\xi, \quad X, Y \in TM,$$

where $(\kappa, \mu) \in \mathbb{R}^2$, $2h$ is the Lie derivative of φ in the direction ξ and R_0 is given by

$$R_0(X, Y)Z = g(Y, Z)X - g(X, Z)Y, \quad X, Y, Z \in TM.$$

A contact Riemannian manifold belonging to this class is called a (κ, μ) -manifold. Characteristic examples of non-Sasakian (κ, μ) -manifolds are the tangent sphere bundles of Riemannian manifolds of constant sectional curvature not equal to one.

On the other hand, S. Bochner introduced a Kähler analogue of the Weyl conformal curvature tensor by purely formal considerations, which is now well known as the Bochner curvature tensor [5]. A geometric

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meaning of the Bochner curvature tensor is given by D. Blair in [1]. By using the Boothby-Wang's fibration [6], M. Matsumoto and G. Chūman constructed *C-Bochner curvature tensor* [10] from the Bochner curvature tensor. In [7], H. Endo defined *E-Bochner curvature tensor* as an extended *C-Bochner curvature tensor* and showed that a *K-contact manifold* with vanishing *E-Bochner curvature tensor* is a Sasakian manifold.

A *K-contact manifold* is always a contact metric manifold, but the converse is not true in general. Thus, it is worthwhile to study *C-Bochner curvature tensor* and *E-Bochner curvature tensor* in contact metric manifolds. Extending the result of H. Endo[7] to a (κ, μ) -manifold we prove the following

THEOREM 1.1. *A (κ, μ) -manifold with vanishing *E-Bochner curvature tensor* is a Sasakian manifold.*

Then, we draw several corollaries of this result to $N(\kappa)$ -contact metric manifolds [15], the unit tangent sphere bundles [4], $N(\kappa)$ -contact space forms [8] and (κ, μ) -space forms [9].

In [12] and [14], contact metric manifolds satisfying $R(X, \xi) \cdot S = 0$ are studied. Motivated by these studies, we classify non-Sasakian (κ, μ) -manifolds with *C-Bochner curvature tensor* B satisfying $B(\xi, X) \cdot S = 0$. In fact, we prove the following

THEOREM 1.2. *Let M^{2n+1} be a non-Sasakian (κ, μ) -manifold. If the *C-Bochner curvature tensor* B satisfies $B(\xi, X) \cdot S = 0$, then we have one of the following:*

- (i) M^{2n+1} is flat and 3-dimensional;
- (ii) M^{2n+1} is locally isometric to $E^{n+1}(0) \times S^n(4)$;
- (iii) M^{2n+1} is an η -Einstein manifold;
- (iv) M^{2n+1} is a 3-dimensional Einstein manifold.

In a recent paper [13], $N(\kappa)$ -contact metric manifolds satisfying $R(X, Y) \cdot B = 0$ are studied. Hence, we classify $N(\kappa)$ -contact metric manifolds satisfying $B(\xi, X) \cdot R = 0$. In fact, we prove the following

THEOREM 1.3. *Let M^{2n+1} ($n > 1$) be an $N(\kappa)$ -contact metric manifold. If the *C-Bochner curvature tensor* B satisfies $B(\xi, X) \cdot R = 0$, then either M^{2n+1} is a Sasakian manifold or M^{2n+1} is locally isometric to the sphere $S^{2n+1}(1)$.*

In the last, motivated by studies of contact metric manifolds satisfying $R(X, \xi) \cdot R = 0$ ([12, 14]), we consider $N(\kappa)$ -contact metric manifolds satisfying $B(\xi, X) \cdot B = 0$. In fact, we prove the following

THEOREM 1.4. *Let M^{2n+1} be an $N(\kappa)$ -contact metric manifold. Then the C -Bochner curvature tensor B satisfies $B(\xi, X) \cdot B = 0$ if and only if M^{2n+1} is a Sasakian manifold.*

2. Contact metric manifolds

A $(2n + 1)$ -dimensional differentiable manifold M is called an almost contact manifold if either its structural group can be reduced to $U(n) \times 1$ or equivalently, there is an almost contact structure (φ, ξ, η) consisting of a $(1, 1)$ tensor field φ , a vector field ξ , and a 1-form η satisfying

$$(1) \quad \varphi^2 = -I + \eta \otimes \xi, \quad \text{and (one of)} \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

An almost contact structure is said to be *normal* if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by

$$(2) \quad J\left(X, \lambda \frac{d}{dt}\right) = \left(\varphi X - \lambda\xi, \eta(X) \frac{d}{dt}\right)$$

is integrable, where X is tangent to M , t the coordinate of \mathbb{R} and λ a smooth function on $M \times \mathbb{R}$. The condition for being normal is equivalent to vanishing of the torsion tensor $[\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ . Let g be a compatible Riemannian metric with (φ, ξ, η) , that is,

$$(3) \quad g(X, Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y)$$

or equivalently,

$$(4) \quad g(X, \varphi Y) = -g(\varphi X, Y) \quad \text{and} \quad g(X, \xi) = \eta(X)$$

for all $X, Y \in TM$. Then, M becomes an *almost contact metric manifold* equipped with an almost contact metric structure (φ, ξ, η, g) .

An almost contact metric structure becomes a *contact metric structure* if

$$(5) \quad g(X, \varphi Y) = d\eta(X, Y), \quad X, Y \in TM.$$

In a contact metric manifold, the $(1, 1)$ -tensor field h is symmetric and satisfies

$$(6) \quad h\xi = 0, \quad h\varphi + \varphi h = 0, \quad \nabla\xi = -\varphi - \varphi h, \quad \text{trace}(h) = \text{trace}(\varphi h) = 0,$$

where ∇ is the Levi-Civita connection.

A normal contact metric manifold is a *Sasakian manifold*. An almost contact metric manifold is Sasakian if and only if

$$(7) \quad \nabla_X \varphi = R_0(\xi, X), \quad X \in TM,$$

while a contact metric manifold M is Sasakian if and only if

$$(8) \quad R(X, Y)\xi = R_0(X, Y)\xi, \quad X, Y \in TM.$$

A contact metric manifold is called a K -contact manifold if ξ is a Killing vector field. An almost contact metric manifold is K -contact if and only if $\nabla\xi = -\varphi$. A K -contact manifold is a contact metric manifold, while converse is true if $h = 0$. A Sasakian manifold is always a K -contact manifold. A 3-dimensional K -contact manifold is a Sasakian manifold. Thus, a 3-dimensional contact metric manifold is Sasakian if and only if $h = 0$.

The (κ, μ) -nullity distribution $N(\kappa, \mu)$ ([4, 12]) of a contact metric manifold M is defined by

$$\begin{aligned} N(\kappa, \mu) : p &\rightarrow N_p(\kappa, \mu) \\ &= \{U \in T_p M \mid R(X, Y)U = (\kappa I + \mu h)R_0(X, Y)U\} \end{aligned}$$

for all $X, Y \in TM$, where $(\kappa, \mu) \in \mathbb{R}^2$. A contact metric manifold with $\xi \in N(\kappa, \mu)$ is called a (κ, μ) -manifold. For a (κ, μ) -manifold, it follows that $h^2 = (\kappa - 1)\varphi^2$. This class contains Sasakian manifolds for $\kappa = 1$ and $h = 0$. In fact, for a (κ, μ) -manifold, the conditions of being Sasakian manifold, K -contact manifold, $\kappa = 1$ and $h = 0$ are all equivalent. If $\mu = 0$, the (κ, μ) -nullity distribution $N(\kappa, \mu)$ is reduced to the κ -nullity distribution $N(\kappa)$ ([15]). If $\xi \in N(\kappa)$, then we call a contact metric manifold M an $N(\kappa)$ -contact metric manifold. For more details we refer to [3].

3. (κ, μ) -manifolds with vanishing E -Bochner curvature tensor

In [10], Matsumoto and Chūman defined the C -Bochner curvature tensor in an almost contact metric manifold as follows:

$$(9) \quad \begin{aligned} B(X, Y) = & R(X, Y) - \frac{m-4}{2n+4}R_0(Y, X) + \frac{1}{2n+4} \left\{ R_0(QY, X) \right. \\ & - R_0(QX, Y) + R_0(Q\varphi Y, \varphi X) - R_0(Q\varphi X, \varphi Y) \\ & + 2g(Q\varphi X, Y)\varphi + 2g(\varphi X, Y)Q\varphi + \eta(Y)R_0(QX, \xi) \\ & \left. + \eta(X)R_0(\xi, QY) \right\} - \frac{m+2n}{2n+4} \left\{ R_0(\varphi Y, \varphi X) \right. \\ & \left. + 2g(\varphi X, Y)\varphi \right\} + \frac{m}{2n+4} \left\{ \eta(Y)R_0(\xi, X) + \eta(X)R_0(Y, \xi) \right\} \end{aligned}$$

where Q is the Ricci operator, r is the scalar curvature and $m = \frac{2n+r}{2n+2}$.

For a (κ, μ) -manifold M^{2n+1} , we have

$$(10) \quad R(X, Y)\xi = (\kappa I + \mu h)R_0(X, Y)\xi,$$

which is equivalent to

$$(11) \quad R(\xi, X) = R_0(\xi, (\kappa I + \mu h)X) = -R(X, \xi).$$

In particular, we get

$$(12) \quad R(\xi, X)\xi = \kappa(\eta(X)\xi - X) - \mu hX = -R(X, \xi)\xi.$$

From (9), (10), and (11), it follows that

$$(13) \quad B(X, Y)\xi = \left(\frac{2(\kappa - 1)}{n + 2}I + \mu h\right)R_0(X, Y)\xi,$$

$$(14) \quad B(\xi, X) = R_0\left(\xi, \left(\frac{2(\kappa - 1)}{n + 2}I + \mu h\right)X\right) = -B(X, \xi).$$

Consequently, we have

$$(15) \quad B(\xi, X)\xi = \frac{2(\kappa - 1)}{n + 2}(\eta(X)\xi - X) - \mu hX = -B(X, \xi)\xi,$$

$$(16) \quad \eta(B(X, Y)\xi) = 0,$$

$$(17) \quad \eta(B(\xi, X)Y) = \frac{2(\kappa - 1)}{n + 2}(g(X, Y) - \eta(X)\eta(Y)) + \mu g(hX, Y).$$

In [7], H. Endo extended the concept of C -Bochner curvature tensor to E -Bochner curvature tensor as follows:

$$(18) \quad \begin{aligned} B^e(X, Y)Z &= B(X, Y)Z - \eta(X)B(\xi, Y)Z \\ &\quad - \eta(Y)B(X, \xi)Z - \eta(Z)B(X, Y)\xi. \end{aligned}$$

Then, he showed that a K -contact manifold with vanishing E -Bochner curvature tensor is a Sasakian manifold. Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Let M^{2n+1} be a (κ, μ) -manifold. If E -Bochner curvature tensor of M^{2n+1} vanishes, then from (15) and (18), we have

$$(19) \quad 0 = B^e(X, \xi)\xi = -B(X, \xi)\xi = \frac{2(\kappa - 1)}{n + 2}(\eta(X)\xi - X) - \mu hX,$$

which, in view of $h^2 = (\kappa - 1)\varphi^2$, implies that

$$(20) \quad h^2 = \frac{(n + 2)\mu}{2}h.$$

Taking the trace of (20), we obtain

$$(21) \quad 0 = \text{trace}(h^2) = 2n(1 - \kappa).$$

From (21), we have $\kappa = 1$. Thus, M^{2n+1} becomes Sasakian.

COROLLARY 3.1. *An $N(\kappa)$ -contact metric manifold with vanishing E-Bochner curvature tensor is a Sasakian manifold.*

The unit tangent sphere bundle T_1M equipped with the standard contact metric structure is a (κ, μ) -manifold if and only if the base manifold M is of constant curvature c with $\kappa = c(2 - c)$ and $\mu = -2c$ ([4]). In case of $c \neq 1$, the unit tangent sphere bundle is non-Sasakian [16]. We denote by $T_1M(c)$ the unit tangent sphere bundle of a space of constant curvature c with standard contact metric structure. Then, applying Theorem 1.1 to $T_1M(c)$, we have

COROLLARY 3.2. *If $T_1M(c)$ is of vanishing E-Bochner curvature tensor, then $c = 1$.*

In an almost contact metric manifold, if X is a unit vector which is orthogonal to ξ , we say that X and φX span a φ -section. If the sectional curvature $c(X)$ of all φ -sections is independent of X , we say that M is of pointwise constant φ -sectional curvature. If an $N(\kappa)$ -contact metric manifold M is of pointwise constant φ -sectional curvature c , then we say it an $N(\kappa)$ -contact space form $M(c)$. The curvature tensor of $M(c)$ is given by [8].

$$(22) \quad \begin{aligned} 4R(X, Y)Z &= (c + 3) \{g(Y, Z)X - g(X, Z)Y\} \\ &+ (c - 1) \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \eta(Y)g(X, Z)\xi - \eta(X)g(Y, Z)\xi \\ &+ g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2(\varphi X, Y)\varphi Z\} \\ &+ 4(\kappa - 1) \{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ &+ \eta(X)g(Y, Z)\xi - \eta(Y)g(X, Z)\xi\} \\ &+ 4 \{g(hY, Z)X - g(hX, Z)Y \\ &+ g(Y, Z)hX - g(X, Z)hY \\ &+ \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX \\ &+ \eta(Y)g(hX, Z)\xi - \eta(X)g(hY, Z)\xi\} \\ &+ 2 \{g(hY, Z)hX - g(hX, Z)hY \\ &+ g(\varphi hX, Z)\varphi hY - g(\varphi hY, Z)\varphi hX\} \end{aligned}$$

for all $X, Y, Z \in TM$, where c is a constant on M if $\dim(M) > 3$.

Now, applying Theorem 1.1 to an $N(\kappa)$ -contact space form, we are able to state the following

COROLLARY 3.3. *An $N(\kappa)$ -contact space form with vanishing E -Bochner curvature tensor is a Sasakian space form.*

Let M be a $(2n + 1)$ -dimensional (κ, μ) -manifold ($n > 1$). If M has constant φ -sectional curvature c then it is called a (κ, μ) -space form and is denoted by $M(c)$. The curvature tensor of $M(c)$ is given by [9]

$$\begin{aligned}
 & R(X, Y)Z \\
 = & \frac{c + 3}{4} \{g(Y, Z) X - g(X, Z) Y\} \\
 & + \frac{c - 1}{4} \{2(X, \varphi Y) \varphi Z + g(X, \varphi Z) \varphi Y - g(Y, \varphi Z) \varphi X\} \\
 & + \frac{c + 3 - 4\kappa}{4} \{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\
 (23) \quad & + g(X, Z) \eta(Y)\xi - g(Y, Z) \eta(X)\xi\} \\
 & + \frac{1}{2} \{g(hY, Z) hX - g(hX, Z) hY \\
 & + g(\varphi hX, Z) \varphi hY - g(\varphi hY, Z) \varphi hX\} \\
 & + g(\varphi Y, \varphi Z) hX - g(\varphi X, \varphi Z) hY \\
 & + g(hX, Z) \varphi^2 Y - g(hY, Z) \varphi^2 X \\
 & + \mu \{\eta(Y)\eta(Z)hX - \eta(X)\eta(Z)hY \\
 & + g(hY, Z) \eta(X)\xi - g(hX, Z) \eta(Y)\xi\}
 \end{aligned}$$

for all $X, Y, Z \in TM$, where $c + 2\kappa = -1 = \kappa - \mu$ if $\kappa < 1$.

Now, applying Theorem 1.1 to a (κ, μ) -contact space form, we are able to state the following

COROLLARY 3.4. *A (κ, μ) -contact space form with vanishing E -Bochner curvature tensor is a Sasakian space form.*

REMARK 3.5. Theorem 1.1, Corollary 3.1, Corollary 3.2, Corollary 3.3 and Corollary 3.4 are valid for vanishing of C -Bochner curvature tensor also.

4. (κ, μ) -manifolds satisfying $B(\xi, X) \cdot S = 0$

Before proving Theorem 1.2, we give some results and a brief introduction to η -Einstein (κ, μ) -manifold.

LEMMA 4.1. [4] In a non-Sasakian (κ, μ) -manifold M^{2n+1} , the Ricci operator Q is given by

$$(24) \quad Q = (2(n-1) - n\mu)I + (2(n-1) + \mu)h + (2(1-n) + n(2\kappa + \mu))\eta \otimes \xi.$$

Consequently, the Ricci tensor S is given by

$$(25) \quad \begin{aligned} S(X, Y) = & (2(n-1) - n\mu)g(X, Y) + (2(n-1) + \mu)g(hX, Y) \\ & + (2(1-n) + n(2\kappa + \mu))\eta(X)\eta(Y). \end{aligned}$$

We also recall the following theorem due to D. Blair.

THEOREM 4.2. [2, 3] A contact metric manifold M^{2n+1} satisfying $R(X, Y)\xi = 0$ is locally isometric to $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$.

THEOREM 4.3. [15] If an $N(k)$ -contact metric manifold of dimension ≥ 5 is Einstein, then it is necessarily Sasakian.

We also need the following definition.

DEFINITION 4.4. [11] A contact metric manifold M is said to be η -Einstein if the Ricci operator Q satisfies

$$Q = aI + b\eta \otimes \xi,$$

where a and b are smooth functions on the manifold. In particular, if $b = 0$, then M is an Einstein manifold.

Now, we prove Theorem 1.2.

Proof of Theorem 1.2. Let M^{2n+1} be a non-Sasakian (κ, μ) -manifold. If $\kappa = 0 = \mu$, then from (10), we have $R(X, Y)\xi = 0$, which in view of Theorem 4.2 implies that M^{2n+1} satisfies one of the statements (i) and (ii). Now, we assume that κ and μ are not simultaneously zero. From (25), we have

$$(26) \quad \begin{aligned} S(hX, Y) = & (2(n-1) - n\mu)g(hX, Y) \\ & - (\kappa - 1)(2(n-1) + \mu)g(X, Y) \\ & + (\kappa - 1)(2(n-1) + \mu)\eta(X)\eta(Y), \end{aligned}$$

where we have used $\eta \circ h = 0$, $h^2 = (\kappa - 1)\varphi^2$ and (3). The condition $B(\xi, X) \cdot S = 0$ gives

$$(27) \quad S(B(\xi, X)Y, \xi) + S(Y, B(\xi, X)\xi) = 0.$$

In view of $Q\xi = 2n\kappa\xi$, we get

$$(28) \quad S(X, \xi) = 2n\kappa\eta(X),$$

which implies that

$$(29) \quad S(B(\xi, X)Y, \xi) = 2n\kappa\eta(B(\xi, X)Y),$$

Using (17) in the above equation, we get

$$(30) \quad \begin{aligned} S(B(\xi, X)Y, \xi) &= 2n\kappa\mu g(hX, Y) \\ &+ \frac{4n\kappa(\kappa - 1)}{n + 2} (g(X, Y) - \eta(X)\eta(Y)). \end{aligned}$$

In view of (15) and (28), we have

$$(31) \quad \begin{aligned} S(B(\xi, X)\xi, Y) &= \frac{4n\kappa(\kappa - 1)}{n + 2} \eta(X)\eta(Y) \\ &- \frac{2(\kappa - 1)}{n + 2} S(X, Y) - \mu S(hX, Y). \end{aligned}$$

From (27), (30), and (31), we have

$$(32) \quad \begin{aligned} \frac{2(\kappa - 1)}{n + 2} S(X, Y) &= \frac{4n\kappa(\kappa - 1)}{n + 2} g(X, Y) \\ &+ 2n\mu g(hX, Y) - \mu S(hX, Y). \end{aligned}$$

Finally, from (25), (26), and (32), we have

$$(33) \quad S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where

$$a = \frac{\frac{(\kappa - 1)(4n\kappa + (n + 2)(2(n - 1) + \mu)\mu)}{n + 2} - \frac{\mu(2 + n\mu)(2(n - 1) - n\mu)}{2(n - 1) + \mu}}{\frac{2(\kappa - 1)}{n + 2} - \frac{\mu(2 + n\mu)}{2(n - 1) + \mu}},$$

and

$$(34) \quad b = -\frac{\frac{\mu((\kappa - 1)(2(n - 1) + \mu)^2 + (2 + n\mu)(n(2\kappa + \mu) - 2(n - 1)))}{2(n - 1) + \mu}}{\frac{2(\kappa - 1)}{n + 2} - \frac{\mu(2 + n\mu)}{2(n - 1) + \mu}}$$

Thus, if $\mu \neq 0$, then M^{2n+1} is an η -Einstein manifold, which is the statement (iii). If $\mu = 0$ and $\kappa \neq 0$, then in view of (34), we get $b = 0$ in (33); thus M^{2n+1} becomes an Einstein manifold. Moreover, if $n > 1$, in view of Theorem 4.3, we conclude that M^{2n+1} is Sasakian, which is a contradiction. Therefore we have the statement (iv). This completes the proof.

In view of Theorem 1.2, we are able to state the following two corollaries.

COROLLARY 4.5. *Let M^{2n+1} be a non-Sasakian $N(\kappa)$ -contact metric manifold of dimension ≥ 5 . If the C -Bochner curvature tensor B satisfies $B(\xi, X) \cdot S = 0$, then M^{2n+1} is locally isometric to $E^{n+1}(0) \times S^n(4)$.*

COROLLARY 4.6. *Let M^{2n+1} be a non-Sasakian $N(\kappa)$ -contact metric manifold with $n > 1$ and $\kappa \neq 0$. Then, the C -Bochner curvature tensor B never satisfies $B(\xi, X) \cdot S = 0$.*

5. $N(\kappa)$ -contact metric manifolds satisfying $B(\xi, X) \cdot R = 0$

In an $N(\kappa)$ -contact metric manifold M^{2n+1} , we have

$$(35) \quad R(X, Y)\xi = \kappa R_0(X, Y)\xi,$$

$$(36) \quad R(\xi, X) = \kappa R_0(\xi, X) = -R(X, \xi).$$

Consequently,

$$(37) \quad B(X, Y)\xi = \frac{2(\kappa - 1)}{n + 2} R_0(X, Y)\xi,$$

$$(38) \quad B(\xi, X) = \frac{2(\kappa - 1)}{n + 2} R_0(\xi, X) = -B(X, \xi).$$

Proof of Theorem 1.3. The condition $B(\xi, X) \cdot R = 0$ gives

$$0 = [B(\xi, X), R(Y, Z)]\xi - R(B(\xi, X)Y, Z)\xi - R(Y, B(\xi, X)Z)\xi,$$

which in view of (38) provides

$$0 = \frac{2(\kappa - 1)}{n + 2} \{g(X, R(Y, Z)\xi)\xi - \eta(R(Y, Z)\xi)X - g(X, Y)R(\xi, Z)\xi + \eta(Y)R(X, Z)\xi - g(X, Z)R(Y, \xi)\xi + \eta(Z)R(Y, X)\xi - \eta(X)R(Y, Z)\xi + R(Y, Z)X\}$$

Using (36), we get

$$\frac{2(\kappa - 1)}{n + 2} (R(Y, Z) - \kappa R_0(Y, Z)) = 0.$$

Therefore, either $\kappa = 1$ or

$$(39) \quad R(Y, Z)X = \kappa (g(X, Z)Y - g(X, Y)Z).$$

It is well known that, except for the flat 3-dimensional case, a contact metric manifold of constant curvature is Sasakian and of constant curvature $+1$. Therefore, in view of (39), M is locally isometric to the sphere $S^{2n+1}(1)$. Thus, the proof is complete.

REMARK 5.1. If the $N(\kappa)$ -contact metric manifold is assumed to be complete and simply connected, then in the Theorem 1.3, local isometry is replaced by global isometry.

6. $N(\kappa)$ -contact metric manifolds satisfying $B(\xi, X) \cdot B = 0$

This section is devoted to the proof of Theorem 1.4.

Proof of Theorem 1.4. The condition $B(\xi, X) \cdot B = 0$ gives

$$0 = [B(\xi, X), B(Y, Z)]\xi - B(B(\xi, X)Y, Z)\xi - B(Y, B(\xi, X)Z)\xi,$$

which in view of (38) provides

$$\begin{aligned} 0 = & \frac{2(\kappa - 1)}{n + 2} \{g(X, B(Y, Z)\xi)\xi - \eta(B(Y, Z)\xi)X - g(X, Y)B(\xi, Z)\xi \\ & + \eta(Y)B(X, Z)\xi - g(X, Z)B(Y, \xi)\xi + \eta(Z)B(Y, X)\xi \\ & - \eta(X)B(Y, Z)\xi + B(Y, Z)X\}. \end{aligned}$$

Using (37), we get

$$\frac{2(\kappa - 1)}{n + 2} \left(R(Y, Z) - \frac{2(\kappa - 1)}{(n + 2)} R_0(Y, Z) \right) = 0.$$

Therefore, either $\kappa = 1$ or

$$B(Y, Z)X = \frac{2(\kappa - 1)}{(n + 2)} (g(X, Z)Y - g(X, Y)Z).$$

Contracting Y in the above equation, we conclude that

$$0 = \frac{2(\kappa - 1)}{(n + 2)} (2ng(X, Z)),$$

which gives $\kappa = 1$. Thus in the both cases M^{2n+1} is a Sasakian manifold.

Conversely, if M^{2n+1} is a Sasakian manifold, then in view of (38) we have $B(\xi, X) \cdot B = 0$.

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